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Symmetric Banach *-algebras: invariance of spectrum

by

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Abstract. Let A be a Banach *-algebra which is a subalgebra of a Banach algebra B. In this paper, assuming that A is symmetric, various conditions are given which imply that A is inverse closed in B.

1. Introduction. Let D be a complex unital algebra. The group of invertible elements in D is denoted $\mathrm{Inv}(D)$. For $d \in D$, $\sigma(d; D)$ denotes the spectrum of d relative to D, and $r(d; D)$ denotes the spectral radius of d relative to D: $r(d; D) \equiv \sup\{|\lambda| : \lambda \in \sigma(d; D)\}$. When D is a *-algebra, D_{sa} is the set of elements $d \in D$ with $d = d^*$.

Throughout, A is always a Banach *-algebra which is a subalgebra of a unital Banach algebra B, and A contains the unit of B (the results in this paper are valid in the nonunital case). Recall that A is *symmetric* if for every $a \in A$, $\sigma(a^*a; A) \subseteq [0, \infty)$. In this paper, assuming that A is symmetric, we study the relationships among the following concepts:

DEFINITION 1. (1) A is *inverse closed* in B if whenever $a \in A$ and $a^{-1} \in B$, then $a^{-1} \in A$.

(2) A is **-inverse closed* in B if whenever $a \in A_{\mathrm{sa}}$ and $a^{-1} \in B$, then $a^{-1} \in A$.

(3) A is SRP in B if $r(a; A) = r(a; B)$ for all $a \in A$ (SRP stands for "spectral radius preserving").

The property "A is inverse closed in B" is a strong property which is obviously equivalent to "$\sigma(a; A) = \sigma(a; B)$ for all $a \in A$". On the other hand, the property "A is *-inverse closed in B" is a fairly weak property. In particular, it does not imply in general that "$\sigma(a; A) = \sigma(a; B)$ for all $a \in A_{\mathrm{sa}}$"; see the example in Section 4.

The two questions listed below remain unanswered. Question II is classical. Question I is more general than Question II, since a C^*-algebra A is

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symmetric [R, Theorem (4.8.9), p. 243], and automatically, A is *-inverse closed in B [R, Theorem (4.8.3), p. 240].

QUESTION I. Let A be a symmetric Banach *-algebra which is a subalgebra of a Banach algebra B. If A is *-inverse closed in B, then is A inverse closed in B?

QUESTION II. Let A be a C^*-algebra which is a subalgebra of a Banach algebra B. Is A inverse closed in B?

REMARKS. R1. It follows from the Shirali–Ford Theorem [BD, Thm. 5, p. 226] that A is symmetric if and only if for all $a \in A_{\mathrm{sa}}$, $\sigma(a; A) \subseteq \mathbb{R}$. The term *hermitian* is often used in place of symmetric.

R2. Assume that A is a C^*-algebra. Then, as mentioned above, A is symmetric [R, Theorem (4.8.9), p. 243]. Also, in this case:

(i) $\sigma(a; A) = \sigma(a; B)$ for all normal elements a in A (normal means $a^*a = aa^*$). This implies that A is *-inverse closed in B.

(ii) A is SRP in B.

Both (i) and (ii) follow from [R, Theorem (4.8.3), p. 240].

R3. Let A be a commutative semisimple symmetric algebra with largest C^*-norm γ, and assume that A has another C^*-norm β with $\gamma \neq \beta$. An example of an algebra with these properties is given in the author's paper [B1, Example 2.8]. Fix $a \in A$ such that $\beta(a) < \gamma(a)$. Then $\beta(a^*a) = \beta(a)^2 < \gamma(a)^2 = \gamma(a^*a)$. Let B be the completion of (A, β). Now $\gamma(a)^2 \leq r(a^*a; A)$ by [PT, (4.5)], and $r(a^*a; B) = \beta(a^*a)$ as B is a C^*-algebra. Therefore,

$$r(a^*a; B) = \beta(a^*a) < \gamma(a^*a) = \gamma(a)^2 \leq r(a^*a; A),$$

and it follows that $\sigma(a^*a; A) \neq \sigma(a^*a; B)$. In particular, this implies that A is not *-inverse closed in B.

R4. The author proves in [B3, Theorem] that when A is symmetric and a closed subalgebra of B, and either (i) the embedding of A into B is continuous, or (ii) A is Jacobson semisimple, then A is inverse closed in B. In particular, when A is a C^*-algebra and A is closed in B, then A is inverse closed in B.

R5. D. Goldstein recently proved in [G, Theorem 3] that when A is a C^*-algebra and $\|a\|_A \leq \|a\|_B$ for all $a \in A$, then A is inverse closed in B.

R6. When the algebra A is commutative the answer to both questions is *yes*. There are a number of ways of seeing this. Corollary 4 of this paper provides one proof. In the case where A is a commutative C^*-algebra, that A is inverse closed in B follows from [R, Theorem (4.8.3)(i), p. 240].

2. Preliminary results. In this brief section, we prove two results without the assumption that A is symmetric. In fact, the next result does not involve an involution at all.

PROPOSITION 2. *Let A^- denote the closure of A in B. The following two statements are equivalent*:

(1) *A is SRP in B*;

(2) *A is inverse closed in A^-.*

Proof. That (2) implies (1) is immediate. Now assume that A is SRP in B. Suppose that $a \in A$ and $a^{-1} \in A^-$. Choose $\{a_k\} \subseteq A$ such that $\|a^{-1} - a_k\|_B \to 0$. Then $\|1 - aa_k\|_B \to 0$. Fix m such that $\|1 - aa_m\|_B < 1$. Then $r(1 - aa_m; A) = r(1 - aa_m; B) < 1$. Therefore $aa_m \in \mathrm{Inv}(A)$ by basic spectral theory. Choose $c \in A$ with $aa_mc = 1$. Then $a^{-1} = a_mc \in A$. This proves that (1) implies (2).

For λ a complex number, we use the notation λ^* to denote the complex conjugate of λ. Also, when $E \subseteq \mathbb{C}$, $E^* = \{\lambda^* : \lambda \in E\}$.

PROPOSITION 3. *In this result, A is a *-algebra which is a subalgebra of an algebra B; there are no topological assumptions. The following two statements are equivalent*:

(1) *A is *-inverse closed in B*;

(2) *for all $a \in A$, $\sigma(a; A) = \sigma(a; B) \cup \sigma(a^*; B)^*$.*

Proof. Assume that A is *-inverse closed in B. First note that $\sigma(a; A) = \sigma(a^*; A)^*$ for all $a \in A$ [R, Lemma (4.1.1)]. Also, $\sigma(a; B) \subseteq \sigma(a; A)$, and $\sigma(a^*; B)^* \subseteq \sigma(a^*; A)^* = \sigma(a; A)$. Therefore, $\sigma(a; A) \supseteq \sigma(a; B) \cup \sigma(a^*; B)^*$. To verify the opposite inclusion, suppose that $\lambda \notin \sigma(a; B) \cup \sigma(a^*; B)^*$. Thus $\lambda - a$ and $(\lambda - a)^*$ are in $\mathrm{Inv}(B)$. Since A is *-inverse closed in B, it follows that $(\lambda - a)(\lambda - a)^*$ and $(\lambda - a)^*(\lambda - a)$ are in $\mathrm{Inv}(A)$. This implies that $\lambda - a \in \mathrm{Inv}(A)$, and so, $\lambda \notin \sigma(a; A)$. This proves that (1)⇒(2).

That (2)⇒(1) is immediate.

3. Results when A is symmetric

COROLLARY 4 (of Proposition 3). *Assume that A is symmetric and *-inverse closed in B. Then*

$$\sigma(a; A) = \sigma(a; B) \quad \textit{for all } a \in A_{\mathrm{sa}}.$$

Also, this equality of spectrum holds for all normal elements ($a^*a = aa^*$) in A.

Proof. For $a = a^* \in A$, $\sigma(a; B) \subseteq \sigma(a; A) \subseteq \mathbb{R}$. Applying Proposition 3, we have $\sigma(a; A) = \sigma(a; B) \cup \sigma(a; B)^* = \sigma(a; B)$.

The equality of spectra for normal elements follows as in the proof of part (1) of Theorem 2.2 in [B2].

It is a basic fact (see [P2, Cor. 2.5.8, p. 253]) that

$$A \text{ SRP in } B \Rightarrow \partial\sigma(a; A) \subseteq \sigma(a; B) \subseteq \sigma(a; A) \text{ for all } a \in A.$$

NOTE 5. Assume that A is symmetric. If $a, b \in A_{\mathrm{sa}}$, then

$$r(ab;A) \le r(a;A)r(b;A) \qquad \text{[PT, (5.3), p. 24]}. \tag{1}$$

It is easy to extend this property to normal elements. For assume a and b are normal elements of A. By [PT, Thm. 5,2, p. 23], $r(ab;A)^2 \le r(b^*a^*ab;A)$. Also

$$\begin{aligned} r(b^*a^*ab;A) &= r(a^*abb^*;A)\\ &\le r(a^*a;A)r(bb^*;A) \qquad \text{(by (1))}\\ &\le r(a^*;A)r(a;A)r(b;A)r(b^*;A) \qquad \text{(by normality)}\\ &= r(a;A)^2 r(b;A)^2. \end{aligned}$$

THEOREM 6. *Assume that A is symmetric and continuously embedded in B. Assume that $a \in A_{\mathrm{sa}}$, and $\{a_k\} \subseteq A_{\mathrm{sa}}$ has the properties $\|a_k - a\|_A \to 0$ and $\sigma(a_k;A) = \sigma(a_k;B)$ for all k. Then $\sigma(a;A) = \sigma(a;B)$.*

Proof. Assume that $(\lambda - a)^{-1} \in B$. Choose $\varepsilon > 0$ such that $\sigma(a;B)$ is disjoint from $\{\mu \in \mathbb{C} : |\mu - \lambda| \le \varepsilon\}$. By the upper semicontinuity of the spectrum, there exists N such that

$$\sigma(a_k;A) = (a_k;B) \subseteq \{\mu \in \mathbb{C} : |\mu - \lambda| > \varepsilon\} \qquad \text{for } k \ge N.$$

Fix $k \ge N$. If $\mu \in \sigma((\lambda - a_k)^{-1};A)$, then $\lambda - \mu^{-1} \in \sigma(a_k;A)$, so $|\mu^{-1}| > \varepsilon$, $|\mu| < \varepsilon^{-1}$. Therefore,

$$r((\lambda - a_k)^{-1};A) \le \varepsilon^{-1}.$$

Now for $k \ge N$,

$$\begin{aligned} r(1 - (\lambda - a)(\lambda - a_k)^{-1};A) &= r((a - a_k)(\lambda - a_k)^{-1};A)\\ &\le r(a - a_k;A)r(\lambda - a_k)^{-1};A) \qquad \text{(by Note 5)}\\ &\le \|a - a_k\|_A \varepsilon^{-1} \to 0 \qquad \text{as } k \to \infty. \end{aligned}$$

This implies by standard spectral theory that $(\lambda - a)(\lambda - a_k)^{-1}$ is invertible in A for sufficiently large k. Thus, $\lambda - a \in \mathrm{Inv}(A)$. This proves $\sigma(a;A) = \sigma(a;B)$.

COROLLARY 7. *Assume that A is symmetric and continuously embedded in B. Assume that D is a *-subalgebra of A (not necessarily a Banach algebra) with the properties that:*

(i) *D_{sa} is dense in A_{sa};*
(ii) *D is *-inverse closed in B.*

Then for all $a \in A_{\mathrm{sa}}$, $\sigma(a;A) = \sigma(a;B)$.

Proof. For $d \in D_{\mathrm{sa}}$, since $\sigma(d;B) \subseteq \sigma(d;A) \subseteq \mathbb{R}$, we have $\sigma(d;B) = \sigma(d;B)^*$. Then applying Proposition 3, we have $\sigma(d;D) = \sigma(d;B) \cup \sigma(d;B)^* = \sigma(d;B)$. Now $D \subseteq A$, so automatically, $\sigma(d;A) \subseteq \sigma(d;D)$. This proves that $\sigma(d;B) = \sigma(d;A)$ for all $d \in D_{\mathrm{sa}}$. By (i) and Theorem 6, $\sigma(a;A) = \sigma(a;B)$ for all $a \in A_{\mathrm{sa}}$.

PROPOSITION 8. *Assume that A is symmetric.*

(1) *If $a, a^* \in A$ and $a^{-1}, (a^*)^{-1} \in B$, then $a^{-1}, (a^*)^{-1} \in A^{-}$.*
(2) *If A is SRP in B, then A is *-inverse closed in B, and $\sigma(a;B) = \sigma(a;A)$ for all $a = a^* \in A$.*

Proof. Assume as in (1) that $a^{-1}, (a^*)^{-1} \in B$. Since A is symmetric, for all $n \ge 1$, $(n^{-1} + a^*a)^{-1} \in A$. Then $\|(n^{-1} + a^*a)^{-1}a^* - a^{-1}\|_B \to 0$; and $\|a(n^{-1} + a^*a)^{-1} - (a^*)^{-1}\|_B \to 0$. This proves (1).

Now assume that A is SRP in B. Let $a \in A_{\mathrm{sa}}$, so $\sigma(a;A) \subseteq \mathbb{R}$. Then using the basic fact above, we have

$$\sigma(a;A) = \partial\sigma(a;A) \subseteq \sigma(a;B) \subseteq \sigma(a;A).$$

Therefore $\sigma(a;B) = \sigma(a;A)$. It follows immediately from this that A is *-inverse closed in B.

In the next theorem and its corollary, we give fairly minimal conditions that imply A is inverse closed in B in the case where A is a symmetric *-subalgebra of a *-algebra B.

THEOREM 9. *Let B be a unital Banach *-algebra, and suppose that A is a unital *-subalgebra of B. Assume that A is a symmetric Banach *-algebra. Also assume*

$$r(a^*a;A) = r(a^*a;B) \qquad \text{for all } a \in A.$$

Then A is inverse closed in B.

Proof. Assume that $a \in A$ is normal. Since A is symmetric, by [PT, Thm. 5.3, p. 23] we have $r(a;A)^2 \le r(a^*a;A)$. Also note that $\sigma(a;B) = \sigma(a^*;B)^*$, so $r(a;B) = r(a^*;B)$. Therefore,

$$r(a;A)^2 \le r(a^*a;A) = r(a^*a;B) \le r(a;B)r(a^*;B) \le r(a;B)^2.$$

Thus, $r(a;A) = r(a;B)$ for all normal elements of A.

Now assume that $a = a^* \in A$. Let C be a maximal commutative *-subalgebra of A with $a \in C$, and let D be a maximal commutative *-subalgebra of B with $C \subseteq D$. It follows that when $c \in C$ and $d \in D$, $\sigma(c;C) = \sigma(c;A)$ and $\sigma(d;D) = \sigma(d;B)$. For any $c \in C$, c is a normal element of A, so

$$r(c;C) = r(c;A) = r(c;B) = r(c;D).$$

Thus, C is SRP in D. By Proposition 8, C is *-inverse closed in D. If $a^{-1} \in B$, then $a^{-1} \in D$, and therefore, $a^{-1} \in A$. This argument proves that A is *-inverse closed in B.

Finally, for a general element $a \in A$ with $a^{-1} \in B$, we have $(a^*)^{-1} \in B$ (since B is a *-algebra). Therefore, setting $d = (aa^*)^{-1} = (a^*)^{-1}a^{-1}$, we have $d \in A$ (since A is *-inverse closed in B), so $a^{-1} = a^*d \in A$.

COROLLARY 10. *Let A and B be as in the statement of Theorem 9. Assume also that A has continuous involution and that A is continuously embedded in B. Then A is inverse closed in B if*

$$r(a^*a; A) = r(a^*a; B) \quad \text{for all } a \text{ in a dense subset } D \subseteq A.$$

Proof. We verify that $r(a^*a; A) = r(a^*a; B)$ for all $a \in A$. For suppose that for some $a \in A$, $r(a^*a; A) > r(a^*a; B)$. Choose β, $r(a^*a; A) > \beta > r(a^*a; B)$. Let U be the open set $U = \{\lambda \in \mathbb{C} : |\lambda| < \beta\}$, so that $\sigma(a^*a; B) \subseteq U$. Now choose a sequence $\{a_k\} \subseteq D$ such that $\|a_k - a\|_A \to 0$. Since the involution is continuous on A, $\|a_k^*a_k - a^*a\|_A \to 0$, so as A is continuously embedded in B, $\|a_k^*a_k - a^*a\|_B \to 0$. By the upper semicontinuity of the spectrum, there exists m such that $\sigma(a_m^*a_m; B) \subseteq U$, and we can choose m so large that

$$r(a_m^*a_m - a^*a; A) \le \|a_m^*a_m - a^*a\|_A < \frac{1}{2}(r(a^*a; A) - \beta).$$

Now using [PT, 5,6, 2°, p. 24], we have

$$\begin{aligned} r(a^*a; A) &\le r(a_m^*a_m; A) + r(a_m^*a_m - a^*a; A) \\ &< \beta + \frac{1}{2}(r(a^*a; A) - \beta). \end{aligned}$$

Thus $r(a^*a; A) < \beta$, a contradiction.

The next theorem is one of the main results of this paper. It provides an answer to Question I in the case where the involution of A is continuous with respect to the B-norm.

THEOREM 11. *Assume that A is symmetric and that there exists $M > 0$ such that $\|a^*\|_B \le M\|a\|_B$ for all $a \in A$.*

(1) *If $a \in A$ and $a^{-1} \in B$, then a^{-1} and $(a^*)^{-1} \in A^-$.*
(2) *For all $a \in A$, $\sigma(a^*; B) = \sigma(a; B)^*$.*
(3) *If A is *-inverse closed in B, or A is SRP in B, then A is inverse closed in B.*

Proof. As in (1), assume that $a \in A$ and $a^{-1} \in B$. Suppose that aa^* is not invertible in B. For $k \ge 1$, let

$$b_k = (k^{-1} + aa^*)^{-1}/\|(k^{-1} + aa^*)^{-1}\|_B.$$

Then $b_k = b_k^* \in A$, and by a standard argument $\|aa^*b_k\|_B \to 0$.

Then $\|a^*b_k\|_B = \|a^{-1}aa^*b_k\|_B \le \|a^{-1}\|_B\|aa^*b_k\|_B \to 0$. Therefore, $\|b_ka\|_B \le M\|a^*b_k\|_B \to 0$. Then $\|b_k\|_B = \|b_kaa^{-1}\|_B \le \|b_ka\|_B\|a^{-1}\|_B \to 0$, a contradiction. It follows that aa^* is invertible in B. This implies that $(a^*)^{-1} \in B$. Then Proposition 8(1) shows that $a^{-1}, (a^*)^{-1} \in A^-$.

(2) follows immediately from (1).

Now assume that A is *-inverse closed in B. Applying Proposition 3, we have for all $a \in A$, $\sigma(a; A) = \sigma(a; B) \cup \sigma(a^*; B)^* = \sigma(a; B)$, where the last equality follows from part (2) of the theorem. If A is SRP in B, then A is *-inverse closed in B by Proposition 8(2).

COROLLARY 12. *If A is a C^*-algebra and there exists $M > 0$ such that $\|a^*\|_B \le M\|a\|_B$ for all $a \in A$, then A is inverse closed in B.*

Corollary 12 is an immediate consequence of Theorem 11. It can also be proved using a result of D. Goldstein, as follows: Since A is a C^*-algebra, using [R, Theorem (4.8.3), p. 240], we have $\|a\|_A^2 \le \|a^*\|_B\|a\|_B$ for all $a \in A$. By the hypothesis of the corollary, $\|a\|_A^2 \le M\|a\|_B^2$ for all $a \in A$. Then Goldstein's result [G, Thm. 3] applies, so A is inverse closed in B.

4. An example. Let X be a Banach space with a bounded inner product (x, y). Define A to be the following Banach *-algebra:

$$A \equiv \{T \in B(X) : \exists T^* \in B(X) \text{ with } (Tx, y) = (x, T^*y) \text{ all } x, y \in X\}.$$

The complete algebra norm on A is

$$\|T\|_A \equiv \max(\|T\|, \|T^*\|).$$

Set $B = B(X)$, so A is a subalgebra of B.

CLAIM. *A is *-inverse closed in B.*

For suppose that $T = T^* \in A$, and that T has an inverse S in B. For all $x, y \in X$,

$$(Sx, y) = (Sx, TSy) = (TSx, Sy) = (x, Sy).$$

Therefore, $S = S^* \in A$.

Now let Q be the rectangle in the complex plane,

$$Q \equiv \{z = x + iy : 0 \le x \le 1;\ -1 \le y \le 2\}.$$

Set $X \equiv A(Q) \equiv$ {all functions f which are continuous on Q and holomorphic on the interior of Q} equipped with the sup-norm. Define the bounded inner product on X by

$$(f, g) \equiv \int_0^1 f(x)g(x)^*\,dx \qquad (f, g \in X).$$

For $h \in A(Q) = X$, let M_h be the multiplication operator,

$$M_h(g) = hg \qquad (g \in X).$$

Then $\sigma(M_h; B) = \sigma(h; A(Q)) = \{h(z) : z \in Q\}$. Clearly, when $h(x)$ is in $\mathbb{R}$ for $0 \le x \le 1$, then $(M_h(f), g) = (f, M_h(g))$ for all f, g in X. In particular, when $h(z) = z$, $M_z = M_z^*$ is in A. Note that:

(1) $\sigma(M_z; B) = Q$, and $Q \neq Q^*$;
(2) $\sigma(M_z; A) = Q \cup Q^* \neq Q = \sigma(M_z; B)$.

[The first equality in (2) follows from Proposition 3.]

This example shows that the hypothesis in Corollary 4 that A is symmetric cannot be omitted.

5. Applications to operators. In this section we apply the results of this paper to a *-algebra of operators which is studied in the author's paper [B2]. Let (Ω, d) be a metric space, and assume that μ is a positive σ-finite regular Borel measure on Ω. For $x \in \Omega$ and $\varepsilon > 0$, let $B_c(x;\varepsilon) = \{y \in \Omega : d(x,y) \le \varepsilon\}$. As in [B2, (4.1)], we assume that there exist $D > 0$ and $\beta > 0$ such that $\mu(B_c(x;m)) \le Dm^\beta$ for all positive integers m and all $x \in \Omega$. For a kernel (measurable function on $\Omega \times \Omega$) $K(x,y)$, define with $j = 1, 2$,

$$n_j(K) = \operatorname*{ess\,sup}_{x\in\Omega} \int |K(x,y)|^j \, d\mu(y)^{1/j}, \quad \|K\|_j = \max(n_j(K), n_j(K^*)),$$

where $K^*(x,y) = K(y,x)^*$. Let A_1 be the Banach *-algebra of all kernels K such that $\|K\|_1 < \infty$, with involution $K \mapsto K^*$, and multiplication

$$(K * J)(x,y) = \int K(x,z) J(z,y) \, d\mu(z).$$

For $K \in A_1$ and $1 \le p \le \infty$, define an integral operator

$$T_{K,p}(f)(x) = \int K(x,y) f(y) \, d\mu(y) \quad (f \in L^p).$$

It is easy to check that $T_{K,1}$ and $T_{K,\infty}$ are in $B(L^1)$ and $B(L^\infty)$, respectively. Therefore, $T_{K,p} \in B(L^p)$ for all $1 \le p \le \infty$ by the Riesz–Thorin Convexity Theorem [DS, Thm. 11, p. 525]. Thus for all such p, $K \mapsto T_{K,p}$ is a continuous embedding of A_1 into $B(L^p)$. We denote the spectrum of $T_{K,p}$ relative to $B(L^p)$ by $\sigma(T_{K,p})$.

For $0 < \delta \le 1$, define $w(x,y) = (1 + d(x,y))^\delta$. Let $A_{w,2}$ be the *-subalgebra of A_1 consisting of all K such that $\|K\|_2 < \infty$ and $K(x,y)w(x,y) \in A_1$. The following results are proved in [B2]:

I. $r(K; A_{w,2}) = r(K; A_1)$ for all $K \in A_{w,2}$ [B2, Lemma 4.6].

II. For $K = K^* \in A_{w,2}$, $\sigma(K; A_{w,2}) = \sigma(T_{K,2})$. Thus, $A_{w,2}$ is symmetric [B2, Theorem 4.7].

III. For all K in a certain closed subalgebra of $A_{w,2}$, and for $1 \le p \le \infty$:

(i) $\sigma(K; A_{w,2}) = \sigma(T_{K,p})$ when K is normal;
(ii) $\sigma(K; A_{w,2}) = \sigma(T_{K,p}) \cup \sigma(T_{K^*,p})^*$ for all K [B2, Theorem 4.8].

Using results in this paper we prove that (i) and (ii) hold when $K \in A_{w,2}$. In addition, the proof of this more general result is simpler than the proof of Theorem 4.8 in [B2] since the complicated result, [B2, Theorem 4.5], is not needed.

NOTE 13. Let $K \in A_{w,2}$, and define $K^\sim(x,y) = K(y,x)$. Assume that $1 \le p \le \infty$ and $1/p + 1/q = 1$. With respect to the usual bilinear form $\langle f, g\rangle = \int fg \, d\mu$, $f \in L^p, g \in L^q$, we have

$$\langle T_{K,p}(f), g\rangle = \langle f, T_{K^\sim,q}(g)\rangle. \tag{1}$$

It follows from this equality that $\sigma(T_{K,p}) = \sigma(T_{K^*,q})^*$.

Proof. (1) follows from a straightforward application of Fubini's Theorem. Now (1) shows that $T_{K^\sim,q}$ is the usual adjoint operator of $T_{K,p}$. Therefore, $\sigma(T_{K,p}) = \sigma(T_{K^\sim,q})$. Since $K^\sim(x,y)^* = K^*(x,y)$, it easily follows that $\sigma(T_{K,p}) = \sigma(T_{K^*,q})^*$.

The algebra $A_{w,2}$ is symmetric, but it need not be unital. The results of this paper extend to the nonunital case (by adjoining a unit).

THEOREM 14. *Let $K \in A_{w,2}$.*

(1) $\sigma(K; A_1) = \sigma(T_{K,2}) = \sigma(K; A_{w,2})$.
(2) *The algebra $\{T_{K,p} : K \in A_{w,2}\}$ is SRP in $B(L^p)$ for all $p, 1 \le p \le \infty$.*
(3) *If K is normal, then $\sigma(K; A_{w,2}) = \sigma(T_{K,p})$ for all p, $1 \le p \le \infty$.*
(4) *For general K, $1 \le p \le \infty$,*

$$\sigma(K; A_{w,2}) = \sigma(T_{K,p}) \cup \sigma(T_{K^*,p})^* = \sigma(T_{K,p}) \cup \sigma(T_{K,q}).$$

Proof. As noted above in (I) and (II), $A_{w,2}$ is symmetric and for all $K = K^* \in A_{w,2}$, $r(K; A_{w,2}) = r(K; A_1) = r(T_{K,2})$ [the spectral radius of $T_{K,2}$ in $B(L^2)$]. Therefore, (1) follows from Theorem 9.

From (1) we see that $r(K; A_{w,2}) = r(T_{K,2})$ for all $K \in A_{w,2}$. Note that we always have $r(K; A_{w,2}) \ge r(T_{K,p})$ for all p. Now let p and q be conjugate indices, $1/p + 1/q = 1$ (where, as usual, $1/\infty = 0$). Choose t, $0 \le t \le 1$, such that $1/2 = t(1/p) + (1-t)(1/q)$. Then by the Riesz–Thorin Convexity Theorem, for all $n \ge 1$,

$$\|(T_{K,2})^n\| \le \|(T_{K,p})^n\|^t \|(T_{K,q})^n\|^{1-t}.$$

Taking the nth root and the limit as $n \to \infty$, we have

$$r(K; A_{w,2}) = r(T_{K,2}) \le r(T_{K,p})^t r(T_{K,q})^{1-t} \le r(T_{K,p})^t r(K; A_{w,2})^{1-t}.$$

Thus, $r(K; A_{w,2}) \le r(T_{K,p})$, and so $r(K; A_{w,2}) = r(T_{K,p})$. This proves (2).

Having proved that $\{T_{K,p} : K \in A_{w,2}\}$ is SRP in $B(L^p)$, (3) follows by applying Proposition 8 and Corollary 4. Also, that $\sigma(K; A_{w,2}) = \sigma(T_{K,p}) \cup \sigma(T_{K^*,p})^*$ follows from Propositions 8 and 3. Finally, the last equality in (4) holds since $\sigma(T_{K,q}) = \sigma(T_{K^*,p})^*$ as shown in Note 13.

COROLLARY 15. *Assume* $K \in A_1$, $\|K\|_2 < \infty$, *and* $K(x,y)d(x,y)^\delta \in A_1$ *for some* $0 < \delta \le 1$. *Then for all* p, $1 \le p \le \infty$,

$$\partial\sigma(K;A_1) \subseteq \sigma(T_{K,p}) \subseteq \sigma(K;A_1).$$

Proof. Since $w(x,y) = (1+d(x,y))^\delta \le 1 + d(x,y)^\delta$, it follows that $K(x,y)w(x,y) \in A_1$. Also, $\|K\|_2 < \infty$, and so $K \in A_{w,2}$. Then the result holds by applying parts (1) and (2) of Theorem 14.

EXAMPLE 16. Let $\Omega = \mathbb{N} = \{1,2,3,\ldots\}$, and let $d(n,m) = |n-m|$ for all n,m in $\mathbb{N}$. Let μ be counting measure. An infinite matrix $\{K(n,m)\}_{n,m\ge1}$ is in A_1 if

$$\|K\|_1 = \max\Big(\sup_n \sum_{m=1}^{\infty} |K(n,m)|, \sup_m \sum_{n=1}^{\infty} |K(n,m)|\Big) < \infty.$$

In this case, automatically $\|K\|_1 \ge \|K\|_2$.

COROLLARY 17. *Let* $K(n,m) \in A_1$ *(as above). Assume in addition that* $K(n,m)|n-m|^\delta \in A_1$ *for some* $0 < \delta \le 1$.

(1) $\sigma(K;A_1) = \sigma(T_{K,2})$.

(2) *If* $K(n,m) = K(m,n)^*$ *for all* n,m, *then for all* p, $1 \le p \le \infty$,

$$\sigma(T_{K,p}) = \sigma(K;A_1) \subseteq \mathbb{R}.$$

(3) *In the general case, for all* p, $1 \le p \le \infty$,

$$\partial\sigma(K;A_1) \subseteq \sigma(T_{K,p}) \subseteq \sigma(K;A_1).$$

Corollary 17 is a consequence of Theorem 14 and Corollary 15.

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