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A variant sharp estimate for multilinear singular integral operators

by

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Abstract. We establish a variant sharp estimate for multilinear singular integral operators. As applications, we obtain the weighted norm inequalities on general weights and certain $L\log^+ L$ type estimates for these multilinear operators.

1. Introduction. We will work on $\mathbb{R}^n$, $n \ge 1$. Let m_1, m_2 be two positive integers and $m = m_1 + m_2$. Suppose that $K \in C^1(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree $-n$ and satisfies

$$|K(x)| \le C|x|^{-n} \quad \text{and} \quad |\nabla K(x)| \le C|x|^{-n-1} \quad \text{for } |x| \ne 0,$$
$$\int_{|x|=1} K(x)x^\gamma\,dx = 0 \quad \text{for any } |\gamma| \le m.$$

Let A_j be a function on $\mathbb{R}^n$ whose derivatives of order m_j belong to the space $\mathrm{BMO}(\mathbb{R}^n)$ for $j = 1, 2$. Define the multilinear singular integral operator T_{A_1,A_2} by

$$T_{A_1,A_2}f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)\frac{\prod_{j=1}^2 P_{m_j+1}(A_j;x,y)}{|x-y|^m} f(y)\,dy, \tag{1}$$

where $P_{m_j+1}(A_j;x,y)$ denotes the (m_j+1)th order Taylor series remainder of A_j at x about y, precisely,

$$P_{m_j+1}(A_j;x,y) = A_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha. \tag{2}$$

It is well known that the operators of this type have been studied by many authors (see [2], [4], [5] and [9]). We point out that the first result in this direction was established by Coifman, Rochberg and Weiss in [5]. The

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result of Cohen and Gosselin [4] says that the operator T_{A_1,A_2} is bounded on $L^p(\mathbb{R}^n)$ with the operator norm no more than

$$C_{n,p}\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)\quad \text{for all } 1<p<\infty.$$

The main purpose of this paper is to establish a variant sharp estimate for these multilinear operators; and then using this sharp estimate, we will obtain some weighted norm inequalities on general weights and certain $L\log^+L$ type estimates for these operators T_{A_1,A_2}, which can be regarded as an endpoint theory for multilinear singular integral operators. We point out that some of our ideas in this paper come from the paper [11] of Pérez. Before stating our results, let us give some notation first.

For any locally integrable function f, we denote by $f^\#$ the sharp function of Fefferman and Stein, that is,

$$f^\#(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q|f(y)-m_Q(f)|\,dy,$$

where Q is a cube with sides parallel to the coordinate axes and $m_Q(f)$ is the mean value of f on Q. In what follows, all the cubes considered have edges parallel to the axes. For $0<r<\infty$, we define $f_r^\#$ by

$$f_r^\#(x)=[(|f|^r)^\#(x)]^{1/r}.$$

Let M be the Hardy–Littlewood maximal operator. For $k\in\mathbb{N}$, we denote by M^k the operator M iterated k times, i. e., $M^1f(x)=Mf(x)$ and

$$M^kf(x)=M(M^{k-1}f)(x)\quad \text{when } k\ge 2;$$

for $0<r<\infty$, we set

$$M_rf(x)=[M(|f|^r)(x)]^{1/r}.$$

For a Young function $\Phi:[0,\infty)\to[0,\infty)$ and a function f on $\mathbb{R}^n$, define the Φ average of f over a cube Q by the Luxemburg norm:

$$\|f\|_{\Phi,Q}=\inf\Big\{\lambda>0:\frac{1}{|Q|}\int_Q\Phi\Big(\frac{|f(y)|}{\lambda}\Big)\,dy\le 1\Big\};$$

see [1]. Let $p\in(0,\infty)$. For any non-negative locally integrable weight function w and any Lebesgue measurable function f, we set

$$\|f\|_{p,w}=\Big(\int_{\mathbb{R}^n}|f(x)|^pw(x)\,dx\Big)^{1/p};$$

and if $w\equiv 1$, we denote $\|f\|_{p,w}$ simply by $\|f\|_p$. Also, C denotes a constant that is independent of the main parameters involved but whose value may differ from line to line.

Now we can state our main theorems.

THEOREM 1. *Let T_{A_1,A_2} be the multilinear operator defined by* (1). *Then for any $0<r<1$, there exists a positive constant $C=C_{n,r}$ such that for any $f\in C_0^\infty(\mathbb{R}^n)$,*

$$(T_{A_1,A_2}f)_r^\#(x)\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)M^3f(x).$$

Theorem 1 is interesting since it implies

THEOREM 2. *Let T_{A_1,A_2} be the multilinear operator defined by* (1) *and $p\in(1,\infty)$. Then there exists a positive constant $C=C_{n,p}$ such that for any non-negative locally integrable weight function w and any Lebesgue measurable function f,*

$$\|T_{A_1,A_2}f\|_{p,w}\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)\|f\|_{p,M^{3([p]+1)}w},$$

where $[p]\in\mathbb{N}\cup\{0\}$ is the greatest integer no more than p.

THEOREM 3. *Let T_{A_1,A_2} be the multilinear operator defined by* (1). *Then there exists a positive constant C depending only on n and $\prod_{j=1}^2(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}})$ such that for each $\lambda>0$,*

$$|\{x\in\mathbb{R}^n:|T_{A_1,A_2}f(x)|>\lambda\}|\le C\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\Big(1+\log^+\Big(\frac{|f(x)|}{\lambda}\Big)\Big)^2dx.$$

As in [11], Theorem 3 follows from Theorem 1; so we only need to prove Theorems 1 and 2, whose proofs will be given, respectively, in Sections 3 and 4. In Section 2, we will investigate the boundedness on Hardy spaces and weak Lebesgue spaces of some relative multilinear operators, which will be used in the proofs of our main theorems and have independent interest; see [3].

Finally, we point out that there is still an interesting open problem to see if our results are best possible.

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2. Some multilinear operators on $L^1(\mathbb{R}^n)$.

We begin with a preliminary lemma.

LEMMA 1 (see [4]). *Let b be a function on $\mathbb{R}^n$ with derivatives of order k in $L^q(\mathbb{R}^n)$ for some $q>n$. Then*

$$|P_k(b;x,y)| \le C_{k,n}|x-y|^k \sum_{|\beta|=k} \bigg(\frac{1}{|\widetilde{Q}(x,y)|} \int_{\widetilde{Q}(x,y)} |D^\beta b(z)|^q\,dz\bigg)^{1/q},$$

where $\widetilde{Q}(x,y)$ *is the cube centered at* x *and having side length* $5\sqrt{n}\,|x-y|$.

Let K, m_1, m_2 and m be as in Section 1, and let B_j have derivatives of order m_j in $L^{q_j}(\mathbb{R}^n)$ for some $q_j > n$, $j=1,2$. Define the multilinear operator T_{B_1,B_2} by

$$(2)\qquad T_{B_1,B_2}f(x) = \mathrm{p.v.}\int_{\mathbb{R}^n} K(x-y)\frac{\prod_{j=1}^2 P_{m_j}(B_j;x,y)}{|x-y|^m}f(y)\,dy.$$

The operators of this type were studied by Cohen and Gosselin [3]. In fact, they established the $L^p(\mathbb{R}^n)$-boundedness of some operators for $p\in(1,\infty)$ whose kernels satisfy weaker smoothness assumptions than the operators in (2). In the following, we will first investigate the boundedness of T_{B_1,B_2} on the Hardy spaces $H^p(\mathbb{R}^n)$ when $p\in(0,1)$. Using this boundedness, we will then prove that T_{B_1,B_2} is of weak type $(1,q)$, which will be used in the proof of Theorem 1.

Proposition 1. *Let* $0<p<1$, *let* B_j *have derivatives of order* m_j *in* $L^{q_j}(\mathbb{R}^n)$ *for some* $n<q_j<\infty$, $j=1,2$, $1/r=1/q_1+1/q_2+1/p$ *and* $r(n+1)>n$. *Then the operator* T_{B_1,B_2} *defined by* (2) *is bounded from* $H^p(\mathbb{R}^n)$ *to* $L^r(\mathbb{R}^n)$ *with the operator norm no more than* $C\prod_{j=1}^2(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j})$.

Proof. By the atomic decomposition of the Hardy space $H^p(\mathbb{R}^n)$ (see [7, Chap. 3]), it suffices to show that for any $(p,\infty,0)$ atom a,

$$\|T_{B_1,B_2}a\|_r \le C$$

with C independent of the atom a. Let a be a $(p,\infty,0)$ atom, that is,

(1) a is supported on a cube Q,
(2) $\|a\|_\infty \le |Q|^{-1/p}$,
(3) $\int_{\mathbb{R}^n} a(x)\,dx = 0$.

Set $Q^* = 8\sqrt{n}\,Q$, the cube having the same center as Q and $8\sqrt{n}$ times the side length of Q. Denote by x_0 and l_Q, respectively, the center and the side length of Q. Write

$$\|T_{B_1,B_2}a\|_r^r = \int_{Q^*}|T_{B_1,B_2}a(x)|^r\,dx + \int_{\mathbb{R}^n\setminus Q^*}|T_{B_1,B_2}a(x)|^r\,dx = \mathrm{I}+\mathrm{II}.$$

Choose $1<p_0,q_0<\infty$ such that $1/q_0 = 1/p_0+1/q_1+1/q_2$. The (L^{p_0},L^{q_0})-boundedness of T_{B_1,B_2} (see [3]) tells us that

$$\begin{aligned}\mathrm{I} &\le \Big(\int_{Q^*}|T_{B_1,B_2}a(x)|^{q_0}\,dx\Big)^{r/q_0}|Q^*|^{1-r/q_0}\\ &\le C\prod_{j=1}^2\Big(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j}\Big)^r\|a\|_{p_0}^r|Q|^{1-r/q_0}\\ &\le C\prod_{j=1}^2\Big(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j}\Big)^r|Q|^{-r/p}|Q|^{r/p_0}|Q|^{1-r/q_0}\\ &\le C\prod_{j=1}^2\Big(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j}\Big)^r.\end{aligned}$$

Now we estimate II. Let $y_0\in Q^*\setminus 4\sqrt{n}\,Q$. The vanishing moment of a gives

$$\begin{aligned}&|T_{B_1,B_2}a(x)|\\ &\quad\le C|x-y_0|^{-n-m}\int_Q\Big|\prod_{j=1}^2 P_{m_j}(B_j;x,y)-\prod_{j=1}^2 P_{m_j}(B_j;x,y_0)\Big||a(y)|\,dy\\ &\qquad + C\int_Q\Big|\frac{K(x-y)}{|x-y|^m}-\frac{K(x-y_0)}{|x-y_0|^m}\Big|\prod_{j=1}^2|P_{m_j}(B_j;x,y)a(y)|\,dy.\end{aligned}$$

Note that for $x\in\mathbb{R}^n\setminus Q^*$ and $y\in Q$, $|x-y|\approx|x-y_0|$. With the aid of the formula (see [4, p. 448])

$$P_k(B_j;x,y)-P_k(B_j;x,y_0) = \sum_{|\alpha|<k}\frac{1}{\alpha!}P_{k-|\alpha|}(D^\alpha B_j;y_0,y)(x-y_0)^\alpha$$

for $j=1,2$, we have

$$\begin{aligned}|T_{B_1,B_2}a(x)| &\le C\sum_{|\alpha|<m_2}|x-y_0|^{-n-m+|\alpha|}\\ &\quad\times\int_Q|P_{m_1}(B_1;x,y)P_{m_2-|\alpha|}(D^\alpha B_2;y_0,y)a(y)|\,dy\\ &\quad + C\sum_{|\alpha|<m_1}|x-y_0|^{-n-m+|\alpha|}|P_{m_2}(B_2;x,y_0)|\\ &\quad\times\int_Q|P_{m_1-|\alpha|}(D^\alpha B_1;y_0,y)a(y)|\,dy\\ &\quad + Cl_Q|Q|^{-1/p}|x-y_0|^{-n-m-1}\int_Q\prod_{j=1}^2|P_{m_j}(B_j;x,y)|\,dy.\end{aligned}$$

Lemma 1 now tells us that for $j=1,2$,

$$|P_{m_j}(B_j;x,y)| \le C \sum_{|\beta_j|=m_j} \|D^{\beta_j}B_j\|_{q_j}|x-y|^{m_j-n/q_j}$$

and

$$|P_{m_j-|\alpha|}(D^\alpha B_j;y_0,y)| \le C \sum_{|\beta_j|=m_j} \|D^{\beta_j}B_j\|_{q_j}|y-y_0|^{m_j-|\alpha|-n/q_j}.$$

Thus, for $x\in\mathbb{R}^n\setminus Q^*$,

$$\begin{aligned}
&|T_{B_1,B_2}a(x)|\\
&\quad\le C\prod_{j=1}^{2}\Big(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j}\Big)\\
&\qquad\times\Big(\sum_{|\alpha|<m_2}|x-y_0|^{-n-m_2+|\alpha|-n/q_1}l_Q^{-n/p+n+m_2-|\alpha|-n/q_2}\\
&\qquad+\sum_{|\alpha|<m_1}|x-y_0|^{-n-m_1+|\alpha|-n/q_2}l_Q^{-p/n+n+m_1-|\alpha|-n/q_1}\Big)\\
&\qquad+C\prod_{j=1}^{2}\Big(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j}\Big)|x-y_0|^{-n-1-n(1/q_1+1/q_2)}l_Q^{-n/p+n+1}\\
&\quad\le C\prod_{j=1}^{2}\Big(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j}\Big)|x-y_0|^{-n-1}l_Q^{n+1-n/p-n(1/q_1+1/q_2)}.
\end{aligned}$$

Recall that $r(n+1)>n$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus Q^*}|T_{B_1,B_2}a(x)|^r\,dx\\
&\quad\le C\prod_{j=1}^{2}\Big(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j}\Big)l_Q^{r+nr-n}\int_{\mathbb{R}^n\setminus Q^*}|x-y_0|^{-r(n+1)}\,dx\\
&\quad\le C\prod_{j=1}^{2}\Big(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j}\Big),
\end{aligned}$$

which together with the estimate for I yields the desired result.

This finishes the proof of Proposition 1.

Now we can establish the weak $(1,q)$ boundedness of the operator T_{B_1,B_2}.

PROPOSITION 2. *Let B_j have derivatives of order m_j in $L^{q_j}(\mathbb{R}^n)$ for some $q_j>2n$, $j=1,2$, $q=q_1q_2/(q_1+q_2+q_1q_2)$. Then the operator T_{B_1,B_2} defined by* (2) *is bounded from $L^1(\mathbb{R}^n)$ to weak $L^q(\mathbb{R}^n)$ with the operator norm no more than $\prod_{j=1}^2(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j})$. That is, there exists a positive constant C such that for each $f\in L^1(\mathbb{R}^n)$ and $\lambda>0$,*

$$|\{x\in\mathbb{R}^n:|T_{B_1,B_2}f(x)|>\lambda\}|\le C\prod_{j=1}^{2}\Big(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j}\Big)(\lambda^{-1}\|f\|_1)^q.$$

Proof. We will follow the same lines as the proof of Theorem 3.37 in [7]. Without loss of generality, we may assume that $\prod_{j=1}^2(\sum_{|\beta_j|=m_j}\|D^{\beta_j}B_j\|_{q_j})=1$. Let $f\in L^1(\mathbb{R}^n)$ and $\|f\|_1=1$. Choose $1<p_1,s_1<\infty$ such that $1/s_1=1/p_1+1/q_1+1/q_2$, and $0<p_2,s_2<1$ such that $1/s_2=1/p_2+1/q_1+1/q_2$ and $(n+1)s_2>n$ (recall that $q_1;q_2>2n$). For each fixed $\lambda>0$, apply the Calderón–Zygmund decomposition to express $f=g+b$ with

$$\|g\|_{p_1}^{p_1}\le C\lambda^{q(p_1-1)}\quad\text{and}\quad\|b\|_{H^{p_2}}^{p_2}\le C\lambda^{q(p_2-1)}$$

(see [7, p. 112]). Write

$$\begin{aligned}
\{x\in\mathbb{R}^n:|T_{B_1,B_2}f(x)|>\lambda\}|&\le|\{x\in\mathbb{R}^n:|T_{B_1,B_2}g(x)|>\lambda/2\}|\\
&\quad+|\{x\in\mathbb{R}^n:|T_{B_1,B_2}b(x)|>\lambda/2\}|.
\end{aligned}$$

The (L^{p_1},L^{s_1})-boundedness of T_{B_1,B_2} gives

$$\begin{aligned}
|\{x\in\mathbb{R}^n:|T_{B_1,B_2}g(x)|>\lambda/2\}|&\le C\lambda^{-s_1}\|T_{B_1,B_2}g\|_{s_1}^{s_1}\le C\lambda^{-s_1}\|g\|_{p_1}^{s_1}\\
&\le C\lambda^{-s_1}\lambda^{q(p_1-1)s_1/p_1}=C\lambda^{-q}.
\end{aligned}$$

On the other hand, by Proposition 1,

$$\begin{aligned}
|\{x\in\mathbb{R}^n:|T_{B_1,B_2}b(x)|>\lambda/2\}|&\le C\lambda^{-s_2}\|T_{B_1,B_2}b\|_{s_2}^{s_2}\le C\lambda^{-s_2}\|b\|_{H^{s_2}}^{s_2}\\
&\le C\lambda^{-s_2}\lambda^{q(p_2-1)s_2/p_2}=C\lambda^{-q}.
\end{aligned}$$

This finishes the proof of Proposition 2.

Let $\widetilde K\in C^1(\mathbb{R}^n\setminus\{0\})$ be homogeneous of degree zero and for some positive constant C,

$$|\widetilde K(x)|\le C|x|^{-n},\quad|\nabla\widetilde K(x)|\le C|x|^{-n-1},\quad|x|>0,$$

and

$$\int_{|x|=1}\widetilde K(x)x^\gamma\,dx=0,\quad|\gamma|=k,$$

where $k\in\mathbb{N}$. Define the multilinear operator

$$T_Bf(x)=\text{p.v.}\int_{\mathbb{R}^n}\widetilde K(x-y)\frac{P_k(B;x,y)}{|x-y|^k}f(y)\,dy. \tag{3}$$

Parallel to Proposition 2, we can prove

PROPOSITION 3. *Let B have derivatives of order k in $L^q(\mathbb{R}^n)$ for some $q > n$, $\widetilde{q} = q/(q+1)$. Then the operator T_B defined by (3) is bounded from $L^1(\mathbb{R}^n)$ to weak $L^{\widetilde{q}}(\mathbb{R}^n)$ with the operator norm no more than $C\sum_{|\beta|=k}\|D^\beta B\|_q$.*

3. Proof of Theorem 1. To prove Theorem 1, we need some lemmas.

LEMMA 2. *Let T be a bounded operator from $L^1(\mathbb{R}^n)$ to weak $L^u(\mathbb{R}^n)$ with the operator norm no more than B for some $u > 0$. Then for any fixed $0 < r < u$, there exists a positive constant C_r such that for any measurable set E and $f \in L^1(\mathbb{R}^n)$,*

$$\int_E |Tf(y)|^r\,dy \le C_r B^r |E|^{1-r/u}\|f\|_1^r.$$

This lemma can be proved in a similar way to the proof of Kolmogorov's inequality (see [8, p. 485]); we omit the details for brevity.

LEMMA 3. *Let f_1, f_2 be two Lebesgue measurable functions on $\mathbb{R}^n$. Then for any measurable set $E \subset \mathbb{R}^n$,*

$$\int_E e^{f_1(x)f_2(x)}\,dx \le \Big(\int_E e^{f_1^2(x)}\,dx\Big)^{1/2}\Big(\int_E e^{f_2^2(x)}\,dx\Big)^{1/2}.$$

Lemma 3 is an easy corollary of the Hölder inequality and the trivial inequality $2ab \le a^2 + b^2$ for any $a, b \in \mathbb{R}$. We omit the details.

Proof of Theorem 1. By homogeneity, we may assume that

$$\sum_{|\alpha_1|=m_1}\|D^{\alpha_1}A_1\|_{\mathrm{BMO}} = \sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}} = 1.$$

For fixed $x \in \mathbb{R}^n$, let $Q = Q(x_0, d)$ be a cube centered at x_0 and having side length d such that $x \in Q(x_0, d)$. Set $Q^* = 10\sqrt{n}\,Q$, and $Q^{**} = 20\sqrt{n}\,Q$. For $f \in C_0^\infty(\mathbb{R}^n)$, we decompose f as

$$f(y) = f(y)\chi_{Q^*}(y) + f(y)\chi_{\mathbb{R}^n\setminus Q^*}(y) = f_1(y) + f_2(y).$$

It is enough to prove that there exists a positive constant C_r such that

$$\Big(\frac{1}{|Q|}\int_Q |T_{A_1,A_2}f_1(y)|^r\,dy\Big)^{1/r} \le C_r M^3 f(x) \tag{4}$$

and for some constant c_Q,

$$\sup_{y\in Q}|T_{A_1,A_2}f_2(y) - c_Q| \le CM^3 f(x), \tag{5}$$

where C_r and C are independent of x, Q and f.

Let $\phi_Q \in C_0^\infty(\mathbb{R}^n)$ have $\operatorname{supp}\phi_Q \subset 40\sqrt{n}\,Q$ and be identically one on Q^{**}. Denote by $m_{Q^*}(b)$ the mean value of the locally integrable function b on Q^*. Let y_0 be a point on the boundary of Q. Set

$$A_j^Q(y) = P_{m_j}\Big(A_j(\cdot) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}m_{Q^*}(D^\alpha A_j)(\cdot)^\alpha; y, y_0\Big)\phi_Q(y).$$

The observation of Cohen and Gosselin in [4, pp. 452–453] shows that for $y \in Q$ and $z \in Q^*$,

$$P_{m_j+1}(A_j; y, z) = P_{m_j+1}(A_j^Q; y, z).$$

Thus for $y \in Q$,

$$\begin{aligned}
T_{A_1,A_2}f_1(y) &= \text{p.v.}\int_{\mathbb{R}^n} K(y-z)\frac{\prod_{j=1}^2 P_{m_j}(A_j^Q; y, z)}{|y-z|^m}f_1(z)\,dz\\
&\quad - \sum_{|\beta_2|=m_2}\Big(\text{p.v.}\int_{\mathbb{R}^n} K(y-z)(y-z)^{\beta_2}\\
&\quad\times\frac{P_{m_1}(A_1^Q; y, z)}{|y-z|^m}D^{\beta_2}A_2^Q(z)f_1(z)\,dz\Big)\\
&\quad - \sum_{|\beta_1|=m_1}\Big(\text{p.v.}\int_{\mathbb{R}^n} K(y-z)(y-z)^{\beta_1}\\
&\quad\times\frac{P_{m_2}(A_2^Q; y, z)}{|y-z|^m}D^{\beta_1}A_1^Q(z)f_1(z)\,dz\Big)\\
&\quad + \sum_{|\beta_1|=m_1,\,|\beta_2|=m_2}\Big(\text{p.v.}\int_{\mathbb{R}^n} K(y-z)\frac{(y-z)^{\beta_1+\beta_2}}{|y-z|^m}\\
&\quad\times D^{\beta_1}A_1^Q(z)D^{\beta_2}A_2^Q(z)f_1(z)\,dz\Big)\\
&\equiv T^{\mathrm{I}}_{A_1,A_2}f_1(y) + T^{\mathrm{II}}_{A_1,A_2}f_1(y) + T^{\mathrm{III}}_{A_1,A_2}f_1(y) + T^{\mathrm{IV}}_{A_1,A_2}f_1(y).
\end{aligned}$$

By Lemma 1 and the identity

$$\begin{aligned}
D^\beta A_j^Q(y) = \sum_{\beta=\mu+\nu} P_{m_j-|\mu|}\Big(\frac{\partial^\mu}{\partial x^\mu}\Big(A_j(\cdot)\\
- \sum_{|\alpha|=m_j}\frac{1}{\alpha!}m_{Q^*}(D^\alpha A_j)(\cdot)^\alpha\Big); y, y_0\Big)D^\nu\phi_Q(y),
\end{aligned}$$

we deduce that for $y \in Q$ and $|\beta_j| = m_j$,

$$|D^{\beta_j}A_j^Q(y)| \le C + C|D^{\beta_j}A_j(y) - m_{Q^*}(D^{\beta_j}A_j)|. \tag{6}$$

For fixed $0 < r < 1$, choose $2n < q < \infty$ such that $0 < r < q/(q+2)$. Observe that $\operatorname{supp} A_j^Q \subset 40\sqrt{n}\, Q$. Therefore,

$$\sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j^Q\|_q \le C|Q|^{1/q}, \quad j = 1, 2.$$

Proposition 2 together with Lemma 2 then gives

$$\begin{aligned}\Big(\frac{1}{|Q|}\int_Q |T^{\mathrm{I}}_{A_1,A_2} f_1(y)|^r\,dy\Big)^{1/r} &\le C|Q|^{-1/r}\prod_{j=1}^{2}\Big(\sum_{|\beta_j|=m_j}\|D^{\beta_j}A_j^Q\|_q\Big)|Q|^{1/r-(q+2)/q}\|f_1\|_1\\ &\le C|Q|^{-1}\|f_1\|_1 \le CMf(x).\end{aligned}$$

On the other hand, by Proposition 3, Lemma 2 and the estimate (6), we have

$$\begin{aligned}\Big(\frac{1}{|Q|}\int_Q |T^{\mathrm{II}}_{A_1,A_2} f_1(y)|^r\,dy\Big)^{1/r} &\le C\sum_{|\beta_1|=m_1}\sum_{|\beta_2|=m_2}\|D^{\beta_1}A_1^Q\|_q|Q|^{-1/r}|Q|^{1/r-(q+1)/q}\|f_1D^{\beta_2}A_2^Q\|_1\\ &\le C\sum_{|\beta_2|=m_2}|Q|^{-1}\|f_1(D^{\beta_2}A_2 - m_{Q^*}(D^{\beta_2}A_2))\|_1 + C|Q|^{-1}\|f\|_1\\ &\le C\sum_{|\beta_2|=m_2}\|f_1\|_{L\log^+L,Q^*}\|D^{\beta_2}A_2 - m_{Q^*}(D^{\beta_2}A_2)\|_{\exp L,Q^*} + CMf(x)\\ &\le CM^2f(x),\end{aligned}$$

where we have invoked the generalized Hölder inequality (see [10, p. 168] or [1]), the fact that

$$\|D^{\beta_2}A_2 - m_{Q^*}(D^{\beta_2}A_2)\|_{\exp L,Q^*} \le C\|D^{\beta_2}A_2\|_{\mathrm{BMO}}$$

and Pérez's estimate (see [11, p. 181])

$$\|f\|_{L(\log L)^k,Q^*} \le CM^{k+1}f(x), \quad k\in\mathbb{N}. \tag{7}$$

Similarly, we have

$$\Big(\frac{1}{|Q|}\int_Q |T^{\mathrm{III}}_{A_1,A_2} f_1(y)|^r\,dy\Big)^{1/r} \le CM^2f(x).$$

As for $T^{\mathrm{IV}}_{A_1,A_2}f_1$, note that for any multi-index β_1 and β_2 with $|\beta_1| = m_1$ and $|\beta_2| = m_2$, $K(x)x^{\beta_1+\beta_2}|x|^{-m}$ is a standard Calderón–Zygmund kernel. Hence, by the Calderón–Zygmund theory, we know that the operator T defined by

$$Th(x) = \mathrm{p.v.}\int_{\mathbb{R}^n} K(x-y)\frac{(x-y)^{\beta_1+\beta_2}}{|x-y|^m}h(y)\,dy, \quad |\beta_1| = m_1,\ |\beta_2| = m_2,$$

is of weak type $(1,1)$. Thus via Lemma 2, Lemma 3 and the inequality (7) imply

$$\begin{aligned}\Big(\frac{1}{|Q|}\int_Q |T^{\mathrm{IV}}_{A_1,A_2} f_1(y)|^r\,dy\Big)^{1/r} &\le C\sum_{|\beta_1|=m_1,\,|\beta_2|=m_2}|Q|^{-1}\int_{Q^*}|D^{\beta_1}A_1^Q(y)D^{\beta_2}A_2^Q(y)f(y)|\,dy\\ &\le C\sum_{|\beta_1|=m_1,\,|\beta_2|=m_2}|Q|^{-1}\Big\|f_1\prod_{j=1}^{2}(D^{\beta_j}A_j - m_{Q^*}(D^{\beta_j}A_j))\Big\|_1\\ &\quad + C\sum_{|\beta_2|=m_2}|Q|^{-1}\|f_1(D^{\beta_2}A_2 - m_{Q^*}(D^{\beta_2}A_2))\|_1\\ &\quad + C\sum_{|\beta_1|=m_1}|Q|^{-1}\|f_1(D^{\beta_1}A_1 - m_{Q^*}(D^{\beta_1}A_1))\|_1 + CMf(x)\\ &\le C\sum_{|\beta_1|=m_1,\,|\beta_2|=m_2}\Big\|\prod_{j=1}^{2}(D^{\beta_j}A_j - m_{Q^*}(D^{\beta_j}A_j))\Big\|_{(\exp L)^{1/2},Q^*}\\ &\quad\times\|f\|_{L(\log L)^2,Q^*} + CM^2f(x)\\ &\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)M^3f(x) + CM^2f(x)\\ &\le cM^3f(x).\end{aligned}$$

Now we turn to estimating $T_{A_1,A_2}f_2$. Let

$$\widetilde{A}_j(y) = A_j(y) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}m_{Q^*}(D^\alpha A_j)y^\alpha.$$

It is easy to see that $P_{m_j+1}(A_j;y,z) = P_{m_j+1}(\widetilde{A}_j;y,z)$. Let $y_1 \in 4Q\setminus 2Q$ such that $|T_{A_1,A_2}f_2(y_1)| < \infty$. Write

$$
\begin{aligned}
|T_{A_1,A_2}f_2(y) - T_{A_1,A_2}f_2(y_1)|
&\le C\int_{\mathbb{R}^n} |y_1-z|^{-n-m}|P_{m_1+1}(\widetilde{A}_1;y,z)f_2(z)| \\
&\quad \times |P_{m_2+1}(\widetilde{A}_2;y,z) - P_{m_2+1}(\widetilde{A}_2;y_1,z)|\,dz \\
&\quad + C\int_{\mathbb{R}^n} |y_1-z|^{-n-m}|P_{m_2+1}(\widetilde{A}_2;y_1,z)f_2(z)| \\
&\quad \times |P_{m_1+1}(\widetilde{A}_1;y,z) - P_{m_1+1}(\widetilde{A}_1;y_1,z)|\,dz \\
&\quad + C\int_{\mathbb{R}^n} \left|\frac{K(y-z)}{|y-z|^m} - \frac{K(y_1-z)}{|y_1-z|^m}\right| \prod_{j=1}^{2}|P_{m_j+1}(\widetilde{A}_j;y,z)|\cdot|f_2(z)|\,dz \\
&\equiv T^{\mathrm{I}}_{A_1,A_2}f_2(y) + T^{\mathrm{II}}_{A_1,A_2}f_2(y) + T^{\mathrm{III}}_{A_1,A_2}f_2(y).
\end{aligned}
$$

Note that for $b\in \mathrm{BMO}(\mathbb{R}^n)$ and $k\in\mathbb{N}$, $|m_{Q^*}(b) - m_{2^kQ^*}(b)| \le Ck\|b\|_{\mathrm{BMO}}$. Let $n<q<\infty$. By Lemma 1, we obtain

$$
\begin{aligned}
|P_{m_j+1}(\widetilde{A}_j;y,z)|
&\le |P_{m_j}(\widetilde{A}_j;y,z)| + C|y-z|^{m_j}\sum_{|\beta_j|=m_j}|D^{\beta_j}A_j(z) - m_{Q^*}(D^{\beta_j}A_j)| \\
&\le C|y-z|^{m_j}\sum_{|\beta_j|=m_j}\left(\frac{1}{|\widetilde{Q}(y,z)|}\|D^{\beta_j}A_j(z) - m_{Q^*}(D^{\beta}A_j)\|^q_{q,\widetilde{Q}(y,z)}\right)^{1/q} \\
&\quad + C|y-z|^{m_j}\sum_{|\beta_j|=m_j}|D^{\beta_j}A_j(z) - m_{Q^*}(D^{\beta}A_j)|.
\end{aligned}
$$

Therefore, for each $y\in Q$ and $z\in 2^kQ^*\setminus 2^{k-1}Q^*$ ($k\in\mathbb{N}$),

$$
\begin{aligned}
(8)\qquad |P_{m_j+1}(\widetilde{A}_j;y,z)|
&\le Ck|y-z|^{m_j}\sum_{|\beta_j|=m_j}(1 + |D^{\beta_j}A_j(z) - m_{2^kQ^*}(D^{\beta_j}A_j)|).
\end{aligned}
$$

This in turn leads to

$$
\begin{aligned}
\prod_{j=1}^{2}|P_{m_j+1}(\widetilde{A}_j;y,z)|
&\le Ck^2|y-z|^m\prod_{j=1}^{2}\Big(1 + \sum_{|\beta_j|=m_j}|D^{\beta_j}A_j(z) - m_{2^kQ^*}(D^{\beta_j}A_j)|\Big).
\end{aligned}
$$

From this, by Lemma 3, the generalized Hölder inequality and (7), we see that

$$
\begin{aligned}
&T^{\mathrm{III}}_{A_1,A_2}f_2(y) \\
&\le Cd\sum_{k=1}^{\infty}k^2(2^kd)^{-n-1}\int_{2^kQ^*}|f(z)|\,dz \\
&\quad + Cd\sum_{|\beta_2|=m_2}\sum_{k=1}^{\infty}k^2(2^kd)^{-n-1}\int_{2^kQ^*}|D^{\beta_2}A_2(z) - m_{2^kQ^*}(D^{\beta_2}A_2)|\cdot|f(z)|\,dz \\
&\quad + Cd\sum_{|\beta_1|=m_1}\sum_{k=1}^{\infty}k^2(2^kd)^{-n-1}\int_{2^kQ^*}|D^{\beta_1}A_1(z) - m_{2^kQ^*}(D^{\beta_1}A_1)|\cdot|f(z)|\,dz \\
&\quad + Cd\sum_{|\beta_1|=m_1}\sum_{|\beta_2|=m_2}\sum_{k=1}^{\infty}k^2(2^kd)^{-n-1} \\
&\quad \times \int_{2^kQ^*}\prod_{j=1}^{2}|D^{\beta_j}A_j(z) - m_{2^kQ^*}(D^{\beta_j}A_j)|\cdot|f(z)|\,dz \\
&\le C\Big(\sum_{k=1}^{\infty}\frac{k^2}{2^k}\Big)Mf(x) \\
&\quad + C\sum_{k=1}^{\infty}\frac{k^2}{2^k}\|f\|_{L\log^+L,2^kQ^*} \\
&\quad \times\Big(\sum_{|\beta_1|=m_1}\|D^{\beta_1}A_1 - m_{2^kQ^*}(D^{\beta_1}A_1)\|_{\exp L,2^kQ^*} \\
&\quad + \sum_{|\beta_2|=m_2}\|D^{\beta_2}A_2 - m_{2^kQ^*}(D^{\beta_2}A_2)\|_{\exp L,2^kQ^*}\Big) \\
&\quad + C\sum_{|\beta_1|=m_1}\sum_{|\beta_2|=m_2}\sum_{k=1}^{\infty}\frac{k^2}{2^k}\Big\|\prod_{j=1}^{2}|D^{\beta_j}A_j - m_{2^kQ^*}(D^{\beta_j}A_j)|\Big\|_{(\exp L)^{1/2},2^kQ^*} \\
&\quad \times\|f\|_{L(\log L)^2,2^kQ^*} \\
&\le CMf(x) + C\Big(\sum_{k=1}^{\infty}\frac{k^2}{2^k}\Big)(M^2f(x) + M^3f(x)) \\
&\le CM^3f(x).
\end{aligned}
$$

The estimates for the terms $T^{\mathrm{I}}_{A_1,A_2}f_2$ and $T^{\mathrm{II}}_{A_1,A_2}f_2$ are very much similar; so we only deal with $T^{\mathrm{I}}_{A_1,A_2}f_2$. Applying the formula (see [4, p. 448])

$$P_k(b;y,z) - P_k(b;y_1,z) = P_{k-1}(b;y,y_1) + \sum_{|\alpha|=k-1} \frac{1}{\alpha!} D^\alpha b(z)(y-y_1)^\alpha + \sum_{0<|\alpha|<k-1} \frac{1}{\alpha!} P_{k-|\alpha|}(D^\alpha b; y_1, z)(y-y_1)^\alpha$$

for any $b \in \mathrm{BMO}$ and $k \in \mathbb{N}$, we may write that for $y \in Q$,

$$\begin{aligned}
&T^{\mathrm{I}}_{A_1,A_2}f_2(y)\\
&\le |P_{m_2}(\widetilde{A}_2;y,y_1)| \int_{\mathbb{R}^n} |y_1-z|^{-n-m}|P_{m_1+1}(\widetilde{A}_1;y,z)f_2(z)|\,dz\\
&\quad + \sum_{0<|\alpha|<m_2} d^{|\alpha|} \int_{\mathbb{R}^n} |y_1-z|^{-n-m}|P_{m_1+1}(\widetilde{A}_1;y,z)|\\
&\quad \times |P_{m_2+1-|\alpha|}(D^\alpha \widetilde{A}_2; y_1,z)|\cdot|f_2(z)|\,dz\\
&\quad + d^{m_2} \sum_{|\beta|=m_2} \int_{\mathbb{R}^n} |y_1-z|^{-n-m}|P_{m_1+1}(\widetilde{A}_1;y,z)||f_2(z)D^{\beta_2}\widetilde{A}_2(z)|\,dz\\
&\equiv \mathrm{U}+\mathrm{V}+\mathrm{W}.
\end{aligned}$$

By Lemma 1, we easily see that for $y \in Q$,

$$|P_{m_2}(\widetilde{A}_2;y,y_1)| \le C|y-y_1|^{m_2}.$$

Thus by the inequality (8),

$$\begin{aligned}
\mathrm{U} &\le Cd^{m_2} \sum_{|\beta|=m_1} \sum_{k=1}^\infty k(2^kd)^{-n-m_2}\\
&\quad \times \int_{2^kQ} |D^\beta A_1(z) - m_{2^kQ^*}(D^\beta A_1)|\cdot|f(z)|\,dz\\
&\quad + Cd^{m_2} \sum_{k=1}^\infty k(2^kd)^{-n-m_2} \int_{2^kQ^*} |f(z)|\,dz\\
&\le CM^2f(x).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathrm{V}+\mathrm{W} &\le Cd\sum_{k=1}^\infty k^2(2^kd)^{-n-1} \int_{2^kQ^*} \prod_{j=1}^2 \Big(1+\sum_{|\beta_j|=m_1} |D^{\beta_1}\widetilde{A}_1(z)|\Big)|f(z)|\,dz\\
&\le CM^3f(x).
\end{aligned}$$

Combining the estimates for $T^{\mathrm{I}}_{A_1,A_2}f_2$, $T^{\mathrm{II}}_{A_1,A_2}f_2$ and $T^{\mathrm{III}}_{A_1,A_2}f_2$ establishes the estimate (4), and completes the proof of Theorem 1.

4. Proof of Theorem 2. Let $\widetilde{K} \in C^1(\mathbb{R}^n \setminus \{0\})$ be homogeneous of degree zero and satisfy, for some positive constant C,

$$|\widetilde{K}(x)| \le C|x|^{-n}, \quad |\nabla\widetilde{K}(x)| \le C|x|^{-n-1}, \quad |x|>0,$$

and

$$\int_{|x|=1} \widetilde{K}(x)x^\gamma\,dx = 0, \quad |\gamma|=k,$$

where k is a positive integer. Define the multilinear operator

$$T_Af(x) = \mathrm{p.v.}\int_{\mathbb{R}^n} \widetilde{K}(x-y)\frac{P_{k+1}(A;x,y)}{|x-y|^k}f(y)\,dy. \tag{9}$$

For a positive integer l and $\mathrm{BMO}(\mathbb{R}^n)$ functions $b_1,\ldots,b_l$, we define the commutator of T_A with $b_1,\ldots,b_l$ by

$$T_{A;b_1,\ldots,b_l}f(x) = \mathrm{p.v.}\int_{\mathbb{R}^n} \widetilde{K}(x-y)\prod_{j=1}^l (b_j(x)-b_j(y))\frac{P_{k+1}(A;x,y)}{|x-y|^k}f(y)\,dy.$$

Repeating the proof of Theorem 1 and combining some techniques of Pérez used in [11], we can prove the following theorem. We omit the details.

THEOREM 1′. *Let A have derivatives of order k in* $\mathrm{BMO}(\mathbb{R}^n)$, *$T_A$ be the operator defined by* (9) *and $0<r<1$. Then there exists a positive constant $C=C_{n,r}$ independent of f such that*

$$(T_Af)^\#_r(x) \le C\sum_{|\beta|=k} \|D^\beta A\|_{\mathrm{BMO}} M^2f(x).$$

Furthermore, let $0<r<\varepsilon<1$ and $l\in\mathbb{N}$, $b_1,\ldots,b_l \in$ $\mathrm{BMO}(\mathbb{R}^n)$. *Then for any $f \in C_0^\infty(\mathbb{R}^n)$, the sharp estimate*

$$\begin{aligned}
(T_{A;b_1,\ldots,b_l})^\#_r(x) &\le C\sum_{|\beta|=k} \|D^\beta A\|_{\mathrm{BMO}} \prod_{j=1}^l \|b_j\|_{\mathrm{BMO}} M^{l+2}f(x)\\
&\quad + C\|b_l\|_{\mathrm{BMO}}\Big(M_\varepsilon(T_Af)(x) + \sum_{j=1}^{l-1} M_\varepsilon(T_{A;b_1,\ldots,b_j}f)(x)\Big)
\end{aligned}$$

holds and the constant C depends only on n, r and ε.

Proof of Theorem 2. Let $A_p(\mathbb{R}^n)$ be the weight function class of Muckenhoupt and $A_\infty(\mathbb{R}^n) = \bigcup_{p\ge1} A_p(\mathbb{R}^n)$ (see [8]). Set $K^*(x) = K(-x)$. Let

$T^*_{A_1,A_2}$ be the dual operator of T_{A_1,A_2}, that is,

$$T^*_{A_1,A_2}f(x) = \text{p.v.} \int_{\mathbb{R}^n} K^*(x-y)\frac{\prod_{j=1}^2 P_{m_j+1}(A_j;y,x)}{|x-y|^m} f(y)\,dy.$$

It is easy to see that K^* enjoys the same size condition and smoothness property as those of K. Thus we may view $T^*_{A_1,A_2}$ as

$$T^*_{A_1,A_2}f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)\frac{\prod_{j=1}^2 P_{m_j+1}(A_j;y,x)}{|x-y|^m} f(y)\,dy.$$

By the observation of Pérez in [10], it suffices to show that for fixed $1 < p < \infty$ and $u \in A_\infty(\mathbb{R}^n)$,

$$\|T^*_{A_1,A_2}f\|_{p,u} \le C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)\|M^3f\|_{p,u},$$

where C is independent of f. Write

$$\begin{aligned}
&T_{A_1,A_2}f(x) - T^*_{A_1,A_2}f(x)\\
&\quad= \text{p.v.} \int_{\mathbb{R}^n} K(x-y)P_{m_1+1}(A_1;x,y)\\
&\qquad\times \frac{(P_{m_2+1}(A_2;x,y) + P_{m_2+1}(A_2;y,x))}{|x-y|^m} f(y)\,dy\\
&\qquad+ \text{p.v.} \int_{\mathbb{R}^n} K(x-y)P_{m_2+1}(A_2;x,y)\\
&\qquad\times (P_{m_1+1}(A_1;x,y) + P_{m_1+1}(A_1;y,x))|x-y|^m f(y)\,dy\\
&\qquad- \text{p.v.} \int_{\mathbb{R}^n} K(x-y)\frac{\prod_{j=1}^2(P_{m_j+1}(A_j;x,y) + P_{m_j+1}(A_j;y,x))}{|x-y|^m} f(y)\,dy\\
&\quad\equiv T^{\mathrm{I}}f(x) + T^{\mathrm{II}}f(x) + T^{\mathrm{III}}f(x).
\end{aligned}$$

For each fixed multi-index α with $|\alpha| \le m_2$, let $K_\alpha(x) = K(x)|x|^{-|\alpha|}x^\alpha$ and

$$T^{\mathrm{I}}_\alpha f(x) = \text{p.v.} \int_{\mathbb{R}^n} K_\alpha(x-y)\frac{P_{m_1+1}(A_1;x,y)P_{m_2+1-|\alpha|}(D^\alpha A_2;x,y)}{|x-y|^{m-|\alpha|}} f(y)\,dy.$$

Note that for $|\alpha| = m_2$, T^{I}_α is just the commutator of the operator T_{A_1} defined by (9) with $D^\alpha A_2$. A straightforward computation shows that for any function b,

$$P_{k+1}(b;x,y) + P_{k+1}(b;y,x) = \sum_{0<|\alpha|\le k}\frac{(-1)^{|\alpha|+1}}{\alpha!}P_{k+1-|\alpha|}(D^\alpha b;x,y)(x-y)^\alpha.$$

So we have

$$|T^{\mathrm{I}}f(x)| \le \sum_{0<|\alpha|\le m_2}|T^{\mathrm{I}}_\alpha f(x)|.$$

Theorem 1 now states that for each $0 < r < 1$ and α with $0 < |\alpha| < m_2$,

$$(T^{\mathrm{I}}_\alpha f)^\#_r(x) \le C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)M^3f(x),$$

which together with the Córdoba–Fefferman inequality in [6] gives

$$\begin{aligned}
\sum_{|\alpha|<m_2}\|T^{\mathrm{I}}_\alpha f\|_{p,u} &= \sum_{|\alpha|<m_2}\|(T^{\mathrm{I}}_\alpha f)^r\|^{1/r}_{p/r,u} \le C\sum_{|\alpha|<m_2}\|(T^{\mathrm{I}}_\alpha f)^\#_r\|_{p,u}\\
&\le C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)\|M^3f\|_{p,u}.
\end{aligned}$$

For fixed $1 < p < \infty$ and $u \in A_\infty(\mathbb{R}^n)$, choose $0 < r < \varepsilon < 1$ such that $u \in A_{p/\varepsilon}(\mathbb{R}^n)$. It follows from Theorem 1′ that

$$\begin{aligned}
&\sum_{|\alpha|=m_2}\|T^{\mathrm{I}}_\alpha f\|_{p,u}\\
&\quad\le C\sum_{|\alpha|=m_2}\|(T^{\mathrm{I}}_\alpha f)^\#_r\|_{p,u}\\
&\quad\le C\|M_\varepsilon(T_{A_1}f)\|_{p,u} + C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)\|M^3f\|_{p,u}\\
&\quad\le C\|T_{A_1}f\|_{p,u} + C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)\|M^3f\|_{p,u}\\
&\quad\le C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)\|M^3f\|_{p,u}.
\end{aligned}$$

Therefore,

$$\|T^{\mathrm{I}}f\|_{p,u} \le C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)\|f\|_{p,u}.$$

Repeating the estimate for T^{I}, we can obtain

$$\|T^{\mathrm{II}}f + T^{\mathrm{III}}f\|_{p,u} \le C\prod_{j=1}^2\Big(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\Big)\|f\|_{p,u}.$$

On the other hand, Theorem 1 and the Córdoba–Fefferman inequality in [6] tell us that

$$\|T_{A_1,A_2}\|_{p,u} \le C \prod_{j=1}^{2} \Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \Big) \|f\|_{p,u};$$

and so

$$\|T^*_{A_1,A_2} f\|_{p,u} \le C \prod_{j=1}^{2} \Big(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \Big) \|f\|_{p,u}.$$

This finishes the proof of Theorem 1.

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Limit laws for products of free and independent random variables

by

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Abstract. We determine the distributional behavior of products of free (in the sense of Voiculescu) identically distributed random variables. Analogies and differences with the classical theory of independent random variables are then discussed.

1. Introduction. The concept of free independence introduced by D. Voiculescu has developed into a powerful noncommutative analogue of the classical notion of independence in probability theory. The book [8] provides an introduction to the area, showing in particular that some results about free random variables parallel in a rather striking fashion classical facts of probability theory. One instance of this parallelism occurs in our earlier work [2], where we studied the limiting behavior of sums of free, identically distributed infinitesimal random variables. More precisely, let $\{X_{ij} : i \ge 1,\ 1 \le j \le n_i\}$ be an array of classical independent random variables, and $\{Y_{ij} : i \ge 1,\ 1 \le j \le n_i\}$ an array of free random variables. Assume that $\lim_{i\to\infty} n_i = \infty$ and the variables $X_{i1}, \ldots, X_{in_i}, Y_{i1}, \ldots, Y_{in_i}$ are identically distributed for every i. The main result of [2] states that the variables $\sum_{j=1}^{n_i} X_{ij}$ have a limit in distribution as $i \to \infty$ if and only if the variables $\sum_{j=1}^{n_i} Y_{ij}$ do. Moreover, the classical and free limits are related in a rather explicit manner.

Our purpose in this paper is to develop a similar result for products of positive random variables. Here the parallelism between freeness and independence is not as perfect. An instance of this phenomenon was already seen in [5], where it was shown that there exist two free multiplicative "Poisson" laws with no commutative analogues.

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