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A Marcinkiewicz type multiplier theorem for H^1 spaces on product domains

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Abstract. It is proved that if $m : \mathbb{R}^d \to \mathbb{C}$ satisfies a suitable integral condition of Marcinkiewicz type then m is a Fourier multiplier on the H^1 space on the product domain $\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k}$. This implies an estimate of the norm $N(m, L^p(\mathbb{R}^d))$ of the multiplier transformation of m on $L^p(\mathbb{R}^d)$ as $p \to 1$. Precisely we get $N(m, L^p(\mathbb{R}^d)) \lesssim (p-1)^{-k}$. This bound is the best possible in general.

1. Introduction. If $X = X(\mathbb{R}^d)$ is a translation invariant function space on n-dimensional Euclidean space $\mathbb{R}^d$, we denote by $M(X)$ the space of all Fourier multipliers for $X(\mathbb{R}^d)$. For $\phi \in M(X)$ the norm of its multiplier transform T_ϕ will be written $N(\phi, X)$. In this paper we show some sufficient conditions for a function ϕ to be a multiplier for the Hardy spaces on product domains. Our main result, Theorem 1, states, roughly speaking, that ϕ is a multiplier on the H^1 space of a product domain if an integral condition of Marcinkiewicz–Hörmander type holds for derivatives of ϕ of sufficiently high order. As an important consequence we get the best possible general estimate of the growth of the norm of ϕ as a multiplier transform on L^p space as $p \to 1$.

A similar, and even stronger, result was proved by R. Fefferman in [F2] for products of two factors. In his proof a highly developed theory of multiparameter Hardy spaces is involved.

Our argument in the proof of Theorem 1 uses only the classical (non-product) theory of H^p spaces and the Sobolev integral representation of functions by means of their derivatives. The main idea is based on the observation that a function which is a "lacunary" sum of suitably rescaled kernels of the Sobolev representation is a Fourier multiplier on the classical H^1 space. A tensor product of such operators is a Fourier multiplier on a multiparameter H^1 space. Using then McCarthy's principle of "tensoring unconditionality" and the H^1 variant of Littlewood–Paley theory, we are

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able to prove that the above holds for a more general class of functions: suitable sums of tensor products of rescaled kernels of the Sobolev representation. This in turn allows us, using the Sobolev formula, to represent (separately in every dyadic parallelepiped) an arbitrary function satisfying the integral condition as a combination of multipliers of this special form.

The origin of this paper was the study of special classes of rational multipliers which occur as entries of the multiplier matrix for the so-called canonical projection of the jet representation of a general anisotropic Sobolev space (cf. [P], [BBPW]). While boundedness of $N(\phi, L^p(\mathbb{R}^d))$ for such a multiplier ϕ could be easily established by means of the multidimensional Marcinkiewicz theorem (cf. [S], Chapt. IV, Th. 6′), the order of growth of $N(\phi, L^p(\mathbb{R}^d))$ as p tends to 1 remains unknown (except some special cases). There is a wide gap between the estimate from above given by the Marcinkiewicz theorem, which actually could not be better than $(p-1)^{-3d/2}$ (cf. [B], Th. 1), and the estimate from below—the worst known ϕ of the above type gives the order of growth $(p-1)^{-d+1}$. It turns out that the attempt to diminish this gap leads to the study of multipliers on Hardy spaces on product domains. Theorem 1 could be regarded as a modification of the Marcinkiewicz theorem: the difference is that we derive a stronger conclusion from stronger assumptions. It is a simple consequence of Theorem 1 that in the case of rational multipliers appearing in the investigation of anisotropic Sobolev spaces the upper bound could be as low as $(p-1)^{-d}$. Moreover, in many particular cases Theorem 1 allows one to improve this bound.

The paper is organized as follows. Section 1 contains preliminaries and the definition of multiparameter H^p spaces. In Section 2 the main results are formulated. In Section 3 we study some properties of multiparameter H^p spaces, in particular we decompose a function from such an H^p as a sum of summands for which the Littlewood–Paley theorem holds. Section 4 contains the Sobolev integral representation of a function by its derivatives. In Section 5 we study the multiplier properties of kernels appearing in the representations from Section 4. The proof of the main result is given in Section 6. In Appendix A we give the proof of Lemma 6 on commuting families of projections.

Elements of $\mathbb{R}^d$ are denoted by x, y, z possibly with superscripts, and Greek characters ξ, η stand for elements of the dual group. Multiindices $\alpha, \beta, \ldots$ are elements of $\mathbb{Z}^d$ with non-negative coordinates and the symbols D^α, x^β have the usual meaning. The meaning of $|\cdot|$ depends on the context: $|x|$ is the Euclidean norm for $x \in \mathbb{R}^d$ while $|\alpha|$ is the l^1-norm for a multiindex α. The symbol $\mathbb{R}^d_+$ stands for elements of $\mathbb{R}^d$ with all coordinates positive, and $\mathbb{Z}^d_+$ for elements of $\mathbb{Z}^d$ with all coordinates non-negative. The notation $a \lesssim b$ for some variable quantities a, b means $a \le Cb$ for some constant $C > 0$; $a \simeq b$ stands for $a \lesssim b \lesssim a$.

We begin with the definition of H^p spaces on products of Euclidean spaces. We choose the definition most suitable for our purpose. For more information and a deeper understanding of those spaces one can consult the papers [ChF2], [F1], [F2].

Let $d_1, \ldots, d_k$ be positive integers. Let $x = (x^1, \ldots, x^k) \in \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k}$ where $x^j = (x^j_1, \ldots, x^j_{d_j})$ for $j = 1, \ldots, k$. For every coordinate $\eta \in \{\xi^j_1, \ldots, \xi^j_{d_j}\}$ of $\mathbb{R}^{d_j}$, we denote by R_η the Riesz transform on $L^1(\mathbb{R}^{d_j})$ with respect to the coordinate variable η, i.e. the translation invariant operator given by the formula

$$(R_\eta f)^\wedge(\xi^j) = \widehat{f}(\xi^j)\frac{\eta}{|\xi^j|}.$$

Then we denote by $\widetilde{R}_\eta$ the operator acting on $L^1(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$ as the tensor product of R_η acting on the coordinates $x^j_1, \ldots, x^j_{d_j}$ with identities on the other coordinates, i.e.

$$(\widetilde{R}_\eta f)(x) = R_\eta(f(x^1, \ldots, x^{j-1}, \cdot, x^{j+1}, \ldots, x^k))(x^j).$$

In other words $\widetilde{R}_\eta$ is given by the formula

$$(\widetilde{R}_\eta f)^\wedge(\xi) = \widehat{f}(\xi)\frac{\eta}{|\xi^j|}.$$

We let R_0 denote the identity operator. Set

$$S = \{(\sigma_1, \ldots, \sigma_k) : \sigma_j \in \{0, \xi^j_1, \ldots, \xi^j_{d_j}\} \text{ for } j = 1, \ldots, k\}.$$

Definition. The space $H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$ is the set of all functions f on $\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k}$ for which the norm

$$\|f\|_{H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})} = \Big(\sum_{\sigma \in S} \|\widetilde{R}_{\sigma_1} \ldots \widetilde{R}_{\sigma_k} f\|_p^p\Big)^{1/p}$$

is finite.

We obtain $H^p(\mathbb{R}^d)$ by specifying $k = 1$ and $d_1 = d$ in the above definition.

2. Results. Let $r = (r_1, \ldots, r_k) \in \mathbb{R}^k_+$. The set $Y_r = \{\xi = (\xi^1, \ldots, \xi^k) \in \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k} : r_j < |\xi^j| \le 2r_j,\ j = 1, \ldots, k\}$ is called the *r-frame* in $\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k}$.

The main result of this paper is the following

Theorem 1. *Let $m \in L^\infty(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$ satisfy*

$$(\mathrm{M}_1) \qquad r_1^{|\alpha^1| - d_1} \ldots r_k^{|\alpha^k| - d_k} \int_{Y_r} |D^\alpha m(\xi)|\, d\xi < C$$

for every $\alpha = (\alpha^1, \ldots, \alpha^k) \in \mathbb{Z}^{d_1}_+ \times \ldots \times \mathbb{Z}^{d_k}_+$ with $|\alpha^j| \le d_j + 1$ and every r-frame $Y_r \subset \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k}$. Then:

(i) *The multiplier transform T_m is a bounded operator on $H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$ for $1 \le p \le 2$. Moreover, its norm $N(m, H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k}))$ is uniformly bounded for $1 \le p \le 2$.*

(ii) *T_m is a bounded operator on $L^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$ with norm*

$$N(m, L^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})) \lesssim \max\left(p, \frac{p}{p-1}\right)^k \quad \text{where } 1 < p < \infty.$$

Part (ii) is an easy consequence of (i). Indeed, let $1 < p < 2$. The identical embedding $I_p : L^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k}) \to H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$ has norm

$$\|I_p\| < C\left(\frac{p}{p-1}\right)^k$$

while I_p^{-1} is a norm one operator. Hence the desired conclusion follows from the factorization $(T_m : L^p \to L^p) = I_p \circ (T_m : H^p \to H^p) \circ I_p^{-1}$.

REMARK 1. As the example of the characteristic function of the set $\mathbb{R}_+^d$ shows, the conclusion of Theorem 1(ii) cannot be strengthened (this multiplier is a tensor product of d different one-dimensional Riesz projections). Also the order of the derivatives for which the integral condition (M_1) has to be checked for Theorem 1 to hold is optimal, at least in the case of $d_1 = \ldots = d_k = 1$. This follows from Theorem 1 of [B], which yields that there exist multipliers $\phi \in \bigcup_{p>1} M(L^p(\mathbb{R}))$ satisfying the original Marcinkiewicz condition (i.e. (M_1) for the first derivative) such that $N(\phi, L^p(\mathbb{R})) \gtrsim (p-1)^{-3/2}$. On the other hand it is very likely that the order of the derivatives considered could be diminished if we assume integrability of their qth powers for some $q > 1$. A result of this type is proved in [F2] for $q = 2$ and for the domain being the product of two factors.

REMARK 2. In fact it is enough to prove Theorem 1 for $p = 1$. This follows from the result of S. Y. Chang and R. Fefferman (cf. [ChF1]; see also [Mu] and [X]) that the H^p spaces on a product domain form an interpolation scale.

REMARK 3. Our method of proof of Theorem 1 could be used to prove the Marcinkiewicz multiplier theorem itself. The difference is that we have to apply the Sobolev representation of a function by means of derivatives of lower order. Since the assumption of Marcinkiewicz's theorem is not sufficient to yield the H^1 boundedness of the multipliers involved, we have to deal with L^p spaces directly. But the constants in the Littlewood–Paley theorem for L^p depend on p. Hence we obtain a weaker estimate on the norm of the multiplier transform. More precisely for $\psi \in L^\infty(\mathbb{R}^d)$ satisfying the assumption of the Marcinkiewicz theorem, we find in this way that $\psi \in M(L^p(\mathbb{R}^d))$ for $1 < p < \infty$ with $N(\psi, L^p(\mathbb{R}^d)) \lesssim \max(p, (p-1)^{-1})^{2d}$. It seems to be unknown whether this estimate could be improved.

3. Some properties of H^p spaces. The next lemmas show the properties of the space $H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$ which will be used in our argument. Let $T : H^p(\mathbb{R}^{d_j}) \to H^p(\mathbb{R}^{d_j})$ be a translation invariant bounded operator. Then we denote by $\widetilde{T}$ the operator acting on $H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$ as the tensor product of T acting on the coordinates $x_1^j, \ldots, x_{d_j}^j$ with the identities on the other coordinates, i.e.

$$(\widetilde{T} f)(x) = T(f(x^1, \ldots, x^{j-1}, \cdot, x^{j+1}, \ldots, x^k))(x^j).$$

LEMMA 1. *Assume that $T : H^p(\mathbb{R}^{d_j}) \to H^p(\mathbb{R}^{d_j})$ is a bounded translation invariant operator for some $j = 1, \ldots, k$. Then the operator $\widetilde{T} : H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k}) \to H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$ is bounded.*

Proof. To simplify notation denote the coordinates $(x_1^j, \ldots, x_k^j)$ by x and the remaining coordinates by y. Let $\alpha \in S$. We get (remembering that translation invariant operators commute)

$$\begin{aligned}
\|\widetilde{R}_{\sigma_1} \ldots \widetilde{R}_{\sigma_k} \widetilde{T} f\|_p^p &= \iint |\widetilde{R}_{\sigma_1} \ldots \widetilde{R}_{\sigma_k} \widetilde{T} f(x,y)|^p \, dx \, dy \\
&= \iint |\widetilde{R}_{\sigma_j} \widetilde{T} \widetilde{R}_{\sigma_1} \ldots \widetilde{R}_{\sigma_{j-1}} \widetilde{R}_{\sigma_{j+1}} \ldots \widetilde{R}_{\sigma_k} f(x,y)|^p \, dx \, dy \\
&\le \int \|T(\widetilde{R}_{\sigma_1} \ldots \widetilde{R}_{\sigma_{j-1}} \widetilde{R}_{\sigma_{j+1}} \ldots \widetilde{R}_{\sigma_k} f(\cdot, y))\|_{H^p(\mathbb{R}^{d_j})}^p \, dy \\
&\lesssim \int \|\widetilde{R}_{\sigma_1} \ldots \widetilde{R}_{\sigma_{j-1}} \widetilde{R}_{\sigma_{j+1}} \ldots \widetilde{R}_{\sigma_k} f(\cdot, y)\|_{H^p(\mathbb{R}^{d_j})}^p \, dy \\
&\le \iint \sum_{\tau \in \{0, x_1^j, \ldots, x_{d_j}^j\}} |\widetilde{R}_\tau \widetilde{R}_{\sigma_1} \ldots \widetilde{R}_{\sigma_{j-1}} \widetilde{R}_{\sigma_{j+1}} \ldots \widetilde{R}_{\sigma_k} f(x,y)|^p \, dx \, dy \\
&\le \|f\|_{H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})}^p
\end{aligned}$$

since $(\sigma_1, \ldots, \sigma_{j-1}, \tau, \sigma_{j+1}, \ldots, \sigma_k) \in S$ for $\sigma \in S$ and $\tau \in \{0, \xi_1^j, \ldots, \xi_{d_j}^j\}$. ∎

The next lemma is a well known property of H^p spaces (cf. [S], Chapt. VII, Th. 9).

LEMMA 2. *The Riesz transforms are bounded operators on $H^p(\mathbb{R}^d)$ with uniformly bounded norm for $1 \le p \le 2$.* ∎.

LEMMA 3. *Every translation invariant bounded operator $T : H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k}) \to L^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$ is actually bounded from $H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$ into itself. Moreover, there exists $C > 0$ such that $\|T : H^p \to H^p\| \le C\|T : H^p \to L^p\|$ for $1 \le p \le 2$.*

Proof. By Lemma 2, the Riesz transforms are bounded on $H^p(\mathbb{R}^d)$. Hence, by Lemma 1, for every coordinate $\tau \in \bigcup_j \{\xi_1^j, \ldots, \xi_{d_j}^j\}$ the operator $\widetilde{R}_\tau$ is bounded on $H^p(\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_k})$. Therefore, since translation invariant operators commute,

$$\|Tf\|^p_{H^p(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})} = \sum_{\sigma\in S}\|\widetilde{R}_{\sigma_1}\ldots\widetilde{R}_{\sigma_k}Tf\|_p^p = \sum_{\sigma\in S}\|T\widetilde{R}_{\sigma_1}\ldots\widetilde{R}_{\sigma_k}f\|_p^p$$
$$\lesssim \sum_{\sigma\in S}\|\widetilde{R}_{\sigma_1}\ldots\widetilde{R}_{\sigma_k}f\|_p^p \lesssim \|f\|^p_{H^p(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})}.$$

The last inequality follows from the boundedness of $\widetilde{R}_\eta$ on $H^p(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})$ while the next to last one from the assumption of the lemma. ■

The considerations of our paper are based on the following classical result of Fefferman and Stein (cf. [FS], Sect. 3) on multipliers on $H^1(\mathbb{R}^n)$. We need the version with $q<2$ which is taken from [TW] (Th. 4.11).

LEMMA 4. *Let $m:\mathbb{R}^d\to\mathbb{C}$ be a bounded measurable function and $1\le q\le 2$. Suppose that there exists $C>0$ such that for every $\alpha\in\mathbb{Z}^d_+$ with $|\alpha|\le[d/q]+1$ and for every $r>0$,*

$$r^{q|\alpha|-d}\int_{\{r<|\xi|<2r\}}|D^\alpha m(\xi)|^q\,d\xi < C. \tag{1}$$

Then T_m is a bounded operator on $H^p(\mathbb{R}^d)$ and $N(m,H^p(\mathbb{R}^d))$ is uniformly bounded for $1\le p\le 2$. ■

Let $a>1$ and $c<(a-1)/(a+1)$. A sequence $\mathfrak{B}=(B(e_\nu,r_\nu))_{\nu\in\mathbb{Z}}$ of balls in $\mathbb{R}^d$ is called *(a,c)-Hadamard* if $|e_{\nu+1}|>a|e_\nu|$, $r_\nu<c|e_\nu|$ for $\nu\in\mathbb{Z}$ and the image of $\mathfrak{B}$ under some orthogonal transformation is contained in $\mathbb{R}^d_+$. A sequence of parallelepipeds indexed by $\mathbb{Z}^k$,

$$(B(e^1_{n_1},r_{n_1})\times\ldots\times B(e^k_{n_k},r_{n_k}))_{n\in\mathbb{Z}^k}\subset\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k},$$

is called *(a,c)-Hadamard* provided for every $j=1,\ldots,k$ the sequence of balls $(B(e^j_\nu,r_\nu))_{\nu\in\mathbb{Z}}\subset\mathbb{R}^{d_j}$ is (a,c)-Hadamard.

DEFINITION. Let $\mathfrak{B}$ be an (a,c)-Hadamard sequence of parallelepipeds in $\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k}$. We denote by $\mathrm{hlp}_p(\mathfrak{B})$ the space of all functions f from $H^p(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})$ such that $\operatorname{supp}\widehat{f}\subset\bigcup_{B\in\mathfrak{B}}B$.

REMARK 4. It follows from the theorem of Carleson (cf. [C], Th. 3) that in the above definition one can replace $H^p(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})$ by $L^p(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})$. The result of Carleson says that $H^1(\mathbb{R}^d)$ coincides with the subspace of $L^1(\mathbb{R}^d)$ generated by the functions whose Fourier transform is supported by the image of $\mathbb{R}^d_+$ under some orthogonal transformation. In particular $\|f\|_{H^p(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})}\simeq\|f\|_p$ for $f\in\mathrm{hlp}_p(\mathfrak{B})$ with equivalence constants independent of $p\in[1,2]$. We will use this equivalence below.

As an easy consequence of Lemma 4 and Remark 4 we get

LEMMA 5. *Let $\mathfrak{B}$ be an (a,c)-Hadamard family of balls in $\mathbb{R}^n$ and $f\in\mathrm{hlp}_p(\mathfrak{B})$. Suppose that m is a bounded function such that $m_{|\Delta}$ is constant for each $\Delta\in\mathfrak{B}$. Then*

$$\|T_mf\|_p\le C\|f\|_p$$

where $C=C(\mathfrak{B})$ does not depend on $p\in[1,2]$. ■

Let $\mathcal{P}$ be a boolean algebra of projections. We define

$$\|\mathcal{P}\| = \sup\Big\{\Big\|\sum_j a_jP_j\Big\| : a_j=\pm1,\ P_j \text{ are disjoint projections from } \mathcal{P}\Big\}.$$

Note that $\|\mathcal{P}\|$ is bounded iff there is a finite upper bound on $\|P\|$, $P\in\mathcal{P}$ (cf. [D]). We will need the following lemma due to McCarthy (cf. [MC]). Its proof is presented in Appendix A.

LEMMA 6. *Let $\mathcal{P}_1,\ldots,\mathcal{P}_d$ be commuting bounded boolean algebras of projections acting on a subspace X of L^p. Then the boolean algebra of projections $\mathcal{Q}$ generated by $\mathcal{P}_1\cup\ldots\cup\mathcal{P}_d$ is bounded and $\|\mathcal{Q}\|\le 2^{d/2}\prod_{j=1}^d\|\mathcal{P}_j\|^2$.*

The next lemma shows that the Littlewood–Paley theorem holds for functions from $\mathrm{hlp}_p(\mathfrak{B})$.

LEMMA 7. *Let $\mathfrak{B}=(\Delta_n)_{n\in\mathbb{Z}^k}$ be an (a,c)-Hadamard family of parallelepipeds in $\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k}$ and $f\in\mathrm{hlp}_p(\mathfrak{B})$. Let $f=\sum_{n\in\mathbb{Z}^k}f_n$ where $\operatorname{supp}\widehat{f}_n\subset\Delta_n$ for $n\in\mathbb{Z}^k$. Then there exist constants $A,B>0$ (depending only on $\mathfrak{B}$) such that for $1\le p\le 2$,*

$$A\|f\|_p^p<\int\Big(\sum_{n\in\mathbb{Z}^k}|f_n|^2\Big)^{p/2}<B\|f\|_p^p. \tag{2}$$

Proof. By Lemmas 5 and 1, for every $j=1,\ldots,k$ and every choice of signs $(c^{(j)}_i)_{i\in\mathbb{Z}}$ the multiplier transform of the function $m:\bigcup_n\Delta_n\to\mathbb{C}$ given by

$$m(\xi)=c^{(j)}_{n_j}\qquad\text{for }\xi\in\Delta_n$$

is a bounded operator on $\mathrm{hlp}_p(\mathfrak{B})$. Hence, by Lemma 6, for every choice of signs $(c_n)_{n\in\mathbb{Z}^k}$ the multiplier transform of the function $m:\bigcup_n\Delta_n\to\mathbb{C}$ given by

$$m(\xi)=c_n\qquad\text{for }\xi\in\Delta_n$$

is a bounded operator on $\mathrm{hlp}_p(\mathfrak{B})$ with norm bounded by some constant independent of $(c_n)_{n\in\mathbb{Z}^k}$ and $p\in[1,2]$. Therefore, by Khinchin's inequality (here $(r_n)_{n\in\mathbb{Z}^k}$ is a sequence of independent Bernoulli variables),

$$\int\Big(\sum_{n\in\mathbb{Z}^k}|f_n|^2\Big)^{p/2}\simeq\int\mathbf{E}_t\Big|\sum_{n\in\mathbb{Z}^k}r_n(t)f_n\Big|^p=\mathbf{E}_t\int\Big|\sum_{n\in\mathbb{Z}^k}r_n(t)f_n\Big|^p$$
$$\simeq\mathbf{E}_t\|f\|_p^p=\|f\|_p^p.\ ■$$

Next we show that every function from $H^p(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})$ admits a representation as a sum of functions from $\mathrm{hlp}_p(\mathfrak{B})$ for suitable families of parallelepipeds. More precisely we have:

LEMMA 8. *Let $a>1$ and $c<(a-1)/(a+1)$. There exist $l=l(a,c)$, $C=C(a,c)$, and Hadamard families $\mathfrak{B}_1,\ldots,\mathfrak{B}_l$ of parallelepipeds in $\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k}$ such that every $f\in H^p(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})$ has a representation $f=f_1+\ldots+f_l$ where $f_j\in\mathrm{hlp}_p(\mathfrak{B}_j)$ for $j=1,\ldots,l$ and*

$$\|f_1\|_p+\ldots+\|f_l\|_p<C\|f\|_p$$

where $C>0$ does not depend on $p\in[1,2]$.

Lemma 8 is an easy consequence of Lemma 1 and the following

LEMMA 9. *For every $a>1$ and $0<c<(a-1)/(a+1)$ there exist $l=l(a,c,d)$ and (a,c)-Hadamard sequences $\mathfrak{B}_1,\ldots,\mathfrak{B}_l$ of balls in $\mathbb{R}^d$ such that $\mathfrak{B}_1\cup\ldots\cup\mathfrak{B}_l$ is a covering of $\mathbb{R}^d\setminus\{0\}$ and there exists a smooth partition of unity $(\psi_B)_{B\in\mathfrak{B}_1\cup\ldots\cup\mathfrak{B}_l}$ subordinate to this covering, with the property that the function $F_j=\sum_{B\in\mathfrak{B}_j}\psi_B$ is a bounded multiplier on $H^p(\mathbb{R}^d)$ for $j=1,\ldots,k$.*

Proof. Pick $b>1$ and $\varrho\in C_0^\infty(\mathbb{R}^d)$ satisfying $\operatorname{supp}\varrho\subset\{\xi:|\xi|\le b\}$ and $\varrho(\xi)=1$ for $|\xi|\le1$. Then for $\nu\in\mathbb{Z}$ put $\varrho_\nu(\xi)=\varrho(b^\nu\xi)-\varrho(b^{\nu+1}\xi)$. Then $\sum_{\nu\in\mathbb{Z}}\varrho_\nu(\xi)=1$ for $\xi\ne0$. Let $(\lambda_j)_{j=1}^s$ be a smooth partition of unity on the unit sphere $\mathbb{S}^{d-1}=\{|\xi|=1\}$ subordinate to some finite covering of $\mathbb{S}^{d-1}$ by sets of diameter less than $b-1$. Let $B_{\nu,j}$ be the smallest ball which contains the support of the function $\psi_{\nu,j}(\xi)=\varrho_\nu(\xi)\lambda_j(\xi/|\xi|)$ ($\nu\in\mathbb{Z}$, $j=1,\ldots,s$). Then if b is sufficiently close to 1 and an integer r is large enough, for every $i=0,1,\ldots,r-1$ and $j=1,\ldots,s$, the sequence of balls $\mathfrak{B}_{i,j}=(B_{\nu r+i,j})_{\nu\in\mathbb{Z}}$ is (a,c)-Hadamard and, by Lemma 4, the function $F_{i,j}=\sum_{\nu\in\mathbb{Z}}\psi_{\nu r+i,j}$ is a bounded multiplier on $H^p(\mathbb{R}^d)$. ■

4. Integral representation. The next three lemmas describe the integral representation which we are going to use in the proof of Theorem 1.

LEMMA 10. *Let $m\in C_0^\infty(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})$ and $l=(l_1,\ldots,l_k)\in\mathbb{Z}_+^k$. Put $A_l=\{\alpha\in\mathbb{Z}^{d_1}\times\ldots\times\mathbb{Z}^{d_k}:|\alpha^j|=l_j$ for $j=1,\ldots,k\}$. There exists a constant K depending only on the numbers l_j and d_j such that*

$$m(\xi)=K\sum_{\alpha\in A_l}\frac{1}{\alpha!}\int D^\alpha m(\eta)\prod_{j=1}^k\frac{(\xi^j-\eta^j)^{\alpha^j}}{|\xi^j-\eta^j|^{d_j}}\,d\eta.$$

We get Lemma 10 by iterating the following well known Sobolev representation of a function by its derivatives (cf. [M], §1.1.10, Th. 2):

LEMMA 11. *Let $m\in C_0^\infty(\mathbb{R}^d)$ and $l\in\mathbb{Z}_+$. There exists a constant $K=K(l,d)$ such that*

$$m(\xi)=K\sum_{|\alpha|=l}\frac{1}{\alpha!}\int D^\alpha m(\eta)\frac{(\xi-\eta)^\alpha}{|\xi-\eta|^d}\,d\eta. \tag{3}$$

Next we state an (obvious) modification of Lemma 10 for functions with compact support. For $j=1,\ldots,k$ let $\phi^j:\mathbb{R}^{d_j}\to\mathbb{R}$ denote a smooth function satisfying $\operatorname{supp}\phi^j\subset B(0,2)$ and $\phi^j(\xi^j)=1$ for $\xi^j\in\mathbb{R}^{d_j}$ with $|\xi^j|\le1$. We put $\phi_s^j(\xi^j)=\phi^j(s^{-1}\xi^j)$ for $s>0$ and $\phi_r(\xi)=\prod_{j=1}^k\phi_{r_j}^j(\xi^j)$ for $r=(r_1,\ldots,r_k)\in\mathbb{R}_+^k$.

LEMMA 12. *Let $m\in C_0^\infty(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})$ satisfy*

$$\operatorname{supp}m\subset\Delta=B(e^1,r_1)\times\ldots\times B(e^k,r_k).$$

There exists a constant $K=K(l_1,\ldots,l_k,d_1,\ldots,d_k)$ such that for $\xi\in\Delta$,

$$m(\xi)=K\sum_{\alpha\in A_l}\frac{1}{\alpha!}\int_\Delta D^\alpha m(\eta)\prod_{j=1}^k\frac{(\xi^j-\eta^j)^{\alpha^j}}{|\xi^j-\eta^j|^{d_j}}\phi_{2r_j}^j(\xi^j-\eta^j)\,d\eta.$$

5. Multiplier properties of the representation kernels. In the next two lemmas we investigate the kernel appearing in the integral representation from Lemma 12. We show that the lacunary sum of suitably rescaled such kernels is a Fourier multiplier on H^p.

LEMMA 13. *Let $a>1$ and $c<\frac12(a-1)/(a+1)$. Let $\mathfrak{B}=(B(e_\nu,r_\nu))_{\nu\in\mathbb{Z}}$ be an (a,c)-Hadamard sequence of balls in $\mathbb{R}^d$. Let $\beta\in\mathbb{Z}_+^d$ satisfy $|\beta|\ge d+1$ and set*

$$F_\beta(\xi)=\sum_{\nu\in\mathbb{Z}}r_\nu^{d-|\beta|}\frac{(\xi-e_\nu)^\beta}{|\xi-e_\nu|^d}\phi_{2r_\nu}(\xi-e_\nu).$$

Then F_β is a bounded multiplier on $H^p(\mathbb{R}^d)$.

Proof. Let $q>1$. We get

$$\int_{B(e_\nu,4r_\nu)}|D^\alpha F_\beta(\xi)|^q\,d\xi=r_\nu^{q(d-|\beta|)}\int_{|\xi|\le4r_\nu}\left|D^\alpha\left(\frac{\xi^\beta}{|\xi|^d}\phi_{2r_\nu}(\xi)\right)\right|^q d\xi. \tag{4}$$

Notice now that $|D^\alpha(\xi^\beta/|\xi|^d)|^q$ is a homogeneous function of degree $q(|\beta|-|\alpha|-d)$ which is greater than $-d$ for $|\alpha|\le d$ and $1<q<d/(d-1)$. Hence the integral is finite. Since we also have $|D^\gamma\phi_{2r_\nu}|\lesssim r_\nu^{-|\gamma|}$, using the formula for the derivative of a product we get an estimate of the right hand side of (4) by $r_\nu^{q(d-|\beta|)}r_\nu^{q(|\beta|-|\alpha|-d)+d}=r_\nu^{d-q|\alpha|}$. This shows that inequality (1) holds for $m=F_\beta$, $1<q<d/(d-1)$, every $\alpha\in\mathbb{Z}_+^d$ with $|\alpha|\le[d/q]+1$ and $r>0$. Hence the lemma follows by Lemma 2. ■

Combining Lemma 1 with Lemma 13 we get

LEMMA 14. *Let $\mathfrak{B} = (B(e^1_{n_1}, r_{n_1}) \times \ldots \times B(e^k_{n_k}, r_{n_k}))_{n\in\mathbb{Z}^k}$ be an (a,c)-Hadamard sequence of parallelepipeds in $\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k}$. Let $\alpha = (\alpha^1,\ldots,\alpha^k) \in \mathbb{Z}^{d_1}_+ \times \ldots \times \mathbb{Z}^{d_k}_+$ satisfy $|\alpha^j| \ge d_j + 1$ and*

$$F_\alpha(\xi) = \sum_{n\in\mathbb{Z}^k} \prod_{j=1}^k r_{n_j}^{d_j - |\alpha^j|} \frac{(\xi^j - e^j_{n_j})^{\alpha^j}}{|\xi^j - e^j_{n_j}|^{d_j}} \phi_{2r_{n_j}}(\xi^j - e^j_{n_j}).$$

Then F_α is a bounded multiplier on $H^p(\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k})$. ■

For $\alpha = (\alpha^1,\ldots,\alpha^k) \in \mathbb{Z}^{d_1}_+ \times\ldots\times \mathbb{Z}^{d_k}_+$, $\eta = (\eta^1,\ldots,\eta^k) \in \mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k}$ and $r = (r_1,\ldots,r_k) \in \mathbb{R}^k_+$ let

$$\Phi^\alpha_{\eta,r}(\xi) = \prod_{j=1}^k r_j^{d_j - |\alpha^j|} \frac{(\xi^j - \eta^j)^{\alpha^j}}{|\xi^j - \eta^j|^{d_j}} \phi_{2r_j}(\xi^j - \eta^j).$$

6. Proof of Theorem 1. The next lemma, crucial to our considerations, provides the link between "general" multipliers and the multipliers of tensor product form.

LEMMA 15. *Let $c < \frac{1}{2}(a-1)/(a+1)$ and let $\mathfrak{B} = (\Delta_n)_{n\in\mathbb{Z}^k}$ be an (a,c)-Hadamard family of parallelepipeds in $\mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k}$, say $\Delta_n = B(e^1_{n_1}, r_{n_1}) \times\ldots\times B(e^k_{n_k}, r_{n_k})$ for $n\in\mathbb{Z}^k$. Let $\eta = (\eta_n)_{n\in\mathbb{Z}^k} \subset \mathbb{R}^{d_1}\times\ldots\times\mathbb{R}^{d_k}$ satisfy $\eta_n \in \Delta_n$ for $n\in\mathbb{Z}^k$ and let α satisfy the assumption of Lemma 14. Then the operator $S^\eta_\alpha : \mathrm{hlp}_p(\mathfrak{B}) \to \mathrm{hlp}_p(\mathfrak{B})$ given for $f = \sum_{n\in\mathbb{Z}^k} f_n$ by*

$$S^\eta_\alpha f = \sum_{n\in\mathbb{Z}^k} S^{\eta_n}_\alpha f_n = \sum_{n\in\mathbb{Z}^k} \widehat{\Phi}^\alpha_{\eta_n,r_n} * f_n \tag{5}$$

has norm bounded uniformly with respect to η and $p\in[1,2]$.

Proof. Let $\mathfrak{B}' = (\Delta'_n)_{n\in\mathbb{Z}^k}$ where $\Delta'_n = B(e^1_{n_1}, 2r_{n_1})\times\ldots\times B(e^k_{n_k}, 2r_{n_k})$ for $n\in\mathbb{Z}^k$. Since $c < \frac{1}{2}(a-1)/(a+1)$, the sequence $\mathfrak{B}'$ is $(a,2c)$-Hadamard. Put $e_n = (e^1_{n_1},\ldots,e^k_{n_k}) \in \Delta'_n$ for $n\in\mathbb{Z}^k$ and notice that for g with $\operatorname{supp}\widehat{g} \subset \Delta_n$ and $\eta_n \in \Delta_n$ we have $\operatorname{supp}(ge^{i\langle\eta_n - e_n,\cdot\rangle})^\wedge \subset \Delta'_n$. Therefore

$$\begin{aligned}\|S^\eta_\alpha f\|^p_p &= \Big\|\sum_{n\in\mathbb{Z}^k} \widehat{\Phi}^\alpha_{\eta_n,r_n} * f_n\Big\|^p_p \\ &= \Big\|\sum_{n\in\mathbb{Z}^k} (\widehat{\Phi}^\alpha_{e_n,r_n} e^{i\langle e_n - \eta_n,\cdot\rangle}) * f_n\Big\|^p_p \\ &= \Big\|\sum_{n\in\mathbb{Z}^k} (\widehat{\Phi}^\alpha_{e_n,r_n} * (f_n e^{i\langle\eta_n - e_n,\cdot\rangle}))e^{i\langle e_n-\eta_n,\cdot\rangle}\Big\|^p_p.\end{aligned}$$

Applying now twice Lemma 7 we get

$$\begin{aligned}\|S^\eta_\alpha f\|^p_p &\lesssim \int\Big(\sum_{n\in\mathbb{Z}^k} |(\widehat{\Phi}^\alpha_{e_n,r_n} * (f_n e^{i\langle\eta_n - e_n,\cdot\rangle}))e^{i\langle e_n-\eta_n,\cdot\rangle}|^2\Big)^{p/2} \\ &= \int\Big(\sum_{n\in\mathbb{Z}^k} |\widehat{\Phi}^\alpha_{e_n,r_n} * (f_n e^{i\langle\eta_n - e_n,\cdot\rangle})|^2\Big)^{p/2} \\ &\lesssim \Big\|\sum_{n\in\mathbb{Z}^k} \widehat{\Phi}^\alpha_{e_n,r_n} * (f_n e^{i\langle\eta_n - e_n,\cdot\rangle})\Big\|^p_p.\end{aligned}$$

Setting $e = (e_n)_{n\in\mathbb{Z}^k}$ and applying Lemma 14 we get

$$\|S^\eta_\alpha f\|^p_p \lesssim \Big\|S^e_\alpha\Big(\sum_{n\in\mathbb{Z}^k} f_n e^{i\langle\eta_n - e_n,\cdot\rangle}\Big)\Big\|^p_p \lesssim \Big\|\sum_{n\in\mathbb{Z}^k} f_n e^{i\langle\eta_n - e_n,\cdot\rangle}\Big\|^p_p.$$

Hence, using Lemma 7 twice we get

$$\|S^\eta_\alpha f\|^p_p \lesssim \int\Big(\sum_{n\in\mathbb{Z}^k} |f_n e^{i\langle\eta_n - e_n,\cdot\rangle}|^2\Big)^{p/2} = \int\Big(\sum_{n\in\mathbb{Z}^k} |f_n|^2\Big)^{p/2} \lesssim \|f\|^p_p.$$

Then the lemma follows by Lemma 3. ■

LEMMA 16. *Let $1 \le p \le \infty$. Assume that for every $\eta = (\eta_j)_{j\in I}$ where $\eta_n \in \Delta_n$ for $n \in I$, the operator $S^\eta = (S^{\eta_n}_n)_{n\in I} : L^p(\Omega, l^2) \to L^p(\Omega, l^2)$ has norm bounded by $C_1 > 0$. Let $\mu = (\mu_n)_{n\in I} \in \bigoplus M(\Delta_n)$ be a sequence of measures with total variations uniformly bounded by C_2. Then the operator $S_\mu : L^p(\Omega,l^2) \to L^p(\Omega,l^2)$ given by*

$$S_\mu f = \Big(\int_{\Delta_n} S^{\eta_n} f_n \, d\mu_n(\eta_n)\Big)_{n\in I}$$

has norm bounded by C_1C_2.

Proof. The formula $\mu \mapsto S_\mu$ defines a linear operator from the space $(\bigoplus M(\Delta_n))_{\ell^\infty}$ to the space of all linear operators on $L^p(\Omega,l^2)$. This operator is bounded since, by the assumption of the lemma, it is bounded on the extremal points of the unit ball of $(\bigoplus M(\Delta_n))_{\ell^\infty}$. ■

Proof of Theorem 1. By Lemma 8 it is enough to check the boundedness for functions $f \in \mathrm{hlp}_p(\mathfrak{B})$ for some (a,c)-Hadamard family of parallelepipeds $\mathfrak{B} = (\Delta_n)_{n\in\mathbb{Z}^k}$ with $a > 1$ and $c < \frac{1}{2}(a-1)/(a+1)$. Let $\Delta_n = B(e^1_{n_1}, r_{n_1}) \times \ldots\times B(e^k_{n_k}, r_{n_k})$ and $\Delta'_n = B(e^1_{n_1}, 2r_{n_1})\times\ldots\times B(e^k_{n_k}, 2r_{n_k})$ for $n\in\mathbb{Z}^k$. Let $\phi_n \in C^\infty_0(\mathbb{R}^d)$ satisfy:

$$\operatorname{supp}\phi_n \subset \Delta'_n, \tag{6}$$

$$\phi_{n|\Delta_n} \equiv 1, \tag{7}$$

and

$$|D^\alpha \phi_n(\xi)| < C \prod_{j=1}^{k} r_{n_j}^{-|\alpha^j|} \tag{8}$$

for $\alpha = (\alpha^1, \ldots, \alpha^k) \in \mathbb{Z}_+^{d_1} \times \ldots \times \mathbb{Z}_+^{d_k}$ with $|\alpha^j| \le d_j + 1$. Let $f = \sum_{n\in\mathbb{Z}^k} f_n$ where $\operatorname{supp} \widehat{f}_n \subset \Delta_n$. Put $m_n = \phi_n m$ for $n \in \mathbb{Z}^k$. Clearly, by (7),

$$T_m f_n = T_{m_n} f_n \tag{9}$$

and, by (8) and (M_1), for every $n \in \mathbb{Z}^k$ and $\alpha = (\alpha^1, \ldots, \alpha^k) \in \mathbb{Z}_+^{d_1} \times \ldots \times \mathbb{Z}_+^{d_k}$ with $|\alpha^j| = d_j + 1$,

$$r_{n_1}^{|\alpha^1| - d_1} \ldots r_{n_k}^{|\alpha^k| - d_k} \int_{\Delta_n'} |D^\alpha m_n(\eta)|\, d\eta < C.$$

Hence setting

$$\mu_n^\alpha = r_{n_1}^{|\alpha^1| - d_1} \ldots r_{n_k}^{|\alpha^k| - d_k} D^\alpha m_n(\eta)\, d\eta \tag{10}$$

we get $\|\mu_n^\alpha\| \le C$ for $n \in \mathbb{Z}^k$. Then, defining $A = \{\alpha \in \mathbb{Z}^{d_1} \times \ldots \times \mathbb{Z}^{d_k} : |\alpha^j| = d_j + 1 \text{ for } j = 1, \ldots, k\}$ and applying Lemma 12 we get, for $\xi \in \Delta_n'$,

$$\begin{aligned} m_n(\xi) &= K \sum_{\alpha\in A} \frac{1}{\alpha!} r_{n_1} \ldots r_{n_k} \int D^\alpha m(\eta) \Phi^\alpha_{\eta, r_n}(\xi)\, d\eta \\ &= K \sum_{\alpha\in A} \frac{1}{\alpha!} \int \Phi^\alpha_{\eta, r_n}(\xi)\, d\mu_n^\alpha(\eta). \end{aligned} \tag{11}$$

Therefore, using subsequently (9), (11), the triangle inequality, Lemma 7, (5), Lemma 16, Lemma 15 and again Lemma 7, we get

$$\begin{aligned} \|T_m f\|_p &= \Big\| \sum_{n\in\mathbb{Z}^k} T_{m_n} f_n \Big\|_p = \Big\| K \sum_{\alpha\in A} \frac{1}{\alpha!} \sum_{n\in\mathbb{Z}^k} \int \widehat{\Phi}^\alpha_{\eta_n, r_n} * f_n\, d\mu_n^\alpha(\eta_n) \Big\|_p \\ &\le K \sum_{\alpha\in A} \frac{1}{\alpha!} \Big\| \sum_{n\in\mathbb{Z}^k} \int \widehat{\Phi}^\alpha_{\eta_n, r_n} * f_n\, d\mu_n^\alpha(\eta_n) \Big\|_p \\ &\lesssim \sum_{\alpha\in A} \Big(\int \Big(\sum_{n\in\mathbb{Z}^k} \Big| \int \widehat{\Phi}^\alpha_{\eta_n, r_n} * f_n\, d\mu_n^\alpha(\eta_n) \Big|^2 \Big)^{p/2} \Big)^{1/p} \\ &= \sum_{\alpha\in A} \Big(\int \Big(\sum_{n\in\mathbb{Z}^k} \Big| \int_{\Delta_n'} S_\alpha^{\eta_n} f_n\, d\mu_n^\alpha(\eta_n) \Big|^2 \Big)^{p/2} \Big)^{1/p} \\ &\lesssim \sum_{\alpha\in A} \sup_{(\eta_n)} \Big(\int \Big(\sum_{n\in\mathbb{Z}^k} |S_\alpha^{\eta_n} f_n|^2 \Big)^{p/2} \Big)^{1/p} \\ &\lesssim \Big(\int \Big(\sum_{n\in\mathbb{Z}^k} |f_n|^2 \Big)^{p/2} \Big)^{1/p} \lesssim \|f\|_p. \end{aligned}$$

By Lemma 3 this ends the proof. ∎

Appendix A

Proof of Lemma 6. Let us recall a variant of Khinchin's inequality (cf. [S], App. D) which will be used below. Let r_k denote the kth Rademacher function ($k = 1, 2, \ldots$). For every sequence $(a_{jk})_{j,k\in\mathbb{Z}_+}$ of complex numbers,

$$\begin{aligned} \frac{1}{2} \Big(\sum_{j,k\in\mathbb{Z}_+} |a_{jk}|^2 \Big)^{1/2} &\le \mathbf{E}_t \mathbf{E}_s \Big| \sum_{j,k\in\mathbb{Z}_+} r_j(t) r_k(s) a_{jk} \Big| \\ &\le \Big(\sum_{j,k\in\mathbb{Z}_+} |a_{jk}|^2 \Big)^{1/2}. \end{aligned} \tag{A1}$$

We show how to prove the assertion for $d = 2$. The general case is similar. Denote the two boolean algebras of projections by $\mathcal{P}$ and $\mathcal{R}$. Clearly computing $\|Q\|$ it suffices to consider operators of the form $\sum_{j,k\le n} a_{jk} P_j R_k$ with $a_{jk} = \pm 1$, $P_j \in \mathcal{P}$ and $R_k \in \mathcal{R}$ for $j, k \le n$, satisfying

$$\sum_{j=1}^n P_j = \mathrm{Id}_X, \qquad \sum_{k=1}^n R_k = \mathrm{Id}_X \tag{A2}$$

(any operator which we use to compute $\|\mathcal{Q}\|$ is in the convex hull of such operators). For every choice of signs $c_1, \ldots, c_n$ the operator $\sum_j c_j P_j$ (which coincides with its inverse) has bound $\|\mathcal{P}\|$, and the same holds for $\mathcal{R}$. Thus

$$\begin{aligned} &\|\mathcal{P}\|^{-1} \|\mathcal{R}\|^{-1} \Big\| \sum_{j,k\le n} c_j d_k a_{jk} P_j R_k f \Big\|_1 \\ &\quad \le \Big\| \sum_{j,k\le n} a_{jk} P_j R_k f \Big\|_1 \le \|\mathcal{P}\| \cdot \|\mathcal{R}\| \cdot \Big\| \sum_{j,k\le n} c_j d_k a_{jk} P_j R_k f \Big\|_1. \end{aligned} \tag{A3}$$

We have

$$\begin{aligned} &\Big\| \sum a_{jk} P_j R_k f \Big\|_1 \\ &\quad = \mathbf{E}_t \mathbf{E}_s \Big\| \sum a_{jk} P_j R_k f \Big\|_1 \\ &\quad \le \|\mathcal{P}\| \cdot \|\mathcal{R}\| \cdot \mathbf{E}_t \mathbf{E}_s \Big\| \sum r_j(t) r_k(s) a_{jk} P_j R_k f \Big\|_1 && \text{by (A3)} \\ &\quad = \|\mathcal{P}\| \cdot \|\mathcal{R}\| \cdot \Big\| \mathbf{E}_t \mathbf{E}_s \Big| \sum r_j(t) r_k(s) a_{jk} P_j R_k f \Big| \Big\|_1 \\ &\quad \le \|\mathcal{P}\| \cdot \|\mathcal{R}\| \cdot \Big\| \Big(\sum |a_{jk} P_j R_k f|^2 \Big)^{1/2} \Big\|_1 && \text{by (A1)} \\ &\quad = \|\mathcal{P}\| \cdot \|\mathcal{R}\| \cdot \Big\| \Big(\sum |P_j R_k f|^2 \Big)^{1/2} \Big\|_1 \\ &\quad = 2\|\mathcal{P}\| \cdot \|\mathcal{R}\| \cdot \Big\| \mathbf{E}_t \mathbf{E}_s \Big| \sum r_j(t) r_k(s) P_j R_k f \Big| \Big\|_1 && \text{by (A1)} \end{aligned}$$

$$= 2\|\mathcal{P}\| \cdot \|\mathcal{R}\| \cdot \mathbf{E}_t \mathbf{E}_s \Big\| \sum r_j(t) r_k(s) P_j R_k f \Big\|_1$$

$$= 2\|\mathcal{P}\|^2 \|\mathcal{R}\|^2 \Big\| \sum P_j R_k f \Big\|_1 \qquad \text{by (A3)}$$

$$= 2\|\mathcal{P}\|^2 \|\mathcal{R}\|^2 \|f\|_1 \qquad \text{by (A2).} \ \blacksquare$$

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