Contents of Volume 138, Number 2

K. KURATA and S. SUGANO, Fundamental solution, eigenvalue asymptotics and eigenfunctions of degenerate elliptic operators with positive potentials . . 101–119
A. I. ALONSO, J. L. HONG and R. OBAYA, Absolutely continuous dynamics and real coboundary cocycles in L^p-spaces, $0 < p < \infty$ 121–134
P. CLÉMENT, B. DE PAGTER, F. A. SUKOCHEV and H. WITVLIET, Schauder decompositions and multiplier theorems 135–163
J. E. GALÉ and T. J. RANSFORD, On the growth of analytic semigroups along vertical lines 165–177
O. KALENDA, Valdivia compacta and equivalent norms 179–191
B. AUPETIT and M. MATHIEU, The continuity of Lie homomorphisms 193–199

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Fundamental solution, eigenvalue asymptotics and eigenfunctions of degenerate elliptic operators with positive potentials

by

KAZUHIRO KURATA and SATOKO SUGANO (Tokyo)

Abstract. We show a weighted version of Fefferman–Phong's inequality and apply it to give an estimate of fundamental solutions, eigenvalue asymptotics and exponential decay of eigenfunctions for certain degenerate elliptic operators of second order with positive potentials.

1. Introduction and main results. In [Sh1] Shen studied L^p boundedness of the operators VL^{-1}, $V^{1/2}L^{-1/2}$, $\nabla^2 L^{-1}$ for the Schrödinger operator $L = -\Delta + V$ with certain positive potentials V. Here L^{-1} is an integral operator with the minimal Green function (or minimal fundamental function) for L as its integral kernel (see e.g. [Mu], [Sm]). In [KS] we extended Shen's results to uniformly elliptic operators L and gave a simple proof of some part of his results. In particular our estimates imply boundedness of several operators on weighted L^p spaces and Morrey spaces. Furthermore, in [Su] Sugano investigated several estimates for the operators $V^\alpha(-\Delta+V)^{-\beta}$ and $V^\alpha\nabla(-\Delta+V)^{-\beta}$ by using arguments as in [Sh1] and [KS]. In [Sh2] Shen also studied eigenvalue asymptotics and exponential decay of eigenfunctions. The main ingredients of his work are the function $m(x,V)$ introduced to control the fundamental solution of L and Fefferman–Phong's inequality associated with this weight $m(x,V)$.

The purpose of this paper is to show a weighted version of Fefferman–Phong's inequality and its applications. The first application is to give an estimate of fundamental solutions of the following degenerate elliptic operators with positive potentials:

$$L = -\nabla(A(x)\nabla) + V(x),$$

where $V(x) \ge 0$, $A(x) = (a_{ij}(x))_{i,j=1}^n$ and $a_{ij}(x)$ is a measurable function

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satisfying

$$\mu w(x)|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \mu^{-1}w(x)|\xi|^2, \qquad \xi\in\mathbb{R}^n, \tag{1}$$

for some positive constant $\mu\in(0,1]$ and a non-negative measurable function w. The second application is to show the eigenvalue asymptotics and exponential decay of eigenfunctions of the operator

$$H=-\frac{1}{w}\operatorname{div}(A(x)\nabla)+U(x)$$

on $L^2(w\,dx)$, where $U\ge 0$. Throughout this paper we assume that w satisfies the so-called A_2 condition of Muckenhoupt.

DEFINITION 1. We say $w\in A_2$ if there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1}\,dx\right)\le C$$

for every ball $B\subset\mathbb{R}^n$.

We show the following estimate for the fundamental solution $\Gamma_w(x,y;V)$ of L under certain conditions on V: for every $k>0$ there exists a constant C_k such that

$$\Gamma_w(x,y;V)\le\frac{C_k}{(1+m_w(x,V/w)|x-y|)^k}\cdot\frac{|x-y|^2}{w(B(x,|x-y|))},$$

where $B(x,r)=\{y\in\mathbb{R}^n: |x-y|<r\}$, $w(B)=\int_B w(z)\,dz$ and $m_w(x,U)$ is defined by

$$\frac{1}{m_w(x,U)}=\sup\left\{r>0:\frac{r^2}{w(B(x,r))}\int_{B(x,r)}U(y)w(y)\,dy\le 1\right\}.$$

This type of estimate was first established by Shen [Sh1] for $L=-\Delta+V$ (see also [Zh]). By using this estimate, we can obtain the boundedness of the operator VL^{-1} on various spaces.

We also assume $U(x)\ge 0$ throughout this paper. We define the potential class $(RH)_q(w)$ which will be considered in this paper.

DEFINITION 2. (1) For $1<q<\infty$ we say $U\in(RH)_q(w)$ if there exists a constant C such that

$$\left(\frac{1}{w(B)}\int_B U(x)^q w(x)\,dx\right)^{1/q}\le\frac{C}{w(B)}\int_B U(x)w(x)\,dx \tag{2}$$

for every ball $B\subset\mathbb{R}^n$. If the inequality only holds for balls with radius less than or equal to 1, then we say $U\in(RH)_{q,\mathrm{loc}}(w)$.

(2) We say $U\in(RH)_\infty(w)$ if there exists a constant C such that

$$\sup_{x\in B}|U(x)|\le\frac{C}{w(B)}\int_B U(x)\,w(x)\,dx$$

for every ball $B\subset\mathbb{R}^n$.

Note that Hölder's inequality yields $(RH)_\infty(w)\subset(RH)_q(w)$ for $1<q<\infty$. It is well known that if $U\in(RH)_q(w)$, then $U\in(RH)_{q+\varepsilon}(w)$ for some $\varepsilon>0$ (see Lemma 1) and satisfies the doubling condition: there exists a constant C_0 such that

$$\frac{1}{w(B(x,2r))}\int_{B(x,2r)}U(y)w(y)\,dy\le C_0\frac{1}{w(B(x,r))}\int_{B(x,r)}U(y)w(y)\,dy$$

for every $x\in\mathbb{R}^n$ and $r>0$. We also note that almost all properties of $(RH)_q(w)$ hold for $(RH)_{q,\mathrm{loc}}(w)$ (see [Sh2]).

DEFINITION 3. (1) We say $w\in D_\gamma$ if there exists a constant $C>0$ such that $w(B(x,tr))\le Ct^\gamma w(B(x,r))$ for every $t>1$.

(2) We say $w\in(RD)_\nu$ if there exists a constant $C>0$ such that $w(B(x,tr))\ge Ct^\nu w(B(x,r))$ for every $t>1$.

We now state the main results of this paper. Let $\Gamma_w(x,y;V)$ be the fundamental solution to the operator $L=-\nabla(A(x)\nabla)+V(x)$ with $A(x)$ satisfying (1) and $V(x)\ge 0$.

THEOREM 1. *Assume $w\in A_2\cap(RD)_\nu\cap D_\gamma$ with $2<\nu\le\gamma$ and $V/w\in(RH)_q(w)$ for some $q>\gamma/2$. Then for each $k>0$ there exists a constant C_k such that*

$$(0\le)\ \Gamma_w(x,y;V)\le\frac{C_k}{(1+m_w(x,V/w)|x-y|)^k}\cdot\frac{|x-y|^2}{w(B(x,|x-y|))}.$$

THEOREM 2. *Let $S=m_w(\cdot,V/w)^2L^{-1}$. Under the same assumptions as in Theorem 1, there exists a constant $C>0$ such that*

$$|Sf(x)|\le CM_w(f/w)(x),$$

where M_w is the Hardy–Littlewood maximal function with respect to the measure $w(x)dx$, i.e., $(M_wf)(x)=\sup_{x\in B}w(B)^{-1}\int_B|f(x)|w(x)\,dx$. Hence we also have

$$\|Sf\|_{L^p(w\,dx)}\le C\|f/w\|_{L^p(w\,dx)}$$

for every $1<p<\infty$.

Theorem 2 is an extension of a result of J. Zhong [Zh, Lemma 3.2] in which only the non-degenerate case $A(x)\equiv I$ and non-negative polynomials V were considered.

THEOREM 3. *Under the same assumptions as in Theorem* 1, *there exists a constant* $C > 0$ *such that*

$$|T^* f(x)| \le C\{M_w(|f|^{q'})(x)\}^{1/q'},$$

where $1/q' + 1/q = 1$ *and* T^* *is the conjugate operator to* $T = VL^{-1}$.

REMARK 1. It is known that $w \in A_2$ implies $w \in D_{2n}$. Hence Theorems 1 and 2 hold for $w \in A_2 \cap (RD)_\nu$ with $2 < \nu$ and $V/w \in (RH)_q(w)$ for some $q > n$.

COROLLARY 1. (a) *Let* $V/w \in (RH)_\infty(w)$ *and* $T = VL^{-1}$. *Then, under the same assumptions as in Theorem* 1, *we have*

$$\|Tf\|_{L^p(\sigma dx)} \le C\|f\|_{L^p(\sigma dx)}$$

for every $1 < p < \infty$, *where* $\sigma(x) = w(x)^{1-p}$.

(b) *Suppose* $V/w \in (RH)_q(w)$ *with* $q > \gamma/2$. *Then*

$$\|Tf\|_{L^p(\sigma dx)} \le C\|f\|_{L^p(\sigma dx)}$$

for every $1 < p < q$, *where* $\sigma(x) = w(x)^{1-p}$.

Next, we consider the operator

$$H = -\frac{1}{w}\operatorname{div}(A(x)\nabla) + U(x)$$

on $L^2(wdx)$, where $U \ge 0, U \in (RH)_q(w)$ for some $q > \gamma/2$. Here we assume that $A(x)$ satisfies (1) with a weight $w \in A_2 \cap D_\gamma$ with $\gamma > 2$. Then the operator H can be realized as a self-adjoint operator on $L^2(wdx)$ by the Friedrichs extension (see e.g. [Da, Theorem 1.2.8]). For $\lambda > 0$, we denote by $N(\lambda, H)$ the number of eigenvalues to H less than or equal to λ. Then we have the following estimate for $N(\lambda, H)$.

THEOREM 4. *Assume* $w \in A_2 \cap D_\gamma \cap (RD)_\nu$ *with* $\gamma \ge \nu > 2$ *and* $U \ge 0$, $U + 1 \in (RH)_{q,\mathrm{loc}}(w)$ *with* $q > \gamma/2$. *Assume also that there exist constants* d_j, $j = 1, 2$, *such that*

$$0 < d_1 \le w(B(x,1)) \le d_2 < \infty \tag{3}$$

for every $x \in \mathbb{R}^n$. *Then there exist positive constants* C_j, $j = 1,2,3,4$, *such that for* $\lambda \ge 1$,

$$\begin{aligned} C_1\lambda^{\nu/2} w(\{x : m_w(x, U+1) \le C_2\sqrt{\lambda}\}) \\ \le N(\lambda, H) \le C_3\lambda^{\gamma/2} w(\{x : m_w(x, U+1) \le C_4\sqrt{\lambda}\}). \end{aligned}$$

Theorem 4 easily implies the following.

COROLLARY 2. *Under the same assumptions as in Theorem* 4, *there exist positive constants* C_j, $j = 1,2,3,4$, *such that for* $\lambda \ge 1$,

$$\begin{aligned} (w(x)dx \times dp)(\{|p|^{2n/\nu} + C_1 m_w(x, U+1)^2 < \lambda\}) \le C_2 N(\lambda, H), \\ C_3 N(\lambda, H) \le (w(x)dx \times dp)(\{|p|^{2n/\gamma} + C_4 m_w(x, U+1)^2 < \lambda\}). \end{aligned}$$

This generalizes Theorem 0.9 of [Sh2] which deals with the non-degenerate case $A(x) = I$. Next, we show the exponential decay of eigenfunctions $u \in L^2(wdx)$ with eigenvalue $\lambda \ge 1$ of H, that is, $Hu = \lambda u$ in $L^2(wdx)$. To state the result, we define the Agmon type distance $d(x,y)$ associated with the potential U and a weight w:

$$d(x,y) = \inf\Big\{\int_0^1 m_w(\gamma(t), U+1))|\gamma'(t)|\,dt : \gamma(0) = x,\ \gamma(1) = y\Big\}.$$

Let $E_\lambda = \{x \in \mathbb{R}^n : m_w(x, U+1) < \sqrt{\lambda}\}$ and define $d_\lambda(x) = \inf\{d(x,y) : y \in E_\lambda\}$.

THEOREM 5. *Under the same assumption as in Theorem* 4, *let* $u \in L^2(wdx)$ *satisfy* $Hu = \lambda u$ *in* $L^2(wdx)$ *for some* $\lambda \ge 1$. *Then for sufficientlly small* $\varepsilon > 0$ *there exist constants* C, C_ε *such that*

$$|u(x)| \le C_\varepsilon \lambda^{\gamma/4} e^{-\varepsilon d_{C\lambda}(x)}\|u\|_{L^2(wdx)} \quad (x \in \mathbb{R}^n).$$

COROLLARY 3. *Let* $u \in L^2(wdx)$ *be an eigenfunction of* H *with eigenvalue* $\lambda \ge 1$. *If* $m_w(x, U+1) \to \infty$, *then there exist constants* C_λ, C'_λ *such that*

$$|u(x)| \le C_\lambda e^{-\varepsilon d(x,0)}\|u\|_{L^2(wdx)} \quad \textit{and} \quad |u(x)| \le C'_\lambda e^{-\varepsilon|x|}\|u\|_{L^2(wdx)}.$$

We use the following notation throughout this paper: $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$ for $x \in \mathbb{R}^n$ and $r > 0$; $f \sim g$ means that there exist positive constants C_1 and C_2 such that $C_1 f \le g \le C_2 f$; $\nabla = (\nabla_1, \ldots, \nabla_n)$, $\nabla_j = \partial/\partial x_j$.

2. Preliminaries.

We collect some properties of the class $(RH)_q(w)$.

LEMMA 1. (i) *If* $U \in (RH)_q(w)$, *then* $U \in A_\infty(w)$.

(ii) *If* $U_1, U_2 \in (RH)_q(w)$, *then* $U = \alpha U_1 + \beta U_2 \in (RH)_q(w)$ *for every* $\alpha, \beta > 0$.

(iii) *If* $U_1, U_2 \in (RH)_{2q}(w)$, *then* $U = U_1 U_2 \in (RH)_q(w)$.

(iv) *If* $U \in (RH)_\infty(w)$, *then* $W = U^\alpha \in (RH)_\infty(w)$ *for every* $\alpha > 0$.

(v) *If* $U \in (RH)_q(w)$, *then* $U \in (RH)_{q+\varepsilon}(w)$ *for some* $\varepsilon > 0$.

Proof. For (i) see [CoF]. (ii) is easy. To show (iii), first by using Hölder's inequality and the assumption $U_j \in (RH)_{2q}(w)$, $j = 1, 2$, we have

$$\left(\frac{1}{w(B)}\int_B (U_1U_2)^q w\,dx\right)^{1/q} \le \left(\frac{1}{w(B)}\int_B U_1^{2q} w\,dx\right)^{1/(2q)}\left(\frac{1}{w(B)}\int_B U_2^{2q} w\,dx\right)^{1/(2q)}$$
$$\le C\left(\frac{1}{w(B)}\int_B U_1 w\,dx\right)\left(\frac{1}{w(B)}\int_B U_2 w\,dx\right)$$

for any ball B. It is known ([CoF]) that $U \in A_\infty(w)$ if and only if

$$\frac{1}{w(B)}\int_B U w\,dx \le C\exp\left(\frac{1}{w(B)}\int_B (\log U) w\,dx\right).$$

Hence by Jensen's inequality we obtain $U_1U_2 \in (RH)_q(w)$. To show (iv), it suffices to consider the case $\alpha \in (0,1)$. Then

$$\sup_B W = (\sup_B U)^\alpha \le \left(C\frac{1}{w(B)}\int_B U w\,dx\right)^\alpha$$
$$\le (\sup_B W^{1-\alpha})\left(\frac{1}{w(B)}\int_B U^\alpha w\,dx\right)^\alpha.$$

Hence we obtain (iv).

The property (v) is well known ([Ge]), but we give its proof based on the weight theory for the sake of completeness (cf. [SW]). It suffices to show that $U \in (RH)_q(w)$ implies $U^q \in A_\infty(w)$, because the weight theory says that for every $W \in A_\infty(w)$ there exists a constant $\delta > 0$ such that $W \in (RH)_{1+\delta}(w)$. Hence we obtain $U \in (RH)_{q(1+\delta)}(w)$. To show $U^q \in A_\infty(w)$ we use the characterization of $A_\infty(w)$ that appeared in the proof of (iii). Note the following identity:

$$\left(\frac{1}{w(B)}\int_B U^q w\,dx\right)^{1/q}\left(\exp\left(\frac{1}{w(B)}\int_B \left(\log\frac{1}{U^q}\right) w\,dx\right)\right)^{1/q}$$
$$= \left(\frac{1}{w(B)}\int_B U w\,dx\right)^{-1}\left(\frac{1}{w(B)}\int_B U^q w\,dx\right)^{1/q}$$
$$\times\left(\frac{1}{w(B)}\int_B U w\,dx\right)\left(\exp\left(\frac{1}{w(B)}\int_B \left(\log\frac{1}{U}\right) w\,dx\right)\right).$$

It follows that $U \in (RH)_q(w)$ implies

$$\left(\frac{1}{w(B)}\int_B U^q w\,dx\right)^{1/q}\left(\exp\left(\frac{1}{w(B)}\int_B \left(\log\frac{1}{U^q}\right) w\,dx\right)\right)^{1/q} \le C,$$

and hence $U^q \in A_\infty(w)$. ∎

The following lemma makes the quantity $m_w(x,U)$ well defined and $0 < m_w(x,U) < \infty$ for $U \in (RH)_q(w)$ with $q > \gamma/2$.

LEMMA 2. *Assume $w \in D_\gamma$ with $\gamma > 0$ and $U \in (RH)_q(w)$ with $q > \gamma/2$. Then*

$$\frac{r^2}{w(B_r(x))}\int_{B_r(x)} U(y)w(y)\,dy \le C_0\left(\frac{r}{R}\right)^\alpha \frac{R^2}{w(B_R(x))}\int_{B_R(x)} U(y)w(y)\,dy$$

for some positive constant C_0 and for every $x \in \mathbb{R}^n$ and $0 < r < R < \infty$, where $\alpha = 2 - \gamma/q > 0$.

Proof. We write $B_r = B_r(x)$ for simplicity. By using Hölder's inequality and (2), we have

$$\frac{r^2}{w(B_r)}\int_{B_r} U(y)w(y)\,dy \le C\left(\frac{r}{R}\right)^2\left(\frac{w(B_R)}{w(B_r)}\right)^{1/q}\frac{R^2}{w(B_R)}\int_{B_R} U(y)w(y)\,dy$$

for every $0 < r < R < \infty$. Since $w \in D_\gamma$, it follows that $w(B_R)/w(B_r) \le C(R/r)^\gamma$. Hence we arrive at the conclusion. ∎

LEMMA 3. *Under the same assumptions as in Lemma 2, if*

$$r^2 w(B_r(x))^{-1}\int_{B_r(x)} U(y)w(y)\,dy \sim 1,$$

then $r \sim m_w(x,U)^{-1}$.

Proof. We use the notation

$$\Phi(x,r) = \frac{r^2}{w(B_r(x))}\int_{B_r(x)} U(y)w(y)\,dy.$$

By assumption there exist positive constants C_1, C_2 such that

$$C_1 \le \frac{r^2}{w(B_r(x))}\int_{B_r(x)} U(y)w(y)\,dy \le C_2.$$

We may assume $0 < C_1 < 1 < C_2$ and that the constant C_0 in Lemma 2 satisfies $C_0 > 1$. Then Lemma 2 and the definition of $m_w(x,U)$ imply

$$(C_0C_2)^{-1/\alpha} r \le \frac{1}{m_w(x,U)} \le (C_0/C_1)^{1/\alpha} r.$$

Indeed, let $R = (C_0C_2)^{-1/\alpha} r$. Then $R < r$ and it follows from Lemma 2 that

$$\Phi(x,R) \le C_0(R/r)^\alpha \Phi(x,r) = \frac{1}{C_2}\Phi(x,r) \le 1.$$

Hence $m_w(x,U)^{-1} \ge (C_0C_2)^{-1/\alpha} r$. On the other hand, for every $R \ge (C_0/C_1)^{1/\alpha} r$, we have $R > r$ and $C_1 \le \Phi(x,r) \le C_0(r/R)^\alpha \Phi(x,R) \le C_1\Phi(x,R)$ by Lemma 2. Hence $m_w(x,U)^{-1} \le (C_0/C_1)^{1/\alpha} r$. ∎

LEMMA 4. *Under the same assumptions as in Lemma 2, we have the following properties.*

(i) *For any constant $C > 0$, we have $m_w(x,U) \sim m_w(y,U)$ if $|x-y| \le C/m_w(x,U)$.*

(ii) *There exist positive constants C_1, C_2, k_0 such that*

$$m_w(y,U) \le C_1(1+m_w(x,U)|x-y|)^{k_0} m_w(x,U),$$
$$m_w(y,U) \ge C_2 \frac{m_w(x,U)}{(1+m_w(x,U)|x-y|)^{k_0/(k_0+1)}}.$$

Proof. Recall that $U(x)w(x)dx$ is a doubling measure for $U \in (RH)_q(w)$ with $q > 1$. Since the proof is similar to that in [Sh1, Lemma 1.4] by using Lemmas 2 and 3, we omit the details. ∎

3. A weighted Fefferman–Phong inequality. In this section we show a weighted version of Fefferman–Phong's inequality. We also recall some estimates for the fundamental solution of $L_0 = -\nabla(A(x)\nabla)$. First we note the following Poincaré type inequality.

LEMMA 5. *Let $w \in A_2$. Then there exists a constant $C > 0$ such that*

$$\int_{B_r(x_0)} \int_{B_r(x_0)} |u(x)-u(y)|^2 w(x)w(y)\,dx\,dy \le Cr^2 w(B_r(x_0)) \int_{B_r(x_0)} |\nabla u(x)|^2 w(x)\,dx$$

for $u \in C^1(\overline{B_r(x_0)})$.

Proof. For simplicity, we write $B_r = B_r(x_0)$ in the proof. The assertion is an easy consequence of the following well known Poincaré inequality: under the assumption $w \in A_2$, we have

$$\int_{B_r} |u(x) - (u)_{B_r}|^2 w(x)\,dx \le Cr^2 \int_{B_r} w(x)|\nabla u(x)|^2\,dx, \tag{4}$$

where $(u)_{B_r} = w(B_r)^{-1}\int_{B_r} u(x)w(x)\,dx$. For the proof of (4) see [FKS]. Actually, for every $y \in B_r$, we have

$$\begin{aligned}\int_{B_r} |u(x)-u(y)|^2 w(x)\,dx &\le 2\int_{B_r} (|u(x)-(u)_{B_r}|^2 + |u(y)-(u)_{B_r}|^2) w(x)\,dx \\ &\le Cr^2 \int_{B_r} w(x)|\nabla u(x)|^2\,dx + 2w(B_r)|u(y)-(u)_{B_r}|^2.\end{aligned}$$

Hence by multiplying with $w(y)$, integrating on $y \in B_r$ and using the inequality (4) again, we arrive at the desired estimate. ∎

Next we show a weighted version of Fefferman–Phong's inequality. See [Sh1] for its original version (see also [Fe] and [Sm]).

LEMMA 6. *Assume $w \in A_2 \cap D_\gamma$ with $\gamma > 0$ and $U \in (RH)_q(w)$ with $q > \gamma/2$. Then*

$$\int_{\mathbb{R}^n} |u(x)|^2 m_w(x,U)^2 w(x)\,dx \le C\int_{\mathbb{R}^n} (|\nabla u(x)|^2 w(x) + |u(x)|^2 U(x)w(x))\,dx. \tag{5}$$

Proof. Let $x_0 \in \mathbb{R}^n$ and $r_0 = m_w(x_0,U)^{-1}$. Since U is an $A_\infty(w)$-weight, there exists a constant $\varepsilon > 0$ such that

$$w\bigg(\bigg\{x \in B_{r_0}(x_0) : U(x) \ge \frac{\varepsilon}{w(B_{r_0}(x_0))}\int_{B_{r_0}(x_0)} U(y)w(y)\,dy\bigg\}\bigg) \ge \frac{1}{2}w(B_{r_0}(x_0)).$$

Put

$$\overline{U} \equiv w(B_{r_0}(x_0))^{-1}\int_{B_{r_0}(x_0)} U(y)w(y)\,dy$$

and $A = \{y \in B_{r_0}(x_0) : U(x) \ge \varepsilon\overline{U}\}$. Then, by the definition of r_0, we have $\overline{U} = r_0^{-2}$ and $w(A) \ge \frac{1}{2}w(B_{r_0})$ with $B_{r_0} = B_{r_0}(x_0)$. Hence we obtain

$$\begin{aligned}\int_{B_{r_0}}\int_{B_{r_0}} \min(\varepsilon/r_0^2, U(y))|u(x)|^2 w(x)w(y)\,dx\,dy &\ge \int_{B_{r_0}} |u(x)|^2 w(x)\Big(\int_A (\varepsilon/r_0^2)w(y)\,dy\Big)\,dx \\ &\ge \frac{\varepsilon}{2r_0^2} w(B_{r_0})\int_{B_{r_0}} |u(x)|^2 w(x)\,dx.\end{aligned} \tag{6}$$

On the other hand, we will show

$$\int_{B_{r_0}}\int_{B_{r_0}} \min(\varepsilon/r_0^2, U(y))|u(x)|^2 w(x)w(y)\,dx\,dy \le Cw(B_{r_0})\Big(\int_{B_{r_0}} w(x)|\nabla u(x)|^2\,dx + \int_{B_{r_0}} |u(x)|^2 U(x)w(x)\,dx\Big). \tag{7}$$

Indeed, since $|u(x)|^2 \le 2(|u(x)-u(y)|^2 + |u(y)|^2)$, we have

$$\int_{B_{r_0}}\int_{B_{r_0}} \min(\varepsilon/r_0^2, U(y))|u(x)|^2 w(x)w(y)\,dx\,dy$$
$$\le 2\int_{B_{r_0}}\int_{B_{r_0}} \frac{\varepsilon}{r_0^2}|u(x)-u(y)|^2 w(x)w(y)\,dx\,dy$$
$$+ 2w(B_{r_0})\int_{B_{r_0}} |u(y)|^2 U(y)w(y)\,dy.$$

By Lemma 5, we arrive at the inequality (7). Therefore, by (6) and (7) we obtain

$$\int_{B_{r_0}} (|\nabla u|^2 + |u|^2 U(x))w(x)\,dx \ge \frac{C}{r_0^2}\int_{B_{r_0}} |u|^2 w\,dx.$$

By Lemma 4(i), we have $m_w(x,U) \sim 1/r_0$ for $x \in B_{r_0}$. Hence

$$\int_{B_{r_0}} m_w(x,U)^n(|\nabla u|^2 + |u|^2 U)w\,dx \ge C\int_{B_{r_0}} |u(x)|^2 m_w(x,U)^{n+2}w(x)\,dx.$$

Integrating this over $x_0 \in \mathbb{R}^n$ and changing the order of integration, we obtain the desired estimate (5). Here we use

$$\int_{|x-x_0|<1/m_w(x_0,U)} dx_0 \sim \int_{|x-x_0|<1/m_w(x,U)} dx_0 \sim m_w(x,U)^{-n}. \ \blacksquare$$

LEMMA 7. *Assume $w \in A_2 \cap (RD)_\nu$ with $\nu > 2$. Let $\Gamma^0(x,y)$ be the fundamental solution of L_0. Then*

$$(0 \le)\ \Gamma^0(x,y) \le C\frac{|x-y|^2}{w(B(x,|x-y|))}.$$

Proof. By the estimate in [FJK], we know

$$(0 \le)\ \Gamma^0(x,y) \le C\int_{|x-y|}^{\infty} \frac{s^2}{w(B(x,s))}\frac{ds}{s}.$$

The assumption $w \in (RD)_\nu$ with $\nu > 2$ implies that

$$\int_r^{\infty} \frac{s^2}{w(B(x,s))}\frac{ds}{s} \le C\frac{r^2}{w(B(x,r))}. \tag{8}$$

In fact, since $w \in (RD)_\nu$ with $\nu > 2$, we obtain

$$\int_r^{\infty} \frac{s^2}{w(B(x,s))}\frac{ds}{s} = \int_1^{\infty} \frac{r^2 s^2}{w(B(x,rs))}\frac{ds}{s}$$
$$\le C\int_1^{\infty} \frac{r^2 s^{2-\nu}}{w(B(x,r))}\frac{ds}{s} = C\frac{r^2}{w(B(x,r))}. \ \blacksquare$$

The observation (8) can also be found in [CSW, p. 316].

4. An estimate of fundamental solutions. First we note the following subsolution estimate.

LEMMA 8. *Assume $w \in A_2$. Let v be a non-negative subsolution of L_0 on $B_{2R} = B_{2R}(y)$. Then for all $\sigma \in (0,1)$ there exists a constant C_σ such that*

$$\sup_{x\in B_{\sigma R}(y)} v(x) \le \frac{C_\sigma}{w(B_R(y))}\int_{B_R(y)} v(x)w(x)\,dx.$$

Proof. We write $B_r = B_r(y)$ for simplicity. The result in [FKS] yields the following estimate:

$$\sup_{x\in B_{\sigma R}} v(x) \le C_\sigma\left(\frac{1}{w(B_R)}\int_{B_R} v(x)^2 w(x)\,dx\right)^{1/2}.$$

We conclude by using the reverse Hölder type estimate:

$$\left(\frac{1}{w(B_R)}\int_{B_R} v(x)^2 w(x)\,dx\right)^{1/2} \le C\frac{1}{w(B_{2R})}\int_{B_{2R}} v(x)w(x)\,dx.$$

This inequality can be shown by using the argument as in [Gu] (see also [CFG], [Ku1,2]). $\blacksquare$

LEMMA 9. *Assume $w \in A_2 \cap D_\gamma$ with $\gamma > 0$ and $V/w \in (RH)_q(w)$ with $q > \gamma/2$. Let u be a solution of $Lu = 0$ on $B_{4R}(x_0)$. Then for every $k > 0$ there exists a constant C_k such that*

$$\sup_{x\in B_R(x_0)} |u(x)|$$
$$\le \frac{C_k}{(1+Rm_w(x_0,V/w))^k}\left(\frac{1}{w(B_{2R}(x_0))}\int_{B_{2R}(x_0)} |u(x)|^2 w(x)\,dx\right)^{1/2}.$$

Proof. We write $B_R = B_R(x_0)$ for simplicity. Since

$$\frac{1}{2}\nabla(A(x)\nabla|u|^2) = V|u|^2 + \sum_{i,j=1}^{n} a_{ij}(x)\nabla_i u\nabla_j u \ge 0,$$

$v = |u|^2$ is a subsolution to $L_0 v = 0$. Hence Lemma 8 shows that

$$\sup_{B_R}|u| \le C\left(\frac{1}{w(B_{3R/2})}\int_{B_{3R/2}} |u|^2 w\,dx\right)^{1/2}. \tag{9}$$

By Caccioppoli's inequality, we have

$$\int_{B_{3R/2}} w|\nabla u|^2\,dx + \int_{B_{3R/2}} V|u|^2\,dx \le \frac{C}{R^2}\int_{B_{2R}} w|u|^2\,dx. \tag{10}$$

Hence (10) and Lemma 6 yield

$$\int_{B_R} m_w(x,V/w)^2|u(x)|^2w(x)\,dx \le \frac{C}{R^2}\int_{B_{2R}} w|u|^2\,dx.$$

By Lemma 4(ii),

$$m_w(x,V/w) \ge Cm_w(x_0,V/w)/(1+Rm_w(x_0,V/w))^{k_0/(k_0+1)}$$

for every $x\in B_R$. Thus

$$\int_{B_R}|u(x)|^2w(x)\,dx \le \frac{C}{(1+Rm_w(x_0,V/w))^{2/(k_0+1)}}\int_{B_{2R}}|u|^2w\,dx.$$

By repeating this argument and using (9) and $w\in D_\gamma$, we arrive at the desired estimate. ∎

Proof of Theorem 1. By the maximum principle (see e.g. [CW]), we have

$$0\le \Gamma_w(x,y;V)\le \Gamma^0(x,y).$$

Let $x_0\ne y\in\mathbb{R}^n$ and put $R=|x_0-y|$. Since $u(x)=\Gamma_w(x,y;V)$ satisfies $Lu=0$ on $B_{R/2}(x_0)$, Lemma 9 shows that

$$\sup_{x\in B_{R/4}(x_0)}|u(x)| \le \frac{C_k}{(1+Rm_w(x_0,V/w))^k}\left(\frac{1}{w(B_{R/2}(x_0))}\int_{B_{R/2}(x_0)}|u(x)|^2w(x)\,dx\right)^{1/2}.$$

Note that $\Gamma^0(x,y)\le C\Gamma^0(x_0,y)$ for $x\in B_{R/2}(x_0)$ by Harnack's inequality (see [FKS, Lemma 2.3.5]). Hence by Lemma 7, we have

$$\left(\frac{1}{w(B_{R/2}(x_0))}\int_{B_{R/2}(x_0)}|u(x)|^2w(x)\,dx\right)^{1/2}\le C\frac{R^2}{w(B(x_0,R))}.$$

Therefore we obtain

$$\Gamma_w(x_0,y;V)=u(x_0)\le\sup_{x\in B_{R/4}(x_0)}|u(x)| \le \frac{C_k}{(1+Rm_w(x_0,V/w))^k}\cdot\frac{R^2}{w(B(x_0,R))}.$$

Since $R=|x_0-y|$, we get the desired estimate. ∎

Once we obtain Theorem 1, we can prove Theorems 2 and 3 in a similar way to [KS].

Proof of Theorem 2. Let $f\in C_0^\infty$. By Theorem 1, we have

$$(11)\qquad |Sf(x)|\le\int\frac{m_w(x,V/w)^2C_k|x-y|^2}{(1+m_w(x,V/w)|x-y|)^kw(B(x,|x-y|))}|f(y)|\,dy.$$

Put $r=1/m_w(x,V/w)$. Then, by the doubling property of $w(x)$, the right hand side of (11) is dominated by

$$\sum_{j=-\infty}^{\infty}\int_{\{2^{j-1}r<|x-y|\le 2^jr\}}\frac{C_k}{r^2}\cdot\frac{(2^jr)^2}{(1+2^{j-1})^kw(B(x,2^jr))}|f(y)|\,dy \le CM_w(f/w)(x),$$

if we take $k\ge 3$. ∎

Proof of Theorem 3. Let $r=1/m_w(x,V/w)$. Then

$$\begin{aligned}
|T^*f(x)| &\le \int_{\mathbb{R}^n}\Gamma_w(y,x;V)V(y)|f(y)|\,dy\\
&\le \int_{\mathbb{R}^n}\frac{C_k|x-y|^2}{(1+m_w(x,V/w)|x-y|)^kw(B(x,|x-y|))}V(y)|f(y)|\,dy\\
&\le C_k\sum_{j=-\infty}^{\infty}\int_{\{2^{j-1}r<|x-y|\le 2^jr\}}\frac{(2^jr)^2}{(1+2^{j-1})^kw(B(x,2^{j-1}r))}V(y)|f(y)|\,dy\\
&\le C_k\sum_{j=-\infty}^{\infty}\frac{(2^jr)^2}{(1+2^{j-1})^k}\left[\frac{1}{w(B(x,2^{j-1}r))}\int_{B(x,2^{j-1}r)}\left(\frac{V(y)}{w(y)}\right)^qw(y)\,dy\right]^{1/q}\\
&\quad\times\left[\frac{1}{w(B(x,2^{j-1}r))}\int_{B(x,2^{j-1}r)}|f(y)|^{q'}w(y)\,dy\right]^{1/q'}\\
&\le C_k[M_w(|f|^{q'})(x)]^{1/q'}\sum_{j=-\infty}^{\infty}\frac{(2^jr)^2}{(1+2^{j-1})^k}\\
&\quad\times\left[\frac{1}{w(B(x,2^{j-1}r))}\int_{B(x,2^{j-1}r)}\left(\frac{V(y)}{w(y)}\right)w(y)\,dy\right].
\end{aligned}$$

Note that $V\,dx$ is a doubling measure because $V/w\in A_\infty(w)$.

For the case $j\ge1$, since $w\in(RD)_\nu$, we obtain

$$\frac{(2^jr)^2}{w(B(x,2^{j-1}r))}\int_{B(x,2^{j-1}r)}V(y)\,dy \le 2^{2j}C_0^{j-1}C2^{-(j-1)\nu}\frac{r^2}{w(B(x,r))}\int_{B(x,r)}V(y)\,dy=4C(2^{k_0})^{j-1},$$

where $k_0=2-\nu+\log_2C_0$. For the case $j\le0$, by Lemma 2 we obtain

$$\frac{(2^j r)^2}{w(B(x,2^{j-1}r))}\int_{B(x,2^{j-1}r)} V(y)\,dy$$
$$\le C\left(\frac{r}{2^{j-1}r}\right)^{\gamma/q-2}\frac{r^2}{w(B(x,r))}\int_{B(x,r)} V(y)\,dy = C(2^{j-1})^{2-\gamma/q}.$$

Hence if we take $k \ge k_0+1$ we conclude that

$$|T^* f(x)| \le C[M_w(|f|^{q'})(x)]^{1/q'}. \ \blacksquare$$

Proof of Corollary 1. Since $V(x)/w(x) \le Cm_w(x,V/w)^2$ by the definition of $m_w(x,V/w)$, Theorem 2 yields

$$\left\|\frac{Tf}{w}\right\|_{L^p(wdx)} \le C\left\|\frac{f}{w}\right\|_{L^p(wdx)}.$$

Part (a) follows from this. Let $1/p+1/p'=1$. For $q'<p'$ it is known that M_w is bounded on $L^{p'/q'}(wdx)$. Hence T^* is bounded on $L^{p'}(wdx)$ for $1<p<q$. By duality we obtain the boundedness of T on $L^p(w^{1-p}\,dx)$. This gives (b). $\blacksquare$

5. Eigenvalue asymptotics and decay of eigenfunctions. In this section we show Theorems 4 and 5. We denote by $Q(x,r)$ the cube with sidelength r and center $x\in\mathbb{R}^n$.

Proof of Theorem 4. We follow the argument of [Sh2].

STEP 1. First, we show the lower bound. Let $E_\lambda = \{x : m_w(x,U+1) \le \sqrt{\lambda}\}$ for $\lambda \ge 1$ and divide $\mathbb{R}^n$ into disjoint cubes $\{Q_l\}$ with sidelength $1/\sqrt{\lambda}$ whose sides are parallel to the coordinate axes. Let m be the number of cubes Q_l such that $Q_l \cap E_\lambda \ne \emptyset$ and let $Q_k = Q_{l_k} = Q(x_k, 1/\sqrt{\lambda})$ be such that $Q_{l_k}\cap E_\lambda \ne \emptyset$. Then

$$w(E_\lambda) = \sum_l w(Q_l\cap E_\lambda) \le \sum_{k=1}^m w(Q_k) \le Cd_2 m\lambda^{-\nu/2}.$$

Here we have used

$$w(Q(x,1/\sqrt{\lambda})) \le C(1/\sqrt{\lambda})^\nu w(Q(x,1))$$

since $w\in(RD)_\nu$ and $\sup_x w(Q(x,1)) \le d_2$ by assumption. Next we show that there exists an m-dimensional subspace $\mathcal{H}$ of $L^2(wdx)$ such that

$$\text{(12)}\qquad \sum_{i,j=1}^n \int a_{ij}(x)\nabla_i u\nabla_j u\,dx + \int Uu^2w\,dx \le C\lambda\int u^2 w\,dx \quad (u\in\mathcal{H}).$$

Now (12) and the min-max principle imply that

$$N(C\lambda, H) \ge m \ge C\frac{1}{d_2}\lambda^{\nu/2}w(E_\lambda).$$

Choose a function $\eta\in C_0^\infty(Q(0,1))$ such that $\eta(x)=1$ on $x\in Q(0,1/2)$ and put $\eta_{l,\lambda}(x) = \lambda^{\nu/4}\eta(\sqrt{\lambda}(x-x_l))$, $l=1,\dots,m$. Let $\mathcal{H}$ be the subspace spanned by $\{\eta_{k,\lambda}\}_{k=1}^m$. Since Q_k are disjoint, $\mathcal{H}$ is an m-dimensional subspace of $L^2(wdx)$. Let $L=\sup(|\nabla\eta(x)|+|\eta(x)|)$ and $r_k = 1/m_w(x_k,U+1)$. Then

$$\sum_{i,j=1}^n\int a_{ij}\nabla_i\eta_{k,\lambda}\nabla_j\eta_{k,\lambda}\,dx \le \mu^{-1}L^2\lambda[\lambda^{\nu/2}w(Q(x_k,1/\sqrt{\lambda}))],$$

and

$$\int U\eta_{k,\lambda}^2 w\,dx \le L^2\lambda^{\nu/2}\int_{Q_k} Uw\,dx$$
$$\le L^2\lambda[\lambda^{\nu/2}w(Q(x_k,1/\sqrt{\lambda}))]\frac{(1/\sqrt{\lambda})^2}{w(Q(x_k,1/\sqrt{\lambda}))}\int_{Q(x_k,1/\sqrt{\lambda})} Uw\,dx$$
$$\le L^2\lambda[\lambda^{\nu/2}w(Q(x_k,1/\sqrt{\lambda}))](\sqrt{\lambda}r_k)^{\gamma/q-2}\frac{r_k^2}{w(Q(x_k,r_k))}\int_{Q(x_k,r_k)} Uw\,dx$$
$$\le L^2\lambda(\lambda^{\nu/2}w(Q(x_k,1/\sqrt{\lambda}))).$$

Here we have used Lemma 3 and $1/\sqrt{\lambda}\le Cr_k$ because $Q_k\cap E_\lambda\ne\emptyset$. Hence it follows from the doubling property of w that

$$\text{(13)}\qquad \sum_{i,j=1}^n\int a_{ij}\nabla_i\eta_{k,\lambda}\nabla_j\eta_{k,\lambda}\,dx + \int U\eta_{k,\lambda}^2 w\,dx \le CL^2\lambda\left[\lambda^{\nu/2}w\left(Q\left(x_k,\frac{1}{2\sqrt{\lambda}}\right)\right)\right].$$

On the other hand, it is easy to see that

$$\text{(14)}\qquad \int\eta_{k,\lambda}^2 w\,dx \ge \lambda^{\nu/2}w\left(Q\left(x_k,\frac{1}{2\sqrt{\lambda}}\right)\right).$$

Thus by (13) and (14) we obtain the desired estimate.

STEP 2. Next, we show the upper bound. It suffices to show the existence of a subspace $\mathcal{H}$ satisfying the following conditions: there exist positive constants C_j, $j=1,2,3$, such that

$$\dim\mathcal{H} \le C_1\lambda^{\gamma/2}w(E_\lambda),$$
$$\int(|\nabla u|^2+Uu^2)w\,dx \ge C_2\lambda\int u^2w\,dx, \quad u\in\mathcal{H}^\perp,\ \lambda\ge C_3.$$

We need the following lemma which follows from Lemma 4.

LEMMA 10 ([Hö, Theorem 1.4.10]). *Let $U\in(RH)_{q,\mathrm{loc}}(w)$ with $q>\gamma/2$. Then there exist a sequence $\{x_l\}\subset\mathbb{R}^n$ and non-negative functions $\phi_l\in C_0^\infty(\mathbb{R}^n)$ satisfying the following conditions:*

(a) $\mathbb{R}^n = \bigcup_{l=1}^{\infty} B_l$, $B_l = B(x_l, m_w(x_l, U+1)^{-1})$;
(b) $\phi_l \in C_0^\infty(B_l)$, $\sum_{l=1}^{\infty} \phi_l(x) \equiv 1$ $(x \in \mathbb{R}^n)$;
(c) $|\partial_x^\beta \phi_l(x)| \le C m_w(x, U+1)^{|\beta|}$ *for every* $|\beta| \le 2$;
(d) $\sum_{l=1}^{\infty} \chi_{B_l}(x) \le C$.

Here the constant C does not depend on l.

First, for each l we have

$$\int |\nabla(u\phi_l)|^2 w\,dx \le 2\int |\nabla u|^2 \phi_l^2 w\,dx + C\int_{B_l} (m_w(x, U+1))^2 u^2 w\,dx.$$

It follows from Lemma 10(d) that

$$\sum_{l=1}^{\infty} \int (|\nabla(u\phi_l)|^2 w + (U+1)(u\phi_l)^2 w)\,dx$$
$$\le CC \sum_{l=1}^{\infty} \int_{B_l} (|\nabla u|^2 \phi_l^2 w + m_w(x, U+1)^2 u^2 w + (U+1)u^2\phi_l^2 w)\,dx$$
$$\le C\int |\nabla u|^2 w\,dx + C\int m_w(x, U+1)^2 u^2 w\,dx + C\int (U+1)u^2 w\,dx.$$

Therefore, Lemma 6 implies

$$\sum_{l=1}^{\infty} \int (|\nabla(u\phi_l)|^2 w + (U+1)(u\phi_l)^2 w)\,dx \le C\int (|\nabla u|^2 + (U+1)u^2) w\,dx. \tag{15}$$

Now, we consider two cases. If $B_l \cap E_\lambda^c \ne \emptyset$, then by the definition of E_λ we have $\sup_{x\in B_l} m_w(x, U+1) \ge \lambda^{1/2}$. Hence, Lemma 4(i) implies $\inf_{x\in B_l} m_w(x, U+1) \ge C\lambda^{1/2}$. Using this and Lemma 6 we obtain

$$\int_{B_l} (|\nabla(u\phi_l)|^2 + (U+1)(u\phi_l)^2) w\,dx \ge C\int_{B_l} m_w(x, U+1)^2 (u\phi_l)^2 w\,dx \ge C'\lambda \int_{B_l} (u\phi_l)^2 w\,dx. \tag{16}$$

Next, suppose $B_l = B(x_l, r_l) \subset E_\lambda$, where $r_l = m_w(x_l, U+1)^{-1}$. Let $Q_l = Q(x_l, 2r_l)$ denote the cube with sidelength $2r_l$ and center x_l. Then $B_l \subset Q_l$. Divide Q_l into disjoint subcubes $\{Q_l^\beta\}$ with sidelength comparable to $\lambda^{-1/2}$. Note that $r_l \ge \lambda^{-1/2}$, since $x_l \in E_\lambda$. Let $\mathcal{H}$ be the subspace generated by $\{\phi_l(x)\chi_{Q_l^\beta}(x) : B_l \subset E_\lambda\}$. If $\int_{Q_l^\beta} u\phi_l w\,dx = 0$ for each Q_l^β, that is, $u \in \mathcal{H}^\perp$, then (4) yields

$$\lambda\int (u\phi_l)^2 w\,dx = \lambda \sum_\beta \int_{Q_l^\beta} (u\phi_l)^2 w\,dx \tag{17}$$
$$\le C\lambda \sum_\beta (1/\sqrt{\lambda})^2 \int_{Q_l^\beta} |\nabla(u\phi_l)|^2 w\,dx$$
$$\le C\int_{Q_l} |\nabla(u\phi_l)|^2 w\,dx + \int_{Q_l} (U+1)(u\phi_l)^2 w\,dx.$$

Thus, (16) and (17) imply

$$\int_{Q_l} |\nabla(u\phi_l)|^2 w\,dx + \int_{Q_l} (U+1)(u\phi_l)^2 w\,dx \ge C\lambda \int (u\phi_l)^2 w\,dx \tag{18}$$

for every $u \in \mathcal{H}^\perp$. From (15) and (18), there exists a constant $C_0 > 0$ such that for every $\lambda \ge C_0$ we obtain

$$C\lambda \int u^2 w\,dx \le C\int (|\nabla u|^2 + Uu^2) w\,dx$$

for $u \in \mathcal{H}^\perp$. Let m_l be the number of Q_l^β associated with $B_l \subset E_\lambda$. Then there exists a positive constant C such that $m_l \le C\lambda^{\gamma/2} w(B_l)$. Indeed, assumptions (3) and $w \in D_\gamma$ imply

$$w(B_l) \ge C(n) m_l w(Q_l^\beta) \ge C(n) m_l (1/\sqrt{\lambda})^\gamma w(Q(x_l, 1)) \ge C m_l \lambda^{-\gamma/2}.$$

Hence by Lemma 10(d) we obtain

$$\dim \mathcal{H} \le C \sum_{B_l \subset E_\lambda} \lambda^{\gamma/2} w(B_l) \le C' \lambda^{\gamma/2} w(E_\lambda).$$

Thus we have proved the upper bound. ∎

Proof of Theorem 5. Since we reason as in [Sh2], we just mention the key estimates of the proof. For an eigenfunction $u \in L^2(w\,dx)$ with eigenvalue $\lambda \ge 1$, we obtain the subsolution estimate: there exists a constant C such that

$$|u(x)| \le C\left(\frac{1}{w(B_r(x))} \int_{B_r(x)} |u(y)|^2 w(y)\,dy\right)^{1/2}$$

for $\lambda r^2 \le 1$. Secondly, we obtain the estimate

$$\|e^{\varepsilon d_{C\lambda}} u\|_{L^2(w\,dx)} \le C' \|u\|_{L^2(w\,dx)}$$

for small $\varepsilon > 0$ and some constants C, C'. Now, by the assumption (7) and $w \in D_\gamma$, we have $Cr^\gamma \le w(B(x,r))$ for some positive constant C. Combining these estimates, we conclude that

$$|u(x)| \le C m_w(x, U+1)^{\gamma/2} e^{-\varepsilon d_{C\lambda}(x)} \|u\|_{L^2(w\,dx)}.$$

Once we obtain this estimate we can prove Theorem 5 in the same way as in Theorem 0.20 of [Sh2]. ∎

Corollary 3 follows from Theorem 5 as in [Sh2].

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