ACTA ARITHMETICA
XCII.2 (2000)

Collatz cycles with few descents

by

T. Brox (Stuttgart)

1. Introduction. Let $T : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $T(x) = x/2$ if x is even, $T(x) = (3x+1)/2$ if x is odd. T is known as the $3x+1$ *function* (see [4], [7]). We are interested in the cycles of T, referred to as $3x+1$ *cycles* (see [3]–[5]). The well known Finite Cycle Conjecture asserts that T has only finitely many cycles. In this note we generalize a theorem due to R. Steiner (see [4], [6]).

In order to state the result in a concise form, we consider instead of T a slightly different function T_1 defined by the unique decomposition

$$2^{k(x)}T_1(x) = 3x + 1 \qquad (x \text{ odd}), \tag{1.1}$$

where $k(x) \ge 1$ is the multiplicity of the prime factor 2 in the number $3x+1$. Note that T_1 is only defined for odd integers. For each odd argument the iteration $T^{k(x)}(x)$ is equal to $T_1(x)$. An odd integer x is called *descending* if $k(x) \ge 2$.

In what follows, a cycle of T_1 is called a *Collatz cycle*. By definition, we can represent a Collatz cycle by a tuple $\Gamma = (x_1, \ldots, x_n)$ consisting of distinct odd integers ($n \ge 1$). By $|\Gamma|$ we denote the number of elements in Γ, i.e. the period of the Collatz cycle. Each Collatz cycle consists of the odd elements in a $3x+1$ cycle, and conversely. If k denotes the sum of $k(x)$ over all elements of a Collatz cycle, then k is the period of the corresponding $3x+1$ cycle. Let $\delta(\Gamma)$ be the number of descending elements in a Collatz cycle Γ.

Theorem 1.1. *The number of Collatz cycles satisfying* $\delta(\Gamma) < 2\log|\Gamma|$ *is finite.*

We will prove Theorem 1.1 in the following sections. In Section 4 we briefly discuss the extension of Theorem 1.1 to $3x+d$ mapping (see [2]). The theorem of R. Steiner states that the fixed point 1 is the only positive

2000 *Mathematics Subject Classification*: 11D61, 11Z05.

Collatz cycle with $\delta(\Gamma) = 1$. It is easy to deduce the following corollary of Theorem 1.1, using Lemma 2.2 in Section 2.

Corollary 1.2. *For each fixed* $\nu \geq 1$ *the number of Collatz cycles with* $\delta(\Gamma) \leq \nu$ *is finite.*

2. A divisibility problem. In this section we express the existence problem of a Collatz cycle by an appropriate periodic sequence (see Sections 2, 5 of [3]). Given an odd integer x_0, consider the iteration $x_m = T_1(x_{m-1})$. Let $k_m = k(x_{m-1})$. Because of (1.1), we get

$$2^{k_m} x_m = 3x_{m-1} + 1 \quad (m \geq 1), \tag{2.1}$$

where it is required that $x_1, x_2, \ldots$ are odd. Proceeding iteratively, we immediately obtain

$$2^{k_1+\ldots+k_m} x_m = 3^m x_0 + \varphi_{m-1}(k_1, \ldots, k_{m-1}). \tag{2.2}$$

Here $\varphi_0 = 1$ and

$$\varphi_m(u_1, \ldots, u_m) = \sum_{l=0}^{m} 3^{m-l} 2^{u_1+\ldots+u_l} \qquad ((u_1, \ldots, u_m) \in \mathbb{R}^m), \tag{2.3}$$

where $u_1 + \ldots + u_l$ is zero if $l = 0$. For further references see Section 2 of [3].

Assume now $(x_0, \ldots, x_{n-1})$ is a Collatz cycle. By construction, $k_i \geq 1$ for $1 \leq i \leq n$. Since $x_n = x_0$, (2.2) implies

$$Mx_0 = \varphi_{n-1}(k_1, \ldots, k_{n-1}),$$

where

$$M = 2^k - 3^n, \qquad k = k_1 + \ldots + k_n. \tag{2.4}$$

Consider $(x_0, \ldots, x_{n-1})$ and $(k_1, \ldots, k_n)$ as n-periodic sequences $\{x_i\}_{i\in\mathbb{Z}}$, $\{k_i\}_{i\in\mathbb{Z}}$. Since we can take each x_i as start value, we obtain

$$Mx_i = \varphi_{n-1}(k_{i+1}, \ldots, k_{i+n-1}) \quad (i \in \mathbb{Z}). \tag{2.5}$$

Hence M divides the right hand side of (2.5) for each i. Since the Collatz cycle consists of distinct elements, n is the smallest period of $\{k_i\}$, by (2.5).

Conversely, let $\{k_i\}_{i\in\mathbb{Z}}$ be any periodic sequence of positive integers. Let n be the smallest period of $\{k_i\}$. Define M and k by (2.4). Let

$$F_i = \varphi_{n-1}(k_{i+1}, \ldots, k_{i+n-1}) \quad (i \in \mathbb{Z}).$$

Note that F_i does not depend on k_i. Plainly $\{F_i\}$ is n-periodic. Because of (2.3) and periodicity, we get by a simple computation

$$2^{k_i} F_i = 3F_{i-1} + M \quad (i \in \mathbb{Z}). \tag{2.6}$$

Since each k_i is positive, we conclude from (2.3) and (2.4) that

$$\gcd(2\cdot 3, F_i) = 1 \quad (i \in \mathbb{Z}), \tag{2.7}$$

$$\gcd(2\cdot 3, M) = 1. \tag{2.8}$$

Because of (2.6) and (2.7), k_i and F_i are uniquely determined by F_{i-1}. If $F_l = F_m$, then $|m-l|$ is a period of $\{k_i\}$. Hence $F_1, \ldots, F_n$ are distinct. We are now ready to prove the following lemmas (see also Section 5 of [3]).

LEMMA 2.1. *Let* $M = 2^k - 3^n$. *Then either* $M \nmid F_i$ $(1 \le i \le n)$ *or* $M \mid F_i$ $(1 \le i \le n)$. *The latter case corresponds to a Collatz cycle* $(x_1, \ldots, x_n)$, *where* $x_i = F_i/M$.

Proof. Assume that $M \mid F_i$ for one i. Then (2.6)–(2.8) imply that for all i, $M \mid F_i$ and $x_i = F_i/M$ is odd. Dividing (2.6) by M, we arrive at $x_i = T_1(x_{i-1})$, since each x_i is odd. Plainly $x_1, \ldots, x_n$ are distinct. ■

LEMMA 2.2. *If* $\{k_i\}$ *generates a Collatz cycle, then* $k = k_1 + \ldots + k_n \le 2n$. *In particular, for each* $n \ge 1$, *the number of Collatz cycles with* $|\Gamma| = n$ *is finite.*

Proof. If $\{k_i\}$ generates a Collatz cycle, then $x_i = F_i/M$, by Lemma 2.1. If we multiply (2.6) from $i = 1$ to $i = n$, a simple reduction yields

$$2^k = \prod_{i=1}^{n} (3 + 1/x_{i-1}) \le 4^n,$$

since $|1/x_i| \le 1$. Hence $k \le 2n$. ■

REMARK 2.3. By Lemma 2.1, since $k_i = 1$ $(i \in \mathbb{Z})$ has period 1, there exists at most one Collatz cycle with $\delta(\Gamma) = 0$, and this Collatz cycle has to be a fixed point. In fact, $-1 = 1/(2-3)$ is a non-descending fixed point.

3. Algebraic reformulation. In order to prove Theorem 1.1, we reformulate the divisibility problem in a more convenient form. As before let $\{k_i\}_{i \in \mathbb{Z}}$ be a periodic sequence of positive integers. Let n be the smallest period of $\{k_i\}$. Define

$$\widetilde{\varphi}_n(u_1, \ldots, u_n) = 2^{u_1} \varphi_{n-1}(u_2, \ldots, u_n) - 2\varphi_{n-1}(u_1, \ldots, u_{n-1}) \qquad ((u_1, \ldots, u_n) \in \mathbb{R}^n).$$

Because of (2.3), we obtain

$$\widetilde{\varphi}_n(u_1, \ldots, u_n) = 2 \sum_{l=0}^{n-1} 3^{n-(l+1)} 2^{u_1 + \ldots + u_l} \{2^{u_{l+1}-1} - 1\}. \tag{3.1}$$

For each $i \in \mathbb{Z}$ define

$$\widetilde{F}_i = \widetilde{\varphi}_n(k_{i+1}, \ldots, k_{i+n}).$$

Then (2.6) implies

$$\widetilde{F}_i = 2^{k_{i+1}} F_{i+1} - 2F_i = F_i + M \qquad (i \in \mathbb{Z}). \tag{3.2}$$

Let ν be the number of indices i such that $1 \le i \le n$ and $k_i \ge 2$. We assume $\nu \ge 1$. Define strictly ascending numbers $\tau(1), \ldots, \tau(\nu)$ by $1 \le \tau(j) \le n$ and $k_{\tau(j)} \ge 2$. Put $\tau(0) = \tau(\nu) - n$. For $1 \le j \le \nu$ let

$$h_j = k_{\tau(j)} - 1, \qquad n_j = \tau(j) - \tau(j-1).$$

Consider $(h_1, \ldots, h_\nu)$, $(n_1, \ldots, n_\nu)$ as ν-periodic sequences $\{h_i\}_{i\in\mathbb{Z}}$, $\{n_i\}_{i\in\mathbb{Z}}$. By construction, the two sequences consist of positive integers, and ν is the smallest common period. Also by construction,

$$n = n_1 + \ldots + n_\nu, \qquad k = h + n, \tag{3.3}$$

where

$$h = h_1 + \ldots + h_\nu. \tag{3.4}$$

Given $(t_0, \ldots, t_{\nu-1}) \in \mathbb{R}^\nu$ and $(u_1, \ldots, u_{\nu-1}) \in \mathbb{R}^{\nu-1}$, put

$$\psi_{2\nu-1}(t_0, u_1, t_1, u_2, \ldots, t_{\nu-1}) = \sum_{l=0}^{\nu-1} 2^{t_0+u_1+\ldots+t_{l-1}+u_l}\{2^{t_l} - 1\}3^{u_{l+1}+\ldots+u_{\nu-1}}, \tag{3.5}$$

where the sum in the 2-exponent (3-exponent) is zero if $l = 0$ ($l = \nu - 1$). For each $j \in \mathbb{Z}$ define

$$H_j = \psi_{2\nu-1}(h_j, n_{j+1}, h_{j+1}, n_{j+2}, \ldots, h_{j+\nu-1}). \tag{3.6}$$

Thus H_j does not depend on n_j. Note that $H_j > 0$ for each $j \in \mathbb{Z}$. Note also that $\{H_j\}$ is ν-periodic. We assert

$$\widetilde{F}_{\tau(j)} = 2^{n_{j+1}} H_{j+1} \qquad (j \in \mathbb{Z}). \tag{3.7}$$

By shift, it is enough to verify (3.7) for $j = 0$. Also by shift, we can assume that $\tau(0) = 0$. Then $\tau(j)$ equals $n_1 + \ldots + n_j$ $(1 \le j \le \nu)$. In (3.1) only those terms can survive which are placed at $l + 1 = \tau(j)$. Hence

$$\widetilde{F}_{\tau(0)} = 2\sum_{j=1}^{\nu} 3^{n-\tau(j)} 2^{h_1+\ldots+h_{j-1}+\tau(j)-1}\{2^{h_j} - 1\}.$$

In the last formula we take the 2-exponent $n_1 - 1$ outside the sum. After some rearrangement, the remaining sum is easily identified as H_1.

Conversely, let $\{h_j\}_{j\in\mathbb{Z}}$, $\{n_j\}_{j\in\mathbb{Z}}$ be any periodic sequences consisting of positive integers. Let ν be the smallest common period of both sequences. Define h, k, n by (3.3) and (3.4). Let H_j be given by (3.6). Up to shift, $\{k_i\}$ is uniquely determined by both sequences, i.e. by $\{h_0, n_1, h_1, n_2, \ldots\}$. Also n is the smallest period of $\{k_i\}$. Because of (2.8), (3.2) and (3.7), we can reformulate Lemma 2.1.

Lemma 3.1. *Let $M = 2^k - 3^n$. Then either $M \nmid H_j$ $(1 \le j \le \nu)$ or $M \mid H_j$ $(1 \le j \le \nu)$. The latter case corresponds to a Collatz cycle with $|\Gamma| = n$ and $\delta(\Gamma) = \nu$.*

4. Minimum of the H-sequence. In this section we prove Theorem 1.1 assuming the truth of Lemma 5.1, which will be stated and proved in Section 5. Consider a pair $\{h_j\}_{j\in\mathbb{Z}}$, $\{n_j\}_{j\in\mathbb{Z}}$ of periodic sequences consisting of positive integers. Define ν, h, k, n as described in the preceding section. By Lemma 3.1, since each $H_j > 0$, the two sequences cannot generate a Collatz cycle if

$$\min_{0\le j\le \nu-1} H_j < |M| = |2^k - 3^n|. \tag{4.1}$$

To prove Theorem 1.1, it is enough, by Lemma 2.2 and Remark 2.3, to show (4.1) holds for all sufficiently large n, assuming that $\nu < 2\log n$.

First we consider the left hand side of (4.1). Let $\theta = \log_2 3$ and $\theta_1 = \theta - 1$. Note that $\theta_1 > 0$. In order to estimate H_j, we replace the bracket inside (3.5) by $2^{h_{j+l}}$. Because of periodicity, an easy rearrangement yields

$$3^{-n} H_j < \sum_{l=0}^{\nu-1} 2^{-n_j + S_{j,l}} \qquad (j \in \mathbb{Z}),$$

where

$$S_{j,l} = \sum_{i=0}^{l} (h_{j+i} - \theta_1 n_{j+i}) \qquad (l \ge 0).$$

As a functional of both sequences define

$$E = \min_{0\le j\le \nu-1} \{-n_j + \max_{0\le l\le \nu-1} S_{j,l}\}.$$

By an elementary estimation,

$$\min_j 3^{-n} H_j < \nu 2^E.$$

We claim that

$$E \le \max(0, \Delta) - A_\nu n, \tag{4.2}$$

where

$$A_\nu = \theta_1/(\theta^\nu - 1), \qquad \Delta = h - \theta_1 n = k - \theta n.$$

But (4.2) is a consequence of Lemma 5.1 below, applied with $x_j = \theta_1 n_j$, $y_j = h_j$, $\alpha = 1/\theta_1$. Hence we obtain

$$\min_j 3^{-n} H_j < \nu 2^{\max(0,\Delta) - A_\nu n}.$$

Let $\gamma = 1 - 2\log\theta = 0.0788\ldots$ If we assume that $\nu < 2\log n$, then

$$(\theta^\nu - 1)^{-1} n > \theta^{-\nu} n = n^{1-\nu\log\theta/\log n} > n^\gamma,$$

and it follows that

$$\min_j 3^{-n} H_j < \log n \cdot 2^{1+\max(0,\Delta)-\theta_1 n^\gamma}. \tag{4.3}$$

We now consider the right hand side of (4.1). By elementary analysis we get

$$3^{-n}|M| = |2^{\Delta} - 1| \geq \min(|\Delta|, 1)2^{\max(0,\Delta)-1}.$$

Note that we have $n \geq 1$ and $k = h + n \geq 2$. By definition, $|\Delta| < 1$ if $k > \theta n - 1$ and $k < \theta n + 1$, otherwise $|\Delta| \geq 1$. A result of A. Baker and N. Feldman (see Theorem 3.1 in [1]) implies the existence of an effectively computable constant $C_0 > 0$ such that

$$|\Delta| > |k \log 2 - n \log 3| > \{\max(k, n)\}^{-C_0} \quad (k \geq 2,\ n \geq 1).$$

Since $\theta n + 1 > n$,

$$|\Delta| > \{\theta n + 1\}^{-C_0},$$

and therefore

$$3^{-n}|M| > \{\theta n + 1\}^{-C_0} 2^{\max(0,\Delta)-1}. \tag{4.4}$$

If n is large enough, the right side of (4.3) is smaller than the right side of (4.4), which gives (4.1).

Remark 4.1. We briefly sketch the extension of Theorem 1.1 to the $3x+d$ mapping, where d is a positive integer prime to 2 and 3. Define T_1 according to (1.1), where the right hand side is replaced by $3x + d$. Now $k(x)$ is the multiplicity of 2 in the number $3x+d$. Define a Collatz cycle, descending and $\delta(\Gamma)$ as in Section 1. After an obvious modification of (2.1), in (2.2) replace φ_{m-1} by $d\varphi_{m-1}$. Note that the definition of φ_m is not affected ($m \geq 0$). Hence multiply the right hand side of (2.5) with d. If (2.6) is multiplied with d, it follows that F_i can be replaced by dF_i in Lemma 2.1. Similarly to the proof of Lemma 2.2, if $\{k_i\}$ generates a Collatz cycle, then

$$2^k = \prod_{i=1}^{n} (3 + d/x_{i-1}) \leq (3 + d)^n.$$

Hence put $k \leq C_1 n$ in Lemma 2.2, where $C_1 = \log_2(3+d)$. In Lemma 3.1 put dH_j instead of H_j. Finally multiply the left hand side of (4.1) with d. The additional factor d causes no harm for the conclusion that the inequality is true if n is large enough and $\nu < 2 \log n$.

5. Upper value of E. Let there be given an integer $\nu \geq 1$ and real numbers $r > 0$, $s > 0$. Let $\{x_j\}_{j\in\mathbb{Z}}$, $\{y_j\}_{j\in\mathbb{Z}}$ be ν-periodic sequences of non-negative real numbers such that

$$x_1 + \ldots + x_\nu = r, \quad y_1 + \ldots + y_\nu = s.$$

For each $j \in \mathbb{Z}$ and $l \geq 0$ define

$$S_{j,l} = \sum_{i=0}^{l} (y_{j+i} - x_{j+i}).$$

LEMMA 5.1. *For any real* $\alpha > 0$,

$$\min_{0 \leq j \leq \nu-1} \{-\alpha x_j + \max_{0 \leq l \leq \nu-1} S_{j,l}\} \leq \max(0, s-r) - r/(\beta^\nu - 1),$$

where $\beta = 1 + 1/\alpha$.

Proof. (a) *Reduction to the case* $r = s$. If $r \neq s$, put $\widetilde{y}_j = ry_j/s$. Then $\widetilde{s} = r$ and

$$S_{j,l} = \sum_{i=0}^{l} (y_{j+i} - \widetilde{y}_{j+i}) + \sum_{i=0}^{l} (\widetilde{y}_{j+i} - x_{j+i}) = \frac{s-r}{s} \sum_{i=0}^{l} y_{j+i} + \widetilde{S}_{j,l}.$$

Hence

$$\max_{0 \leq l \leq \nu-1} S_{j,l} \leq \max(0, s-r) + \max_{0 \leq l \leq \nu-1} \widetilde{S}_{j,l}.$$

Thus we have to prove the assertion for the case $r = s$.

(b) *Elimination of* $\{y_j\}$. We suppose $r = s$. Then $\{S_{0,l}\}_{l \geq 0}$ is ν-periodic. By a simple shift, we can assume

$$S_{0,\nu-1} = \max_{l \geq 0} S_{0,l} = 0.$$

If $1 \leq j \leq \nu - 1$, then

$$\max_{0 \leq l \leq \nu-1} S_{j,l} = \sum_{i=j}^{\nu-1} (y_i - x_i) = \sum_{i=0}^{j-1} (x_i - y_i) \leq \sum_{i=0}^{j-1} x_i.$$

Note that the last inequality is sharp if $y_j = 0$ $(0 \leq j \leq \nu-2)$ and $y_{\nu-1} = r$. Now we have to estimate the minimum of

(5.1) $\quad -\alpha x_0, -\alpha x_1 + x_0, -\alpha x_2 + x_0 + x_1, \ldots, -\alpha x_{\nu-1} + x_0 + \ldots + x_{\nu-2}.$

(c) *Linear optimization*. Consider $x = (x_0, \ldots, x_{\nu-1})$ as a vector in $\mathbb{R}^\nu$. Define linear functionals $L_0, \ldots, L_{\nu-1}$ according to (5.1). Let $f(x)$ be the minimum of $L_0 x, \ldots, L_{\nu-1} x$. Let B be the affine subspace $x_0 + \ldots + x_{\nu-1} = r$. We assert

$$\sup_{x \in B} f(x) = -r/(\beta^\nu - 1). \tag{5.2}$$

In order to prove (5.2), consider the vector

$$b = r(\beta - 1)(\beta^\nu - 1)^{-1}(\beta^0, \beta^1, \ldots, \beta^{\nu-1}).$$

Plainly $b \in B$. An easy computation shows

$$L_0 b = L_1 b = \ldots = L_{\nu-1} b. \tag{5.3}$$

Next, $f(b)$ equals the right hand side of (5.2). Let U be the linear subspace where the sum of coordinates equals zero. If $x \in B$, then $x = b + u$ $(u \in U)$. By (5.3),

$$f(x) = f(b) + \min_j L_j u = f(b) + f(u) \qquad (x \in B).$$

We have to show $f(u) \leq 0$. Assume the contrary, that is, each number $L_0 u, \ldots, L_{\nu-1} u$ is positive. Since $L_0 u > 0$, we get $u_0 < 0$. Since $u_0 < 0$ and $L_1 u > 0$, we get $u_1 < 0$. Repeating the argument, each u_j is negative, which contradicts $u \in U$. ∎

Acknowledgements. We thank the referee for useful hints and improvements.

References

[1] A. Baker, *Transcendental Number Theory*, Cambridge Univ. Press, 1975.
[2] E. G. Belaga and M. Mignotte, *Embedding the* $3x+1$ *conjecture in a* $3x+d$ *context*, Experiment. Math. 7 (1998), 145–152.
[3] L. Halbeisen and N. Hungerbühler, *Optimal bounds for the length of rational Collatz cycles*, Acta Arith. 78 (1997), 227–239.
[4] J. C. Lagarias, *The* $3x+1$ *problem and its generalizations*, Amer. Math. Monthly 92 (1985), 3–23.
[5] —, *The set of rational cycles for the* $3x+1$ *problem*, Acta Arith. 56 (1990), 33–53.
[6] R. P. Steiner, *A theorem on the Syracuse Problem*, in: Proc. 7th Manitoba Conf. on Numerical Mathematics, 1977, Winnipeg, 1978, 553–559.
[7] G. J. Wirsching, *The Dynamical System Generated by the* $3n+1$ *Function*, Lecture Notes in Math. 1681, Springer, Berlin, 1998.

Helblingstr. 21
D-70565 Stuttgart, Germany
E-mail: thomas.brox@daimlerchrysler.com

Received on 12.2.1999
and in revised form on 15.9.1999 (3554)