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BLOW-UP BEHAVIOR IN NONLOCAL VS LOCAL HEAT EQUATIONS

PHILIPPE SOUPLET

Département de Mathématiques, Université de Picardie
INSSET, 02109 St-Quentin, France
and
Laboratoire de Mathématiques Appliquées, UMR CNRS 7641
Université de Versailles, 45 avenue des Etats-Unis, 78035 Versailles, France
E-mail: souplet@math.uvsq.fr

Abstract. We present some recent results on the blow-up behavior of solutions of heat equations with nonlocal nonlinearities. These results concern blow-up sets, rates and profiles. We then compare them with the corresponding results in the local case, and we show that the two types of problems exhibit "dual" blow-up behaviors.

1. Introduction. The aim of this paper is to present some recent results on the blow-up behavior of solutions of heat equations with nonlocal nonlinearities, and to compare them with known results in the case of local nonlinearities.

We shall focus our discussion on the following two typical problems:

$$(NRD) \qquad \begin{cases} u_t - \Delta u = \displaystyle\int_\Omega u^p(t,y)\,dy, \quad t>0, \quad x\in\Omega \\ u(t,x)=0, \quad t>0,\ x\in\partial\Omega \\ u(0,x)=\phi(x)\ge 0, \quad x\in\Omega, \end{cases}$$

and

$$(LRD) \qquad \begin{cases} u_t - \Delta u = u^p(t,x), \quad t>0, \quad x\in\Omega \\ u(t,x)=0, \quad t>0,\ x\in\partial\Omega \\ u(0,x)=\phi(x)\ge 0, \quad x\in\Omega. \end{cases}$$

((LRD) and (NRD) respectively mean "Local Reaction-Diffusion" and "Nonlocal Reaction-Diffusion".) We point out that similar analysis and comparison can be carried out

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if the nonlinear function in (LRD) and (NRD) is replaced with, for instance, e^u. Also, what will be said on (NRD) can be transposed to different types of nonlocal problems (see [S1, S2]).

In what follows, we assume that $p > 1$ and that Ω is a smoothly, say C^2, bounded domain of $\mathbb{R}^N$. We suppose that the initial data ϕ is nonnegative and belongs to $C_0(\Omega)$, so that both (LRD) and (NRD) admit a unique, maximal in time, classical solution $u \geq 0$. We also assume that u blows up in finite time, that is,

$$\lim_{t \to T} \|u(t)\|_\infty = \infty$$

for some finite $T > 0$. This is known to occur, for instance, if ϕ is a sufficiently large multiple of any positive function.

In section 2, we state the main results on blow-up behavior for the nonlocal equation (NRD). Some sketch of proof is given in section 3. Section 4 is then devoted to the comparison between nonlocal and local blow-up behaviors.

2. Nonlocal blow-up behavior. The existence of blowing-up solutions of (NRD) and of related nonlocal equations, and some of their properties, such as blow-up set, were studied by several authors (see, e.g., [BBL, D, WW, S1, S3] and the references therein). In [S2], we investigate the sharp blow-up behavior of solutions of different classes of nonlocal problems, including (NRD). First, we prove that the solutions have global blow-up, we determine the blow-up rate of $|u(t)|_\infty$, and we prove that the blow-up rate of $u(t,x)$ is uniform in all compact subsets of the domain. In rough terms, all happens as if the diffusion term could be neglected in the interior, and the asymptotics is formally given by balancing u_t with the nonlocal source term, resulting in a flat blow-up profile, except for a boundary layer whose thickness vanishes as t goes to T. Furthermore, we derive estimates on the size of the boundary layer and on the asymptotic behavior of the solution in the boundary layer. By the boundary layer, we mean the region near $\partial\Omega$ where the solution follows a fast transition between the blow-up regime and the assigned zero boundary condition. We prove that the size of the boundary layer decays like $\sqrt{T-t}$, and that in this region the solution $u(t,x)$ behaves like

$$|u(t)|_\infty \frac{d(x)}{\sqrt{T-t}},$$

where

$$d(x) = \operatorname{dist}(x, \partial\Omega)$$

denotes the distance to the boundary.

Our main results in the case of (NRD) read as follows.

THEOREM 1. *Let u be a blowing-up solution of (NRD). Then we have*

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(t,x) = \lim_{t \to T} (T-t)^{\frac{1}{p-1}} |u(t)|_\infty = \big[(p-1)|\Omega|\big]^{-\frac{1}{p-1}},$$

uniformly on compact subsets of Ω.

The next result describes the behavior of the solution u near the blow-up time in the boundary layer.

THEOREM 2. *Let u be a blowing-up solution of (NRD). Then, for all $K > 0$, there exist some constants $C_2 \geq C_1 > 0$ and some $t_0 \in (0, T)$, such that u satisfies*

$$C_1 \frac{d(x)}{\sqrt{T-t}}\ |u(t)|_\infty \leq u(t,x) \leq C_2 \frac{d(x)}{\sqrt{T-t}}\ |u(t)|_\infty, \tag{1}$$

for all (t, x) in $[t_0, T) \times \Omega$ such that $d(x) \leq K\sqrt{T-t}$,

From the right hand side of (1), one deduces that the size of the boundary layer is at least of order $\sqrt{T-t}$ near the blow-up time, in the sense that $u(t, x) = o(|u(t)|_\infty)$, as $t \to T$ and $d(x)/\sqrt{T-t} \to 0$. However, estimate (1) is not enough to conclude that the size of the boundary layer is exactly of order $\sqrt{T-t}$, in the sense that $u(t, x)/|u(t)|_\infty \to 1$, as $t \to T$ and $d(x)/\sqrt{T-t} \to \infty$. The estimate (2) in the following theorem, though not very sharp regarding the actual behavior of the solution in the boundary layer, enables one to conclude that this is indeed true. (We recently proved that the restriction $p < 2$ below can be removed. This will appear in a forthcoming publication.)

THEOREM 3. *Let u be a blowing-up solution of (NRD). Assume $1 < p < 2$. Then u satisfies the estimate*

$$\Big(1 - C_3 \frac{T-t}{d^2(x)}\Big)|u(t)|_\infty \leq u(t,x), \tag{2}$$

in $[t_0, T) \times \Omega$, for some constant $C_3 > 0$ and some $t_0 \in (0, T)$. Therefore, we have

$$\frac{u(t,x)}{|u(t)|_\infty} \longrightarrow \begin{cases} 1, & \text{as } t \to T \text{ and } \dfrac{d(x)}{\sqrt{T-t}} \to \infty \\ 0, & \text{as } t \to T \text{ and } \dfrac{d(x)}{\sqrt{T-t}} \to 0. \end{cases} \tag{3}$$

In other words, the size of the boundary layer decays like $\sqrt{T-t}$.

The method of proof of Theorems 1, 2 and 3 is very different from those previously used in blow-up profile studies. It is based on a combination of the following ingredients:

(i) an eigenfunction argument to obtain the interior averaged asymptotics of u;

(ii) the use of the mean value inequality for subharmonic functions and of some related integral inequalities, to deduce the interior uniform asymptotics from the averaged one, and to obtain the upper estimates on the size of the boundary layer;

(iii) suitable sub- and supersolutions and interpolation arguments, to derive the asymptotic boundary behavior of the solutions.

3. Sketch of proof of Theorem 1. (See [S2] for details.) We denote

$$g(t) = \int_\Omega u^p(t,y)\,dy, \quad G(t) = \int_0^t g(s)\,ds \quad \text{and} \quad H(t) = \int_0^t G(s)\,ds.$$

By using maximum principle arguments, we first obtain the estimates

$$G(T) = \infty, \tag{4}$$

$$\Delta u(t,x) \leq C_1 \tag{5}$$

and

$$-C_2 \le u(t,x) \le C_2 + G(t) \tag{6}$$

in $[T/2, T) \times \Omega$, for some C_1, $C_2 > 0$.

Denote by λ the first eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet conditions, and by φ the corresponding eigenfunction, with $\varphi > 0$ in Ω and $\int_\Omega \varphi = 1$. Next define $z(t,x) = G(t) - u(t,x)$ and $\beta(t) = \int_\Omega z(t,y)\,\varphi(y)\,dy$. Using Green's formula, we obtain $\beta'(t) = -\lambda\beta(t) + \lambda G(t)$. This equation is immediately integrated into

$$\beta(t) = \beta(0)\,e^{-\lambda t} + \lambda \int_0^t e^{\lambda(s-t)} G(s)\,ds \le C(1 + H(t)). \tag{7}$$

On the other hand, (6) implies that

$$\inf_\Omega z(t,x) \ge -C_2,$$

for all $t \in [T/2, T)$, which combined with (7) implies

$$\int_\Omega |z(t,y)|\,\varphi(y)\,dy \le C'(1 + H(t)). \tag{8}$$

Moreover, by (5), we have

$$-\Delta z \le C_1. \tag{9}$$

By using the mean-value inequality for subharmonic functions, we then prove the following lemma. (A more accurate version of this Lemma enables one to obtain Theorem 3.)

LEMMA. *Let $K_\rho = \{y \in \Omega;\ \operatorname{dist}(y, \partial\Omega) \ge \rho\}$. Under the assumptions (8) and (9), we have*

$$\sup_{x \in K_\rho} z(t,x) \le \frac{C}{\rho^{N+1}}\,(1 + H(t)).$$

Using the Lemma, (6), (4), and the fact that G is nondecreasing, we show that

$$\lim_{t\to T} \frac{u(t,x)}{G(t)} = \lim_{t\to T} \frac{|u(t)|_\infty}{G(t)} = 1, \tag{10}$$

uniformly on compact subsets of Ω. Using (10), together with (6) and Lebesgue's dominated convergence theorem, it then follows in particular that $G'(t) = g(t) \sim |\Omega| G^p(t)$, as $t \to T$. After integrating between t and T, we obtain $G(t) \sim \big[(p-1)|\Omega|(T-t)\big]^{-1/(p-1)}$, and the conclusion finally follows by returning to (10).

4. Comparison with the local case. Let us recall some results concerning the local equation (LRD). Under the assumptions Ω convex and $p < (N+2)/(N-2)$ (if $N \ge 3$), it is known that the blow-up rate of $|u(t)|_\infty$ satisfies

$$\lim_{t\to T} (T-t)^{1/(p-1)} |u(t)|_\infty = \kappa \equiv (p-1)^{-1/(p-1)}.$$

(This was recently proved in [MZ2], improving previous results of [W2, FM, GK2].) On the other hand, under the same assumptions, Giga and Kohn [GK1, GK2, GK3] proved that for any blow-up point a, one has:

$$\lim_{t\to T} (T-t)^{1/(p-1)} u(t, a + y\sqrt{T-t}) = \kappa,$$

uniformly on any set $|y| \leq K$. It was further proved that in many cases (with $\Omega = \mathbb{R}^N$),

$$\lim_{t\to T}(T-t)^{1/(p-1)}u(t, a + z\sqrt{(T-t)|\log(T-t)|}) = \kappa(1 + C|z|^2)^{-1/(p-1)},$$

uniformly on any set $|z| \leq K$ (see [HV, V1, V3]). In some cases, this is even known to hold uniformly in $z \in \mathbb{R}^N$ (see [BK, MZ1]), so that in particular:

$$\frac{u(t,x)}{|u(t)|_\infty} \longrightarrow \begin{cases} 1, & \text{as } t \to T \text{ and } \dfrac{|x-a|}{\sqrt{(T-t)|\log(T-t)|}} \to 0 \\ 0, & \text{as } t \to T \text{ and } \dfrac{|x-a|}{\sqrt{(T-t)|\log(T-t)|}} \to \infty. \end{cases} \tag{11}$$

On the other hand, it is known that the blow-up set of u is "thin". For instance, in dimension $N = 1$, the blow-up points are isolated, and for radial solutions in any dimension, under certain assumptions, one has single-point blow-up [W1, MW, CM]. More generally, the blow-up set has finite $(N-1)$–dimensional Hausdorf measure [V2].

As a consequence, if we roughly define the "hot zone" to be the set where $u(t, x)$ behaves like its maximum, we see that the solution remains "cold" almost everywhere, and that the hot zone corresponds, up to a logarithmic correction, to space-time parabolas based at the points (T, a), where a is a blow-up point.

On the contrary, for (NRD), it appears from Theorems 1, 2 and 3 above, that the solution becomes "hot" everywhere in Ω, and that the "cold layer" corresponds to space-time parabolas based at each point (T, a), where a is any boundary point.

Thus, in some sense, problems (NRD) and (LRD) exhibit "dual" blow-up behaviors, the comparison of formulas (3) and (11) being especially suggestive in this respect.

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