ANNALES
POLONICI MATHEMATICI
LXXV.1 (2000)

Invariant measures for iterated function systems

by Tomasz Szarek (Katowice and Rzeszów)

Abstract. A new criterion for the existence of an invariant distribution for Markov operators is presented. Moreover, it is also shown that the unique invariant distribution of an iterated function system is singular with respect to the Hausdorff measure.

1. Introduction. The main aim of our paper is to present sufficient conditions for the existence of an invariant measure for general Markov operators. These operators are defined on a Polish space X. When the Markov operators are defined on a compact space, the proof of the existence goes as follows. First we construct a positive, invariant functional defined on the space of all continuous and bounded functions $f : X \to \mathbb{R}$ and then using the Riesz representation theorem we define an invariant measure. This method was extended by A. Lasota and J. Yorke to the case when X is a locally compact and σ-compact metric space [11]. When X is a Polish space this idea breaks down, since a positive functional may not correspond to a measure. Therefore in our considerations we base on the concept of tightness.

Further, we apply our criteria to iterated function systems (ifs for short). Iterated function systems are closely related to the construction of fractals. By a fractal set we mean a fixed point of the operator

$$H(A) = \bigcup_{i=1}^{N} S_i(A), \qquad A \subset X,$$

where $S_i : X \to X$, $i = 1, \ldots, N$, are continuous transformations and X is a metric space. It is well known that if all S_i are Lipschitzian with Lipschitz constants $L_i < 1$, then the operator H admits a fixed point F. Moreover, F is compact and unique. Under some additional assumptions on the transformations S_i (see [3, 4, 7, 8]) we are able to calculate the Hausdorff dimension

2000 *Mathematics Subject Classification*: Primary 47A35; Secondary 28D15.
Key words and phrases: Markov operator, iterated function system, invariant measure.

of F. In this case it is known that

$$0 < \mathcal{H}^s(F) < \infty,$$

where s is equal to the Hausdorff dimension of F.

The last inequalities are the starting point for our considerations. We will study the Markov operator

$$P\mu(A) = \sum_{i=1}^{N} \int_{S_i^{-1}(A)} p_i(x)\,\mu(dx),$$

which maps Borel measures on X into Borel measures on X. Here $S_i : X \to X$ are again continuous transformations and $p_i : X \to [0,1]$ are continuous and such that $\sum_{i=1}^{N} p_i(x) = 1$. A pair (S_i, p_i) is called an *iterated function system*. In fact, we will be interested in the unique invariant distribution μ_* of P. The main purpose of this paper is to give conditions which ensure that this unique measure μ_* supported on the fractal set F is singular with respect to $\mathcal{H}^s$. Similar problems were studied by many authors (see for example [1, 6]).

The material is divided into two sections. The existence of an invariant measure for a general Markov operator P is discussed in Section 1. This part of the paper generalizes our earlier results (see [12]). In fact, we formulated the definitions of globally and locally concentrating Markov operators, which seemed to be a very useful tool in studying iterated function systems and stochastic differential equations. Now we are going to weaken these definitions. In Section 2 we study the invariant distribution of P corresponding to an iterated function system and prove its singularity with respect to the Hausdorff measure $\mathcal{H}^s$.

2. Invariant measures for Markov operators. Let (X, ϱ) be a Polish space, i.e. a separable, complete metric space. This assumption will not be repeated in the statements of theorems.

By $K(x, r)$ we denote the closed ball with center x and radius r.

By $\mathcal{B}(X)$ and $\mathcal{B}_{\mathrm{b}}(X)$ we denote the family of all Borel sets and all bounded Borel sets, respectively. Further, by $B(X)$ we denote the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$ with the supremum norm.

By $\mathcal{M}_{\mathrm{fin}}$ and $\mathcal{M}_1$ we denote the sets of Borel measures (non-negative, σ-additive) on X such that $\mu(X) < \infty$ for $\mu \in \mathcal{M}_{\mathrm{fin}}$ and $\mu(X) = 1$ for $\mu \in \mathcal{M}_1$. The elements of $\mathcal{M}_1$ are called *distributions*.

We say that $\mu \in \mathcal{M}_{\mathrm{fin}}$ is *concentrated* on $A \in \mathcal{B}(X)$ if $\mu(X \setminus A) = 0$. By $\mathcal{M}_1^A$ we denote the set of all distributions concentrated on A.

Given $\mu \in \mathcal{M}_{\mathrm{fin}}$ we define the *support* of μ by the formula

$$\operatorname{supp}\mu = \{x \in X : \mu(K(x,r)) > 0 \text{ for every } r > 0\}.$$

An operator $P : \mathcal{M}_{\mathrm{fin}} \to \mathcal{M}_{\mathrm{fin}}$ is called a *Markov operator* if it satisfies the following two conditions:

(i) *positive linearity*:

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$$

for $\lambda_1, \lambda_2 \ge 0$ and $\mu_1, \mu_2 \in \mathcal{M}_{\mathrm{fin}}$,

(ii) *preservation of the norm*:

$$P\mu(X) = \mu(X) \quad \text{for } \mu \in \mathcal{M}_{\mathrm{fin}}.$$

It is easy to prove that every Markov operator can be extended to the space of signed measures

$$\mathcal{M}_{\mathrm{sig}} = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}_{\mathrm{fin}}\}.$$

Namely for every $\nu \in \mathcal{M}_{\mathrm{sig}}$, $\nu = \mu_1 - \mu_2$, we set

$$P\nu = P\mu_1 - P\mu_2.$$

To simplify the notation we write

$$\langle f, \nu\rangle = \int_X f(x)\,\nu(dx) \quad \text{for } f \in B(X),\ \nu \in \mathcal{M}_{\mathrm{sig}}.$$

In the space $\mathcal{M}_{\mathrm{sig}}$ we introduce the *Fortet–Mourier norm* (see [5])

$$\|\nu\| = \sup\{|\langle f, \nu\rangle| : f \in F\},$$

where $F \subset B(X)$ consists of all functions such that $|f| \le 1$ and $|f(x) - f(y)| \le \varrho(x,y)$.

A Markov operator P is called *non-expansive* if

$$\|P\mu_1 - P\mu_2\| \le \|\mu_1 - \mu_2\| \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1.$$

Let P be a Markov operator. A measure $\mu_* \in \mathcal{M}_{\mathrm{fin}}$ is called *stationary* or *invariant* if $P\mu_* = \mu_*$.

To ensure existence and uniqueness of an invariant measure for a Markov operator P we assume conditions similar to those in [12]. More precisely, we assume that the Markov operator P is *locally concentrating* but the *globally concentrating* property is replaced by a weaker one. The main aim of this part of our paper is to give general and at the same time explicit conditions implying existence and uniqueness of an invariant measure for P. It is worthwhile to add that the proofs presented in this paper are more elegant than those in [12].

We call P a *locally concentrating Markov operator* if for every $\varepsilon > 0$ there is $\alpha > 0$ such that for every $A \in \mathcal{B}_{\mathrm{b}}(X)$ there exists $C \in \mathcal{B}_{\mathrm{b}}(X)$ with

$\operatorname{diam} C < \varepsilon$ and $n_0 \in \mathbb{N}$ satisfying

$$P^{n_0}\mu(C) > \alpha \quad \text{for } \mu \in \mathcal{M}_1^A. \tag{2.1}$$

Now for every $A \in \mathcal{B}_{\mathrm{b}}(X)$ and $\eta \in [0,1]$ we set

$$\mathcal{M}_1^{A,\eta} = \{\mu \in \mathcal{M}_1 : P^n\mu(A) \geq 1-\eta \text{ for } n \in \mathbb{N}\}.$$

Define the function $\varphi : \mathcal{B}_{\mathrm{b}}(X) \times [0,1] \to [0,2] \cup \{-\infty\}$ by

$$\varphi(A,\eta) = \limsup_{n\to\infty} \sup\{\|P^n\mu_1 - P^n\mu_2\| : \mu_1,\mu_2 \in \mathcal{M}_1^{A,\eta}\}.$$

As usual, we assume that the supremum of an empty set is equal to $-\infty$.

We start with an easy but very useful lemma.

Lemma 2.1. *Assume that P is a non-expansive and locally concentrating Markov operator. Let $\varepsilon > 0$ and let $\alpha > 0$ be such that, for $\varepsilon > 0$, the locally concentrating property holds. If $\eta < 1/2$ then*

$$\varphi(A, \eta(1-\alpha/2)) \leq (1-\alpha/2)\varphi(A,\eta) + \alpha\varepsilon/2 \tag{2.2}$$

for $A \in \mathcal{B}_{\mathrm{b}}(X)$.

Proof. Fix $\varepsilon > 0$, $A \in \mathcal{B}_{\mathrm{b}}(X)$ and $\eta < 1/2$. Let $\alpha > 0$, $n_0 \in \mathbb{N}$ and $C \in \mathcal{B}_{\mathrm{b}}(X)$ be such that the locally concentrating property holds. We see at once that if $\mathcal{M}_1^{1,\eta(1-\alpha/2)} = \emptyset$, then (2.2) is satisfied. Fix $\mu_1,\mu_2 \in \mathcal{M}_1^{A,\eta(1-\alpha/2)}$. As $\eta < 1/2$ we have

$$\mu_i \geq \tfrac{1}{2}\mu_i^A,$$

where $\mu_i^A \in \mathcal{M}_1^A$ is of the form

$$\mu_i^A(B) = \frac{\mu_i(A\cap B)}{\mu_i(A)} \quad \text{for } B \in \mathcal{B}(X),\ i = 1,2.$$

By the linearity of P we get

$$P^{n_0}\mu_i(C) \geq \tfrac{1}{2}P^{n_0}\mu_i^A(C) > \alpha/2 \quad \text{for } i = 1,2.$$

Hence for $i = 1,2$ we have

$$P^{n_0}\mu_i = (1-\alpha/2)\overline{\mu}_i + (\alpha/2)\nu_i, \tag{2.3}$$

where $\nu_i \in \mathcal{M}_1^C$ is defined by

$$\nu_i(B) = \frac{P^{n_0}\mu_i(B\cap C)}{P^{n_0}\mu_i(C)} \quad \text{for } B \in \mathcal{B}(X)$$

and $\overline{\mu}_i$ is defined by equation (2.3). Since $\nu_1,\nu_2 \in \mathcal{M}_1^C$ and $\operatorname{diam} C < \varepsilon$, we check at once that $\|\nu_1 - \nu_2\| < \varepsilon$. From (2.3) we conclude that

$$\begin{aligned} P^n\overline{\mu}_i(A) &\geq (1-\alpha/2)^{-1}\{P^{n_0+n}\mu_i(A) - \alpha/2\} \\ &\geq (1-\alpha/2)^{-1}\{1-\eta(1-\alpha/2)-\alpha/2\} \\ &= 1-\eta \quad \text{for } n \in \mathbb{N} \text{ and } i = 1,2. \end{aligned}$$

This gives $\overline{\mu}_1, \overline{\mu}_2 \in \mathcal{M}_1^{A,\eta}$ and consequently, since P is non-expansive, we have

$$\begin{aligned}
&\|P^{n_0+n}\mu_1 - P^{n_0+n}\mu_2\| \\
&\qquad \le (1-\alpha/2)\|P^n\overline{\mu}_1 - P^n\overline{\mu}_2\| + (\alpha/2)\|P^n\nu_1 - P^n\nu_2\| \\
&\qquad \le (1-\alpha/2)\sup\{\|P^n\mu_1 - P^n\mu_2\| : \mu_1, \mu_2 \in \mathcal{M}_1^{A,\eta}\} + \alpha\varepsilon/2.
\end{aligned}$$

By the above we get

$$\varphi(A, \eta(1-\alpha/2)) \le (1-\alpha/2)\varphi(A,\eta) + \alpha\varepsilon/2. \ \blacksquare$$

THEOREM 2.1. *Assume that P is a non-expansive and locally concentrating Markov operator. Moreover, assume that there exists $\mu_0 \in \mathcal{M}_1$ such that for every $\varepsilon > 0$ there is $A \in \mathcal{B}_{\mathrm{b}}(X)$ satisfying*

$$\liminf_{n\to\infty} P^n\mu_0(A) \ge 1 - \varepsilon. \tag{2.4}$$

Then P admits a unique invariant distribution.

Proof. The proof falls naturally into two parts, concerning existence and uniqueness.

Let $\mu_0 \in \mathcal{M}_1$ be as in the statement of the theorem. Since the space $\mathcal{M}_1$ with the Fortet–Mourier distance is a complete metric space (for more details see [2]), to prove the existence of an invariant distribution it is enough to show that $(P^n\mu_0)_{n\ge1}$ satisfies the Cauchy condition. In fact, according to the non-expansiveness of P, if $P^n\mu_0 \to \mu_* \in \mathcal{M}_1$, then $P^n\mu_0 \to P\mu_*$ as $n \to \infty$. Hence $P\mu_* = \mu_*$.

Fix $\varepsilon > 0$. Let $\alpha > 0$ be such that, for $\varepsilon/2$, the locally concentrating property holds. Let $k \in \mathbb{N}$ be such that $4(1-\alpha/2)^k < \varepsilon$. Choose $A \in \mathcal{B}_{\mathrm{b}}(X)$ satisfying

$$P^n\mu_0(A) \ge 1 - \tfrac{1}{3}(1-\alpha/2)^k \quad \text{for } n \in \mathbb{N}.$$

Using (2.2) it is easy to verify by an induction argument that

$$\begin{aligned}
\varphi\big(A, \tfrac{1}{3}(1-\alpha/2)^k\big) &\le (1-\alpha/2)^k\varphi(A, 1/3) + \alpha\varepsilon/4 \\
&\quad + \alpha\varepsilon(1-\alpha/2)/4 + \ldots + \alpha\varepsilon(1-\alpha/2)^{k-1}/4 \\
&\le 2(1-\alpha/2)^k + \varepsilon/2 < \varepsilon.
\end{aligned} \tag{2.5}$$

It is clear that $P^m\mu_0, P^n\mu_0 \in \mathcal{M}_1^{A,\frac{1}{3}(1-\alpha/2)^k}$ for $m, n \in \mathbb{N}$ and from (2.5) it follows that there exists $n_0 \in \mathbb{N}$ such that

$$\|P^{n_0}P^n\mu_0 - P^{n_0}P^m\mu_0\| < \varepsilon.$$

Therefore

$$\|P^n\mu_0 - P^m\mu_0\| < \varepsilon \qquad \text{for } m, n \ge n_0,$$

which finishes the proof of the existence of an invariant measure.

To prove uniqueness suppose, contrary to our claim, that $\overline{\mu}_1, \overline{\mu}_2 \in \mathcal{M}_1$ are two different invariant measures.

Set

$$\varepsilon := \|\overline{\mu}_1 - \overline{\mu}_2\| > 0. \tag{2.6}$$

As in the first part let $\alpha > 0$ be such that, for $\varepsilon/2$, the locally concentrating property holds. Choose $k \in \mathbb{N}$ such that $4(1-\alpha/2)^k < \varepsilon$. Since $P^n\overline{\mu}_i = \overline{\mu}_i$, $i = 1, 2$, $n \in \mathbb{N}$, we conclude that $\overline{\mu}_1, \overline{\mu}_2 \in \mathcal{M}_1^{A,\frac{1}{3}(1-\alpha/2)^k}$ for some $A \in \mathcal{B}_{\mathrm{b}}(X)$. From (2.5) it follows that

$$\|\overline{\mu}_1 - \overline{\mu}_2\| = \lim_{n\to\infty} \|P^n\overline{\mu}_1 - P^n\overline{\mu}_2\| \le \varphi\big(A, \tfrac{1}{3}(1-\alpha/2)^k\big) < \varepsilon,$$

contrary to (2.6), and the proof is complete. ∎

3. Singularity of an invariant measure. A mapping $S : X \to X$ is called a *contraction* if there is a constant L with $0 < L < 1$ such that $\varrho(S(x), S(y)) \le L \cdot \varrho(x, y)$ for all $x, y \in X$, and is called a *similarity* if there is a constant L with $0 < L < 1$ such that $\varrho(S(x), S(y)) = L \cdot \varrho(x, y)$ for all $x, y \in X$. In both cases the constant L is called the *Lipschitz constant*.

Let $S_1, \ldots, S_N$ be contractions. We call a subset F of X *invariant* for the transformations S_i if

$$F = \bigcup_{i=1}^{N} S_i(F). \tag{3.1}$$

Recall that if U is any non-empty subset of X, the *diameter* of U is defined as $|U| = \sup\{\varrho(x, y) : x, y \in U\}$.

If $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most δ that cover F, i.e. $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 < |U_i| \le \delta$ for each $i \in \mathbb{N}$, we say that $\{U_i\}$ is a *δ-cover* of F.

Suppose that F is a subset of X and s is a non-negative number. For any $\delta > 0$ we define

$$\mathcal{H}^s_\delta(F) = \inf\Big\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\Big\}. \tag{3.2}$$

As δ decreases, the class of permissible covers of F in (3.2) is reduced. Therefore, the infimum $\mathcal{H}^s_\delta(F)$ increases, and so approaches a limit as $\delta \to 0$. We write

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F). \tag{3.3}$$

We call $\mathcal{H}^s(F)$ the *s-dimensional Hausdorff outer measure* of F. The restriction of $\mathcal{H}^s$ to the σ-field of $\mathcal{H}^s$-measurable sets, which includes $\mathcal{B}(X)$, is called the *Hausdorff s-dimensional measure* on $\mathcal{B}(X)$.

A measure $\mu \in \mathcal{M}_{\mathrm{fin}}$ is called *absolutely continuous with respect to* $\mathcal{H}^s$ if $\mu(A) = 0$ for every $A \in \mathcal{B}(X)$ such that $\mathcal{H}^s(A) = 0$, and is called *singular with respect to* $\mathcal{H}^s$ if there is $Y \in \mathcal{B}(X)$ such that $\mathcal{H}^s(Y) = 0$ and $\mu(Y) = \mu(X)$.

Let $F \subset X$. The value

$$\dim_{\mathrm{H}} F = \inf\{s > 0 : \mathcal{H}^s(F) = 0\}$$

is called the *Hausdorff dimension* of F.

It can be proved (see [3, 4]) that

$$\dim_{\mathrm{H}} F = \sup\{s > 0 : \mathcal{H}^s(F) = \infty\}.$$

Moreover, if $s = \dim_{\mathrm{H}} F$, then $\mathcal{H}^s(F)$ may be zero or infinite, or may satisfy

$$0 < \mathcal{H}^s(F) < \infty.$$

In the case where $S_1, \ldots, S_N : X \to X$ are similarities with constants $L_1, \ldots, L_N$, respectively, a theorem proved by M. Hata (see Theorem 10.3 of [7] and Proposition 9.7 of [3]) allows us to calculate the Hausdorff dimension of the invariant set for $S_1, \ldots, S_N$. Namely, if we assume that F is an invariant set for the similarities $S_1, \ldots, S_N$ and $S_i(F) \cap S_j(F) = \emptyset$ for $i \neq j$, then $\dim_{\mathrm{H}} F = s$, where s is given by

$$\sum_{i=1}^{N} L_i^s = 1. \tag{3.4}$$

Moreover,

$$0 < \mathcal{H}^s(F) < \infty. \tag{3.5}$$

Let an ifs $(S_i, p_i)_{i=1}^N$ be given. We say that $(S_i, p_i)_{i=1}^N$ has a *stationary distribution* if the corresponding Markov operator P has a stationary distribution.

Now assume that

$$\sum_{i=1}^{N} |p_i(x) - p_i(y)| \le \omega(\varrho(x, y)), \tag{3.6}$$

and

$$\sum_{i=1}^{N} p_i(x)\varrho(S_i(x), S_i(y)) \le r \cdot \varrho(x, y) \quad \text{for } x, y \in X, \tag{3.7}$$

where $r < 1$ and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, $\mathbb{R}_+ = [0, \infty)$, is a non-decreasing and concave function which satisfies the Dini condition, i.e.

$$\int_0^a \frac{\omega(t)}{t}\, dt < \infty \quad \text{for some } a > 0.$$

It has been proved (see Theorem 4.2 in [12] and Theorem 3.2 in [10]) that under the conditions stated above we are able to change the metric ϱ in the Polish space (X,ϱ) in such a way that the new space remains Polish and the Markov operator P is non-expansive and satisfies (2.4) for every measure $\mu_0 \in \mathcal{M}_1$. Moreover, the new metric will define the same space of continuous functions. If we assume that for some i_0, $1 \le i_0 \le N$, S_{i_0} is a contraction and $p_{i_0}(x) \ge \sigma$ for $x \in X$ and some $\sigma > 0$, then P is locally concentrating. Theorem 2.1 now shows that P admits a unique invariant distribution.

In our further considerations we will study an ifs $(S_i,p_i)_{i=1}^N$ such that $S_1,\ldots,S_N$ are contractions with Lipschitz constants $L_1,\ldots,L_N$, respectively. It is obvious that it satisfies the above conditions and its unique invariant distribution is supported on the invariant set F for the transformations S_i, $i=1,\ldots,N$.

Following Lasota and Myjak (see [9]) we prove an easy lemma.

LEMMA 3.1. *Let* $(S_i,p_i)_{i=1}^N$ *satisfy conditions* (3.6), (3.7). *Then its unique invariant measure* $\mu_* \in \mathcal{M}_1$ *is either absolutely continuous or singular with respect to* $\mathcal{H}^s$, *where* $s>0$ *is given by*

$$\sum_{i=1}^N L_i^s = 1. \tag{3.8}$$

Proof. Let $s>0$ satisfy (3.8) and let P be the corresponding Markov operator for $(S_i,p_i)_{i=1}^N$. We first prove that $P\mu$ is singular with respect to $\mathcal{H}^s$ for every $\mu \in \mathcal{M}_{\rm fin}$ singular with respect to $\mathcal{H}^s$. Fix such a $\mu \in \mathcal{M}_{\rm fin}$. Let $Y \in \mathcal{B}(X)$ be such that $\mu(Y)=\mu(X)$ and $\mathcal{H}^s(Y)=0$. By a theorem due to Ulam we can assume that $Y=\bigcup_{n=1}^\infty K_n$, where $K_n \subset X$, $n \in \mathbb{N}$, are compact. Since S_i, $i=1,\ldots,N$, are continuous, we see at once that $\bigcup_{i=1}^N S_i(Y) \in \mathcal{B}(X)$. We have

$$\begin{aligned} P\mu\Big(\bigcup_{i=1}^N S_i(Y)\Big) &= \sum_{i=1}^N \int\limits_{S_i^{-1}(\bigcup_{i=1}^N S_i(Y))} p_i(x)\,\mu(dx) \ge \sum_{i=1}^N \int\limits_Y p_i(x)\,\mu(dx) \\ &= \mu(Y) = \mu(X) = P\mu(X) \end{aligned}$$

and

$$\mathcal{H}^s\Big(\bigcup_{i=1}^N S_i(Y)\Big) \le \sum_{i=1}^N \mathcal{H}^s(S_i(Y)) \le \sum_{i=1}^N L_i^s \mathcal{H}^s(Y) = \mathcal{H}^s(Y)\sum_{i=1}^N L_i^s = \mathcal{H}^s(Y),$$

which finishes the first part of the proof.

Let $\mu_* \in \mathcal{M}_1$ be the unique invariant measure of P. By the Lebesgue Decomposition Theorem $\mu_* = \mu_{\rm a} + \mu_{\rm s}$, where $\mu_{\rm a}$ is absolutely continuous and μ_s is singular with respect to $\mathcal{H}^s$. Clearly,

$$\mu_* = P\mu_* = P\mu_{\mathrm{a}} + P\mu_{\mathrm{s}}.$$

By the above the measure $P\mu_{\mathrm{s}}$ is singular with respect to $\mathcal{H}^s$. From this and the uniqueness of the Lebesgue decomposition it follows that $P\mu_{\mathrm{s}} \le \mu_{\mathrm{s}}$. On the other hand, by the preservation of the norm by Markov operators, we have $P\mu_{\mathrm{s}}(X) = \mu_{\mathrm{s}}(X)$. These two conditions imply $P\mu_{\mathrm{s}} = \mu_{\mathrm{s}}$. Consequently, we also have $P\mu_{\mathrm{a}} = \mu_{\mathrm{a}}$. In order to complete the proof it is sufficient to show that either μ_{a} or μ_{s} is identically equal to zero. For this, suppose that both μ_{a} and μ_{s} are non-trivial. Then $\mu_1 = \mu_{\mathrm{a}}/\mu_{\mathrm{a}}(X)$ and $\mu_2 = \mu_{\mathrm{s}}/\mu_{\mathrm{s}}(X)$ are two different invariant distributions of P, which is impossible. ∎

LEMMA 3.2. *Let* $(S_i, p_i)_{i=1}^N$ *satisfy conditions* (3.6), (3.7) *and let* P *be the corresponding Markov operator. Then for every* $\delta > 0$ *and every* $\mu \in \mathcal{M}_{\mathrm{fin}}$ *supported on a compact set we have*

$$\mathcal{H}^s_\delta(\operatorname{supp} P\mu) \le \mathcal{H}^s_\delta(\operatorname{supp} \mu),$$

where $s > 0$ *is given by* (3.8).

Proof. Let $s > 0$ satisfy (3.8). Fix $\delta > 0$ and $\mu \in \mathcal{M}_{\mathrm{fin}}$ concentrated on a compact set. Since S_i, $i = 1, \ldots, N$, are continuous, we check at once that

$$\operatorname{supp} P\mu \subset \bigcup_{i=1}^N S_i(\operatorname{supp} \mu).$$

Let $\{U_i\}$ be a δ-cover of $\operatorname{supp} \mu$. Then

$$\operatorname{supp} P\mu \subset \bigcup_{j=1}^\infty \bigcup_{i=1}^N S_i(U_j).$$

Since $0 < |S_i(U_j)| \le L_i |U_j| < \delta$, we have

$$\begin{aligned}\mathcal{H}^s_\delta(\operatorname{supp} P\mu) &\le \sum_{j=1}^\infty \sum_{i=1}^N |S_i(U_j)|^s \le \sum_{j=1}^\infty \sum_{i=1}^N L_i^s |U_j|^s \\ &= \sum_{i=1}^N L_i^s \sum_{j=1}^\infty |U_j|^s = \sum_{j=1}^\infty |U_j|^s\end{aligned}$$

and consequently

$$\mathcal{H}^s_\delta(\operatorname{supp} P\mu) \le \mathcal{H}^s_\delta(\operatorname{supp} \mu). \ ∎$$

THEOREM 3.1. *Let* $(S_i, p_i)_{i=1}^N$ *satisfy conditions* (3.6), (3.7). *Assume that*

$$\inf_{x \in X} p_k(x) > L_k^s \quad \text{for some } k,\ 1 \le k \le N,$$

where $s > 0$ *is given by* (3.8). *Then the unique invariant distribution of* $(S_i, p_i)_{i=1}^N$ *is singular with respect to* $\mathcal{H}^s$.

Proof. Let $\mu_* \in \mathcal{M}_1$ be the unique invariant measure of $(S_i, p_i)_{i=1}^N$. Let k, $1 \le k \le N$, be such that $\inf_{x\in X} p_k(x) > L_k^s$ for s given by (3.8). Let $F \subset X$ be the unique invariant compact set for S_i, $i = 1\ldots, N$. Obviously, $\operatorname{supp}\mu_* \subset F$. Let P be the corresponding Markov operator for $(S_i, p_i)_{i=1}^N$. Using the properties of Markov operators we get

$$\begin{aligned}(3.9)\quad & P^n\mu(B) \\ &= \sum_{i_1=1}^{N}\cdots\sum_{i_n=1}^{N}\ \int\limits_{S_{i_1}^{-1}\circ\ldots\circ S_{i_n}^{-1}(B)} p_{i_n}(S_{i_{n-1}}\circ\ldots\circ S_{i_1}(x))\cdot\ldots\cdot p_{i_1}(x)\,\mu(dx) \\ &\ge (\inf_{x\in X} p_k(x))^n\mu(S_k^{-n}(B)) \quad \text{for } B\in\mathcal{B}(X),\ \mu\in\mathcal{M}_1,\ n\in\mathbb{N}.\end{aligned}$$

From this

$$(3.10)\qquad P^n\mu(S_k^n(F)) \ge (\inf_{x\in X} p_k(x))^n\mu(S_k^{-n}(S_k^n(F))) = (\inf_{x\in X} p_k(x))^n$$

for $\mu \in \mathcal{M}_1^F$ and $n\in\mathbb{N}$. Set $B_n = S_k^n(F)$ and $\alpha_n = (\inf_{x\in X} p_k(x))^n$ for $n\in\mathbb{N}$. For every $n\in\mathbb{N}$ we will define by an induction argument two sequences of distributions $(\mu_l^n)_{l\ge 0}$, $(\nu_l^n)_{l\ge0}$, $\mu_l^n, \nu_l^n \in \mathcal{M}_1^F$, $l = 0,1,\ldots$

If $l = 0$ we define $\mu_0^n = \nu_0^n = \mu_*$. If $l\ge 1$ is fixed and μ_{l-1}^n, ν_{l-1}^n are given we define, according to (3.10),

$$(3.11)\qquad \nu_l^n(C) = \frac{P^n\mu_{l-1}^n(C\cap B_n)}{P^n\mu_{l-1}^n(B_n)},$$

$$(3.12)\qquad \mu_l^n(C) = \frac{1}{1-\alpha_n}\{P^n\mu_{l-1}^n(C) - \alpha_n\nu_l^n(C)\}$$

for $C\in\mathcal{B}(X)$. Observe that $\nu_l^n \in \mathcal{M}_1^{B_n}$ for $l\ge 1$. Using equations (3.11) and (3.12) it is easy to verify that

$$\begin{aligned}(3.13)\quad \mu_* = P^{l\cdot n}\mu_* &= \alpha_n P^{(l-1)n}\nu_1^n + \alpha_n(1-\alpha_n)P^{(l-2)n}\nu_2^n \\ &\quad + \ldots + \alpha_n(1-\alpha_n)^{l-1}\nu_l^n + (1-\alpha_n)^l\mu_l^n \quad \text{for } l>1.\end{aligned}$$

Set $\delta_n = L_k^n\cdot\operatorname{diam} F$ for $n\in\mathbb{N}$; we see that $\delta_n\to 0$ as $n\to\infty$. By Lemma 3.2 we have

$$\begin{aligned}(3.14)\quad \mathcal{H}_{\delta_n}^s\Big(\bigcup_{i=1}^{l}\operatorname{supp} P^{(l-i)n}\nu_i^n\Big) &\le \sum_{i=1}^{l}\mathcal{H}_{\delta_n}^s(\operatorname{supp}\nu_i^n) \\ &\le l\cdot(L_k^n)^s(\operatorname{diam} F)^s \quad \text{for } l\in\mathbb{N},\ n\in\mathbb{N}.\end{aligned}$$

On the other hand, by (3.13) we have

$$(3.15)\qquad \mu_*\Big(\bigcup_{i=1}^{l}\operatorname{supp} P^{(l-i)n}\nu_i^n\Big) \ge 1-(1-\alpha_n)^l \quad \text{for } l\in\mathbb{N},\ n\in\mathbb{N}.$$

Define

$$A_n = \bigcup_{i=1}^{l_n}\operatorname{supp} P^{(l_n-i)n}\nu_i^n \quad \text{for } n\in\mathbb{N},$$

where $l_n = [1/\alpha_n]$. (We use $[a]$ to denote the integer part of a, i.e. the largest integer not larger than a.) Then by (3.14) and (3.15),

$$\mathcal{H}^s_{\delta_n}(A_n) \le (L^s_k / \inf_{x\in X} p_k(x))^n (\operatorname{diam} F)^s \tag{3.16}$$

and

$$\mu_*(A_n) \ge 1 - (1-\alpha_n)^{1/\alpha_n - 1}. \tag{3.17}$$

Thus

$$\lim_{n\to\infty} \mathcal{H}^s_{\delta_n}(A_n) = 0$$

and

$$\liminf_{n\to\infty} \mu_*(A_n) \ge 1 - 1/e > 0.$$

Since $\mathcal{H}^s_\delta(A) \le \mathcal{H}^s_{\delta'}(A)$ for $\delta' < \delta$, $A \in \mathcal{B}(X)$, we have

$$\lim_{n\to\infty} \mathcal{H}^s_{\delta_m}(A_n) = 0 \quad \text{for } m \in \mathbb{N}. \tag{3.18}$$

We will define by an induction argument sequences of sets $(A^i_n)_{n\ge 1}$, $i = 1, 2, \ldots$ If $i = 0$ we define $A^0_n = A_n$ for $n = 1, 2, \ldots$ If $i \ge 1$ is fixed and $(A^{i-1}_n)_{n\ge 1}$ is given, we choose according to (3.18) a subsequence $(A^i_n)_{n\ge 1}$ of $(A^{i-1}_n)_{n\ge 1}$ such that

$$\sum_{n=1}^{\infty} \mathcal{H}^s_{\delta_i}(A^i_n) \le 1/i. \tag{3.19}$$

Setting

$$E_n := A^n_n \quad \text{for } n \in \mathbb{N}$$

we define

$$E = \bigcap_{i=1}^{\infty} \bigcup_{n=i}^{\infty} E_n.$$

Then

$$\mu_*(E) = \lim_{i\to\infty} \mu_*\Big(\bigcup_{n=i}^{\infty} E_n\Big) \ge 1 - 1/e > 0$$

and by (3.19) we get

$$\begin{aligned}\mathcal{H}^s(E) &= \lim_{\delta\to 0} \mathcal{H}^s_\delta(E) = \lim_{i\to\infty} \mathcal{H}^s_{\delta_i}(E) \le \limsup_{i\to\infty} \mathcal{H}^s_{\delta_i}\Big(\bigcup_{n=i}^{\infty} E_n\Big) \\ &\le \limsup_{i\to\infty} \sum_{n=i}^{\infty} \mathcal{H}^s_{\delta_i}(E_n) \le \limsup_{i\to\infty} \sum_{n=i}^{\infty} \mathcal{H}^s_{\delta_i}(A^i_n) \le \lim_{i\to\infty} \frac{1}{i} = 0.\end{aligned}$$

By Lemma 3.1, μ_* must be singular with respect to $\mathcal{H}^s$, which is the desired conclusion. ∎

Our next remark shows that in the statement of the last theorem the condition $\inf_{x\in X} p_k(x) > L^s_k$ for some k is essential to the proof.

REMARK. Consider the ifs $(S_i, p_i)_{i=1}^2$, where $S_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are of the form

$$S_1(x) = \tfrac{1}{2}x, \qquad S_2(x) = \tfrac{1}{2}x + \tfrac{1}{2}$$

and $p_1 = p_2 = \frac{1}{2}$. An easy computation shows that the interval $[0,1]$ is an invariant set for the transformations S_1, S_2, i.e.

$$[0,1] = S_1([0,1]) \cup S_2([0,1]).$$

Further, it is easy to prove that $l_1|_{[0,1]}$ (l_1 denotes the Lebesgue measure on $\mathbb{R}$) is the unique invariant measure of $(S_i, p_i)_{i=1}^2$. Since $L_1 + L_2 = 1$, we get $s = 1$. This gives $\mathcal{H}^s = l_1$, which contradicts the fact that $\mathcal{H}^s$ and l_1 are mutually singular and shows that the condition $\inf_{x\in X} p_k(x) > L_k^s$ for some k cannot be omitted.

References

[1] M. Arbeiter and N. Patzschke, *Random self-similar multifractals*, Math. Nachr. 181 (1996), 5–42.

[2] R. M. Dudley, *Probabilities and Matrices*, Aarhus Universitet, 1976.

[3] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, New York, 1990.

[4] —, *The Geometry of Fractal Sets*, Cambridge Univ. Press, Cambridge, 1985.

[5] R. Fortet et B. Mourier, *Convergence de la répartition empirique vers la répartition théorétique*, Ann. Sci. École Norm. Sup. 70 (1953), 267–285.

[6] J. S. Geronimo and D. P. Hardin, *An exact formula for the measure dimensions associated with a class of piecewise linear maps*, Constr. Approx. 5 (1989), 89–98.

[7] M. Hata, *On the structure of self-similar sets*, Japan J. Appl. Math. 2 (1985), 381–414.

[8] J. E. Hutchinson, *Fractals and self-similarities*, Indiana Univ. Math. J. 30 (1981), 713–747.

[9] A. Lasota and J. Myjak, *Generic properties of fractal measures*, Bull. Polish Acad. Sci. Math. 42 (1994), 283–296.

[10] —, —, *Semifractals on Polish spaces*, ibid. 46 (1998), 179–196.

[11] A. Lasota and J. A. Yorke, *Lower bound technique for Markov operators and iterated function systems*, Random Comput. Dynam. 2 (1994), 41–77.

[12] T. Szarek, *Markov operators acting on Polish spaces*, Ann. Polon. Math. 67 (1997), 247–257.

Institute of Mathematics
Polish Academy of Sciences
Bankowa 14
40-007 Katowice, Poland
E-mail: szarek@ux2.math.us.edu.pl

Department of Mathematics
Technical University of Rzeszów
W. Pola 6
35-959 Rzeszów, Poland

Reçu par la Rédaction le 28.2.2000
Révisé le 28.5.2000