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Non-solvability of the tangential $\overline{\partial}$-system in manifolds with constant Levi rank

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Abstract. Let M be a real-analytic submanifold of $\mathbb{C}^n$ whose "microlocal" Levi form has constant rank $s_M^+ + s_M^-$ in a neighborhood of a prescribed conormal. Then local non-solvability of the tangential $\overline{\partial}$-system is proved for forms of degrees s_M^-, s_M^+ (and 0).

This phenomenon is known in the literature as "absence of the Poincaré Lemma" and was already proved in case the Levi form is non-degenerate (i.e. $s_M^- + s_M^+ = n - \operatorname{codim} M$). We owe its proof to [2] and [1] in the case of a hypersurface and of a higher-codimensional submanifold respectively. The idea of our proof, which relies on the microlocal theory of sheaves of [3], is new.

1. Main statement. Let M be a real-analytic generic submanifold of $X = \mathbb{C}^n$ of codimension l, and denote by $\overline{\partial}_M$ the antiholomorphic tangential differential on M. Let $\mathcal{B}_M$ be the Sato hyperfunctions on M, denote by $\mathcal{B}_M^j$ the forms of type $(0,j)$ with coefficients in $\mathcal{B}_M$, and consider the tangential $\overline{\partial}$-complex:

$$0 \to \mathcal{B}_M^0 \stackrel{\overline{\partial}_M}{\to} \mathcal{B}_M^1 \stackrel{\overline{\partial}_M}{\to} \ldots \stackrel{\overline{\partial}}{\to} \mathcal{B}_M^n \to 0. \tag{1}$$

We shall denote by $\mathrm{H}^j_{\overline{\partial}_M}$ the jth cohomology of (1) (which is denoted by $\mathrm{H}^j(\mathbb{R}\mathcal{H}om(\overline{\partial}_M, \mathcal{B}_M))$ in the language of D-modules). In particular $(\mathrm{H}^j_{\overline{\partial}_M})_z$ (z a point of M) are the germs at z of $\overline{\partial}_M$-closed $(0,j)$-forms modulo $\overline{\partial}_M$-exact ones.

It is crucial for our approach to note that the cohomology of (1) is the same as that of $\mathbb{R}\Gamma_M(\mathcal{O}_X)[l]$ ($\mathcal{O}_X$ denoting the holomorphic functions on X); this point of view will always be adopted in our proofs. For an open set $U \subset M$, we shall also consider the analogue of (1) with the sheaves $\mathcal{B}_M^j$ replaced by the spaces $\mathcal{B}_M^j(U)$ of their sections on U. We shall denote by

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$\mathrm{H}^j_{\overline{\partial}_M}(U)$ the cohomology of this complex. Note that $(\mathrm{H}^j_{\overline{\partial}_M})_z = \varinjlim_U \mathrm{H}^j_{\overline{\partial}_M}(U)$ for U ranging through the family of open neighborhoods of z.

Let $T^*_M X \xrightarrow{\pi} M$ be the conormal bundle to M in X, fix $p \in \dot{T}^*_M X$ $(= T^*_M X \setminus \{0\})$ with $\pi(p) = z$, and choose a real function r which vanishes identically on M and such that $dr(z) = p$. If $T^{\mathbb{C}}M$ is the complex tangent bundle to M (i.e. $T^{\mathbb{C}}M = TM \cap \sqrt{-1}\,TM$), we set $L_M(p) = \partial\overline{\partial}r(z)|_{T^{\mathbb{C}}_z M}$ and call it the *Levi form* of M at p. Let $s^+_M(p), s^-_M(p)$ denote the numbers of respectively positive and negative eigenvalues of $L_M(p)$.

THEOREM 1. *Let M be a generic real-analytic submanifold of X. Assume there exists a neighborhood V of p such that*

$$s^{\pm}_M(p') \equiv s^{\pm}_M(p) \quad \textit{for any } p' \in V. \tag{2}$$

Then

$$(\mathrm{H}^j_{\overline{\partial}_M})_z \neq 0 \quad \textit{for } j = s^-_M(p), s^+_M(p), 0. \tag{3}$$

2. Proof of Theorem 1. Let T^*X denote the cotangent bundle to X endowed with the symplectic 2-form $\sigma = \sigma^{\mathbb{R}} + \sigma^{\mathbb{I}}$. Let M be a generic submanifold of X and denote by $\mu_M(\mathcal{O}_X)$ the microlocalization of $\mathcal{O}_X$ along M in the sense of [3]. We shall use the following result by Kashiwara and Schapira:

THEOREM 2 [3, Ch. 11]. *Let $p \in \dot{T}^*_M X$ and assume $s^-_M(p') \equiv s^-_M(p)$ for any p' in a neighborhood of p. We can then find a symplectic complex transformation χ from a neighborhood of p to a neighborhood of $\widetilde{p} = \chi(p)$ which interchanges $T^*_M X$ and $T^*_{\widetilde{M}} X$ where $\widetilde{M}$ is a hypersurface. Denote by $\widetilde{M}^{\pm}$ the closed half-spaces with boundary $\widetilde{M}$ and outward conormals $\pm\widetilde{p}$; we can arrange that $\mathring{\widetilde{M}}{}^-$ is pseudoconvex. Moreover such a transformation can be quantized so that it gives a correspondence*

$$\mu_M(\mathcal{O}_X)_p[l + s^-_M] \xrightarrow{\sim} \mathbb{R}\Gamma_{\widetilde{M}^+}(\mathcal{O}_X)_{\widetilde{z}}[1]. \tag{4}$$

We note that

$$\mathrm{H}^j(\mathbb{R}\Gamma_{\widetilde{M}^+}(\mathcal{O}_X)_{\widetilde{z}}) = \begin{cases} \varinjlim_B \mathcal{O}_X((\mathring{\widetilde{M}}{}^+) \cap B)/\mathcal{O}_X(B) & \text{for } j = 1, \\ 0 & \text{for } j > 1. \end{cases} \tag{5}$$

We also note that the pseudoconvexity of $\mathring{\widetilde{M}}{}^-$ implies that the cohomology of degree 1 is $\neq 0$.

(In the preceding theorem M need not be real-analytic or satisfy $\operatorname{rank} L_M(p') \equiv$ const; only $s^-_M(p') \equiv$ const is required.)

THEOREM 3 ([6]). *Let M be generic real-analytic and assume* $\operatorname{rank} L_M \equiv$ const *at p (i.e.* (2) *of* §1 *holds). Then we may find a complex homogeneous* *symplectic transformation* $\chi : T^*X \xrightarrow{\sim} T^*X' \times T^*Y$ $(\dim X' = \operatorname{rank} L_M)$ *such that*

$$T^*_M X \underset{\chi}{\xrightarrow{\sim}} T^*_{M'}X' \times Y,$$

where M' *is the boundary of a strictly pseudoconvex domain of* X'.

Proof. By a complex symplectic homogeneous transformation, we can interchange $T^*_M X$ with the conormal bundle to a hypersurface. Hence we may assume from the beginning that M is a hypersurface. Since $\operatorname{rank} L_M \equiv$ const, M is foliated by complex leaves tangent to $\operatorname{Ker} L_M$ (cf. [4]). These leaves can be represented as fibers of a real-analytic projection $M \to M_1 = M/\sim$ where $\sim$ identifies all points on the same leaf. Due to the constant rank assumption, it is easy to check that this complex foliation is induced by another foliation $\Lambda \xrightarrow{\varrho_1} \Lambda_1$ where $\Lambda = T^*_M X$ and $\Lambda_1 = \Lambda/\sim$. Note that $\operatorname{Ker} \varrho_1 = T\Lambda \cap \sqrt{-1}\, T\Lambda = \operatorname{Ker} \sigma|_{T\Lambda}$, and therefore $\sigma^{\mathbb{I}}$ induces naturally on Λ_1 a non-degenerate form $\sigma_1^{\mathbb{I}}$. On the other hand, since Λ is a real-analytic CR submanifold of T^*X (due again to the constant rank assumption), there is a complex submanifold $\widetilde{\Lambda}$ of T^*X which contains Λ as a generic submanifold. It is easy to see that $\widetilde{\Lambda}$ is an involutive submanifold of T^*X and that ϱ (which is CR and real-analytic) can be complexified to $\widetilde{\Lambda} \xrightarrow{\varrho_1^{\mathbb{C}}} \Lambda_1^{\mathbb{C}}$, the projection along the bicharacteristic leaves of $\widetilde{\Lambda}$.

We can now conclude the proof. First we make a transformation of T^*X which puts $\widetilde{\Lambda}$ in the canonical form $\widetilde{\Lambda} = T^*X' \times Y$. Then we make a transformation in T^*X' so that Λ' is interchanged with $T^*_{M'}X'$ where the closed half-space M'^{+} with boundary M' and inward conormal $\chi(\varrho(p))$ is the complement of a pseudoconvex domain. ■

Let $V = V' \times Y$ be an open conic neighborhood of p in $\dot{T}^*_M X$ where the conclusions of Theorems 2 and 3 hold. Let $Z = Z' \times Y$ be a relatively closed (in V) conic neighborhood of p such that $Z' \subset\subset V'$. Define $\mathcal{F} := \mu_M(\mathcal{O}_X)[l + s^-_M]$, and let $f \in \Gamma(V, \mathrm{H}^0(\mathcal{F}))$.

THEOREM 4. *For any open neighborhood* W *of* p *with* $W \subset\subset \operatorname{int} Z$ *we may find* $\widetilde{f} \in \Gamma_Z(V, \mathrm{H}^0(\mathcal{F}))$ *with* $\widetilde{f}|_W = f|_W$.

Proof. It is not restrictive to assume W has the form $W = W' \times Y_1$ with $W' \subset\subset \operatorname{int} Z'$ and $Y_1 \subset\subset Y$. If $\mathrm{H}^0(\mathcal{F})$ were soft (e.g. if $\operatorname{rank} L_M \equiv n - l$ in which case it is even flabby), then the theorem would be immediate. In fact given $f \in \Gamma(V, \mathrm{H}^0(\mathcal{F}))$ one would define $\widetilde{f}$ to be an extension to V of the section which takes the value f in a neighborhood of $\overline{W}$ and 0 in a neighborhood of $V \setminus \mathring{Z}$. This section exists because $\overline{W} \cap (V \setminus \mathring{Z}) = \emptyset$ and extends by softness.

In general, let $f \in \Gamma(V, \mathrm{H}^0(\mathcal{F})) = \Gamma(\pi(V), \mathcal{H}^1_{M'^{+} \times Y}(\mathcal{O}_X))$. Also write $\pi(V) = \omega' \times Y$ and take Y_2 with $Y_1 \subset\subset Y_2$, $Y_2 \subset\subset Y$. We remark that f is the boundary value $b(F)$ of $F \in \Gamma(\Omega'^{-} \times Y_2, \mathcal{O}_X)$ where $\Omega'^{-} = \Omega' \cap M'^{-}$ for a neighborhood Ω' of ω' in X'.

Let ω'_1 be an open neighborhood of $\pi(\overline{W}') \cup \pi(V' \setminus \mathring{Z}')$ in X', let Ω'_1 be a neighborhood of ω'_1 in X', and write $\Omega_1'^{-} = \Omega'_1 \cap M'^{-}$. We suppose that Ω'_1 is the union $\Omega'_1 = \Omega'_2 \cup \Omega'_3$ with Ω'_2 and Ω'_3 disjoint neighborhoods of $\omega'_2 := \pi(\overline{W}')$ and $\omega'_3 := \pi(V' \setminus \mathring{Z}')$ respectively. We define $\widetilde{F}$ on $\Omega_1'^{-} \times Y_2$ to be F on $\Omega_2'^{-} \times Y_2$ and 0 on $\Omega_3'^{-} \times Y_2$ (the meaning of the superscript "$-$" being now clear). Note that we may choose Ω'_1 in such a way that

$$\Omega'_4 := \Omega'_1 \cup M'^{-} \quad \text{is still strictly pseudoconvex.}$$

Since $\mathrm{H}^1(\Omega'_4 \times Y_2, \mathcal{O}_X) = 0$, by the Mayer–Vietoris long exact sequence $\widetilde{F}$ decomposes as

$$\widetilde{F} = I + J, \quad I \in \Gamma(\Omega'_1 \times Y_2, \mathcal{O}_X),\ J \in \Gamma(M'^{-} \times Y_2, \mathcal{O}_X).$$

The following equalities then hold in $\mathrm{H}^1(\mathbb{R}\Gamma_{M'^{+} \times Y_2}(\mathcal{O}_X))$:

$$b(J)|_{\omega'_2 \times Y_2} = f, \quad b(J)|_{\omega'_3 \times Y_2} = 0.$$

In particular $b(J)$ has support in $Z' \times Y_2$ and coincides with f in $W' \times Y_2$. Thus $\widetilde{f} := b(J)$ meets all requirements in the statement of Theorem 4. ∎

End of proof of Theorem 1. By a choice of a system of equations $r_h = 0$, $h = 1, \ldots, l$, for M, we identify

$$M \times \mathbb{R}^l \xrightarrow{\sim} T^*_M X, \quad (z;\lambda) \overset{\sim}{\mapsto} (z; \lambda \cdot \partial(r_h)(z)).$$

We fix $p = (z;\lambda) \in M \times \dot{\mathbb{R}}^l$ (where $\dot{\mathbb{R}}^l = \mathbb{R}^l \setminus \{0\}$), and consider a neighborhood V of p where the conclusions of Theorems 2–4 hold. In the coordinates of Theorem 3 we assume $V = V' \times Y$ and take $Z = Z' \times Y$. We recall that the projection ϱ along the complex leaves $\{p'\} \times Y$ is transversal to π. Therefore for a suitable neighborhood U_0 of z, $\dot{\pi}^{-1}(U_0) \cap Z$ is closed in $\dot{\pi}^{-1}(U_0)$. Let A be a closed cone of $\dot{\mathbb{R}}^l$ such that $U_0 \times A \supset \dot{\pi}^{-1}(U_0) \cap Z$. Then we have a natural morphism

$$\Gamma_Z(V, \mathrm{H}^0(\mathcal{F})) \xrightarrow{\alpha} \Gamma_{U_0 \times A}(U_0 \times \dot{\mathbb{R}}^l, \mathrm{H}^0(\mathcal{F})).$$

We also have an isomorphism

$$\mathrm{H}^0(\mathbb{R}\Gamma_{U_0 \times A}(U_0 \times \dot{\mathbb{R}}^l, \mathcal{F})) \xrightarrow{\sim} \Gamma_{U_0 \times A}(U_0 \times \dot{\mathbb{R}}^l, \mathrm{H}^0(\mathcal{F})).$$

Let $\{U_\nu\}$ be a system of neighborhoods of z with $U_\nu \subset U_0$, let B be an open cone with $B \subset\subset \operatorname{int} A$, and define $W_\nu := U_\nu \times B$. We have a commutative diagram

$$
\begin{tikzcd}
\mathrm{H}^0(\mathbb{R}\Gamma_{U_0\times A}(U_0\times\dot{\mathbb{R}}^l,\mathcal{F})) \arrow[r, "\beta"] \arrow[d, equal] & \mathrm{H}^0(\mathbb{R}\Gamma(U_\nu\times\dot{\mathbb{R}}^l,\mathcal{F})) \arrow[r, "\gamma"] & \mathrm{H}^0(\mathbb{R}\Gamma(W_\nu,\mathcal{F})) \arrow[dd, equal] \\
\Gamma_{U_0\times A}(U_0\times\dot{\mathbb{R}}^l,\mathrm{H}^0(\mathcal{F})) & & \\
\Gamma_Z(V,\mathrm{H}^0(\mathcal{F})) \arrow[u, "\alpha"] \arrow[rr, "\delta"] & & \Gamma(W_\nu,\mathrm{H}^0(\mathcal{F})),
\end{tikzcd}
\tag{6}
$$

where β is induced by the morphism $\mathbb{R}\Gamma_{U_0\times A}(U_0\times\dot{\mathbb{R}}^l,\cdot)\to\mathbb{R}\Gamma(U_\nu\times\dot{\mathbb{R}}^l,\cdot)$, and γ (resp. δ) by the restriction from $U_0\times\dot{\mathbb{R}}^l$ (resp. V) to W_ν.

Let $f\in\mathrm{H}^0(\mathcal{F})_p$, $f\neq 0$. According to Theorem 4, we may modify f to a section $\widetilde{f}\in\Gamma_Z(V,\mathrm{H}^0(\mathcal{F}))$ such that $\delta(\widetilde{f})\neq 0$ for any W_ν. Thus

$$\beta\circ\alpha(\widetilde{f})\neq 0 \quad \text{in } \mathrm{H}^0(\mathbb{R}\Gamma(U_\nu\times\dot{\mathbb{R}}^l,\mathcal{F}))=\mathrm{H}^{s_M^-}(\mathbb{R}\Gamma(U_\nu\times\dot{\mathbb{R}}^l,\mu_M(\mathcal{O}_X))[l]).$$

Observe now that since $\varinjlim_{U_\nu}\mathrm{H}^j_{\overline{\partial}_X}(U_\nu)=0$ for all $j>0$, we have

$$
\begin{aligned}
\varinjlim_{U_\nu}\mathrm{H}^j(\mathbb{R}\Gamma(U_\nu\times\dot{\mathbb{R}}^l,\mu_M(\mathcal{O}_X))[l]) &\simeq \varinjlim_{U_\nu}\mathrm{H}^j(\mathbb{R}\Gamma(U_\nu,\mathbb{R}\Gamma_M(\mathcal{O}_X))[l])\\
&\simeq \varinjlim_{U_\nu}\mathrm{H}^j_{\overline{\partial}_M}(U_\nu)=(\mathrm{H}^j_{\overline{\partial}_M})_z.
\end{aligned}
$$

In conclusion $\beta\circ\alpha(\widetilde{f})\neq 0$ in $(\mathrm{H}^{s_M^-}_{\overline{\partial}_M})_z$.

To prove the non-vanishing of the cohomology of (1) in degree $s_M^+(p)$, one just applies the above argument with p replaced by $-p$, and remarks that $s_M^+(p)=s_M^-(-p)$. ■

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