

On CLUR points of Orlicz spaces

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Abstract. Criteria for compactly locally uniformly rotund points in Orlicz spaces are given.

Let $[X, \|\cdot\|]$ be a Banach space. $S(X)$ and $B(X)$ are the unit sphere and unit ball of X respectively. A point x on $S(X)$ is said to be a *locally uniformly rotund point* (*LUR point*) provided that $\{x_n\} \subset X$, $\|x_n\| \rightarrow 1$ and $\|x_n + x\| \rightarrow 2$ imply $\|x_n - x\| \rightarrow 0$. Recently Y. A. Cui, H. Hudzik and C. Meng [2] introduced the concept of compactly locally uniformly rotund point (CLUR point for short). $x \in S(X)$ is said to be a *CLUR point* provided that $\{x_n\} \subset X$, $\|x_n\| \rightarrow 1$ and $\|x_n + x\| \rightarrow 2$ imply that $\{x_n\}$ is a compact set. Obviously, if every point on $S(X)$ is LUR (CLUR) then X is a LUR (CLUR) space. In 1975, B. B. Panda and O. P. Kapoor [4] proved that CLUR implies the Kadec–Klee property and X is LUR iff X is CLUR and strictly rotund. In 1984, J. Y. Fu and W. Y. Zhang [3] showed that if X^{**} is CLUR and K is a non-null closed, convex Chebyshev set in X then the projection on K is continuous. Y. A. Cui *et al.* [2] obtained a criterion for a point to be CLUR in Orlicz sequence spaces equipped with the Luxemburg norm.

In this paper we will discuss the CLUR points in Orlicz sequence spaces equipped with the Orlicz norm and in Orlicz function spaces equipped with both the Luxemburg and Orlicz norm.

A mapping $M : (-\infty, \infty) \rightarrow [0, \infty)$ is said to be an *N-function* if it is even, convex, vanishing only at zero and such that $\lim_{u \rightarrow 0} M(u)/u = 0$ and $\lim_{u \rightarrow \infty} M(u)/u = \infty$. An interval $[a, b]$ is called a *structurally affine interval* (SAI for short) of M provided that M is affine on $[a, b]$ and it is not affine on $[a - s, b]$ or $[a, b + s]$ for any $s > 0$. For an *N-function* M we

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define its *complementary function* by

$$N(v) = \sup_{u>0} \{u|v| - M(u)\} \quad (v \in (-\infty, \infty)).$$

This is again an N -function (see [1]). Denote by $p(u), p_-(u)$ the right and left derivative of $M(u)$ respectively. We have [1] the so-called *Young inequality*

$$uv \leq M(u) + N(v)$$

for any $u, v \geq 0$, and $uv = M(u) + N(v)$ iff $p_-(u) \leq v \leq p(u)$.

M is said to satisfy the Δ_2 -condition ($M \in \Delta_2$, for short) provided there are $k > 2$ and $u_0 > 0$ such that $M(2u) \leq kM(u)$ for u satisfying $u \geq u_0$. $M \in \nabla_2$ means that $N \in \Delta_2$.

Let (T, Σ, μ) be a nonatomic finite measure space and L^0 the set of all Σ -measurable functions $x : T \rightarrow (-\infty, \infty)$. For a given N -function M , the *modular* of x with respect to M is $\varrho_M(x) = \int_T M(x(t)) d\mu$. The linear set

$$\{x \in L^0 : \exists \lambda > 0 \text{ such that } \varrho_M(\lambda x) < \infty\}$$

equipped with the *Luxemburg norm*

$$\|x\|_M = \inf\{\lambda > 0 : \varrho_M(x/\lambda) \leq 1\}$$

or *Orlicz norm*

$$\|x\|_M^0 = \sup \left\{ \int_T x(t)y(t) d\mu : \varrho_N(y) \leq 1 \right\} = \inf_{k>0} \frac{1}{k} (1 + \varrho_M(kx))$$

is a Banach space and is called an *Orlicz space* denoted by L_M or L_M^0 respectively.

For $0 \neq x \in L_M^0$, define

$$k_x^* = \inf \left\{ k > 0 : \varrho_N(p(k|x|)) = \int_T N(p(k|x(t)|)) d\mu \geq 1 \right\},$$

$$k_x^{**} = \sup \{k > 0 : \varrho_N(p(k|x|)) \leq 1\}.$$

It is known that $\|x\|_M^0 = k^{-1}(1 + \varrho_M(kx))$ iff $k \in [k_x^*, k_x^{**}]$ (see [1], Th. 1.31).

Similarly we denote by I_M and I_M^0 the Orlicz sequence spaces equipped with the Luxemburg norm and Orlicz norm respectively. We only need to notice that $M \in \Delta_2$ in the sequence case means that there exist $k > 2$ and $u_0 > 0$ such that $M(2u) \leq kM(u)$ for all u satisfying $|u| \leq u_0$.

The following is known ([2], Th. 6): $x \in S(I_M)$ is a CLUR point iff (a) $M \in \Delta_2$, (b) $M \in \nabla_2$ or $\{i : |x(i)| \in (a, b)\} = \emptyset$ for any SAI $[a, b]$ of M .

A criterion for CLUR points in I_M^0 is different.

THEOREM 1. *Let $x \in S(I_M^0)$. Then x is a CLUR point iff (a) $M \in \Delta_2$, (b) $M \in \nabla_2$.*

Proof. Necessity. Suppose that $1 = \|x\|_M^0 = k^{-1}(1 + \varrho_M(kx))$. Assume $M \notin \Delta_2$. If $\lim_{n \rightarrow \infty} \|x - [x]_n\|_M^0 = 0$, we take

$$z \in B(I_M), \quad \lim_{n \rightarrow \infty} \left\| \frac{z - [z]_n}{k} \right\|_M^0 > 0;$$

if $\lim_{n \rightarrow \infty} \|x - [x]_n\|_M^0 > 0$, we take $z = 0$. Then we always have

$$\lim_{n \rightarrow \infty} \left\| \left(x - \frac{z}{k} \right) - \left[x - \frac{z}{k} \right]_n \right\|_M^0 > 0,$$

where $[x]_n = (x(1), \dots, x(n), 0, 0, \dots)$. Let

$$x_n = \left(x(1), \dots, x(n), \frac{z(n+1)}{k}, \frac{z(n+2)}{k}, \dots \right) \quad (n = 1, 2, \dots).$$

Then

$$\begin{aligned} \|x_n\|_M^0 &\leq \frac{1}{k}(1 + \varrho_M(kx_n)) \\ &= \frac{1}{k} \left(1 + \sum_{i=1}^n M(kx(i)) + \sum_{i>n} M(z(i)) \right) \\ &\rightarrow \frac{1}{k}(1 + \varrho_M(kx)) = 1 \end{aligned}$$

and

$$\|x_n + x\|_M^0 \geq 2\|[x]_n\|_M^0 \rightarrow 2\|x\|_M^0 = 2.$$

Clearly if $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$, then (x_{n_j}) has to converge to x . But

$$\lim_{j \rightarrow \infty} \|x - x_{n_j}\|_M^0 = \lim_{j \rightarrow \infty} \left\| \left(x - \frac{z}{k} \right) - \left[x - \frac{z}{k} \right]_{n_j} \right\|_M^0 > 0,$$

which is a contradiction. Thus (a) is true.

If $M \notin \nabla_2$, then there exist

$$v_n < N^{-1} \left(\frac{1}{n+1} \right), \quad N \left(\left(1 + \frac{1}{n} \right) v_n \right) > (n+1)N(v_n) \quad (n = 1, 2, \dots).$$

Take the positive integer m_n satisfying

$$\frac{1}{n+1} < m_n N(v_n) \leq \frac{1}{n} \quad (n = 1, 2, \dots).$$

Let

$$y_n = \left(\overbrace{0, \dots, 0}^{m_1 + \dots + m_{n-1}}, \overbrace{v_n, \dots, v_n}^{m_n}, 0, 0, \dots \right) \quad (n = 1, 2, \dots).$$

Then $\text{supp } y_n \cap \text{supp } y_j = \emptyset$ ($n \neq j$). Since $\varrho_N(y_n) = m_n N(v_n) < 1$ and

$$\varrho_N((1 + 1/n)y_n) = m_n N((1 + 1/n)v_n) > (n+1)m_n N(v_n) > 1,$$

we have

$$1 \geq \|y_n\|_N \geq \frac{1}{1+1/n} \quad (n = 1, 2, \dots).$$

From [1], Prop. 1.83, y_n has a supporting function x_n on $S(I_M^0)$, i.e. $\|x_n\|_M^0 = 1$,

$$\langle x_n, y_n \rangle = \sum_i x_n(i)y_n(i) = \|y_n\|_N > \frac{1}{1+1/n}$$

and $\text{supp } x_n = \text{supp } y_n$ ($n = 1, 2, \dots$).

From [1], Th. 1.45, there is $y \in S(L_N)$ with $\langle x, y \rangle = \|x\|_M^0 = 1$. Let

$$z_n = \frac{1}{1+1/n} \left(y(1), \dots, y\left(\sum_{k=1}^{n-1} m_k\right), \overbrace{v_n, \dots, v_n}^{m_n}, y\left(\sum_{k=1}^n m_k + 1\right), \dots \right).$$

Then

$$\begin{aligned} \varrho_N(z_n) &\leq \frac{1}{1+1/n} (\varrho_N(y) + m_n N(v_n)) \\ &\leq \frac{1}{1+1/n} (1 + 1/n) = 1 \quad (n = 1, 2, \dots). \end{aligned}$$

Hence

$$\begin{aligned} \|x + x_n\|_M^0 &\geq \langle x + x_n, z_n \rangle \\ &= \frac{1}{1+1/n} \left(\sum_{i=1}^{\infty} x(i)y(i) \right. \\ &\quad \left. - \sum_{i=m_1+\dots+m_{n-1}+1}^{m_1+\dots+m_n} (x(i)y(i) - x(i)v_n - x_n(i)y_n(i)) \right) \\ &\geq \frac{1}{1+1/n} \left(\langle x, y \rangle + \langle x_n, y_n \rangle \right. \\ &\quad \left. - \sum_{i=m_1+\dots+m_{n-1}+1}^{m_1+\dots+m_n} (|x(i)y(i)| + |x(i)v_n|) \right) \\ &\geq \frac{1}{1+1/n} \left(1 + \frac{1}{1+1/n} \right. \\ &\quad \left. - \sum_{i=m_1+\dots+m_{n-1}+1}^{m_1+\dots+m_n} (2M(x(i)) + N(y(i))) - m_n N(v_n) \right) \\ &\rightarrow 1 \quad (n \rightarrow \infty). \end{aligned}$$

But since $\text{supp } x_n \cap \text{supp } x_j = \emptyset$ ($n \neq j$), we have $\|x_n - x_j\|_M^0 \geq \|x_n\|_M^0 = 1$. This means $\{x_n\}$ is not a compact set. Thus (b) is true.

Sufficiency. Assume

$$1 = \|x\|_M^0 = \frac{1}{k}(1 + \varrho_M(kx)) = \|x_n\|_M^0 = \frac{1}{k_n}(1 + \varrho_M(k_n x_n)) \quad (n = 1, 2, \dots)$$

and $\|x_n + x\|_M^0 \rightarrow 2$. Firstly we will prove that

$$(1) \quad \lim_{j_0 \rightarrow \infty} \sup_n \sum_{j > j_0} M(k_n x_n(j)) = 0.$$

Otherwise there exist $j_n \rightarrow \infty$ and $\varepsilon > 0$ such that $\sum_{j > j_n} M(k_n x_n(j)) \geq \varepsilon$ ($n = 1, 2, \dots$). From $M \in \nabla_2$ and [1], Th. 1.35, $\bar{k} = \max\{k, \sup_n k_n\} < \infty$. Again from $M \in \nabla_2$ and [1], Th. 1.13, there exists $\delta \in (0, 1)$ such that

$$(2) \quad M(\lambda u) \leq (1 - \delta)\lambda M(u) \quad \left(\lambda \leq \frac{\bar{k}}{1 + \bar{k}}, |u| \leq \bar{k}M^{-1}(1) \right).$$

From $M \in \Delta_2$ and [1], Th. 1.40, there exists $\eta > 0$ such that for $\varrho_M(x) \leq \bar{k}$ and $\varrho_M(y) \leq \eta$,

$$(3) \quad |\varrho_M(x + y) - \varrho_M(x)| \leq \frac{\delta\varepsilon}{2(1 + \bar{k})}.$$

Since $kk_n/(k + k_n) \leq \bar{k}/2$ and $M \in \Delta_2$,

$$(4) \quad \sum_{j > j_n} M\left(\frac{kk_n}{k + k_n}x(j)\right) < \sum_{j > j_n} M\left(\frac{\bar{k}}{2}x(j)\right) < \eta$$

for n large enough. Hence for n large enough, by (2)–(4),

$$\begin{aligned} 0 &\leftarrow \|x_n\|_M^0 + \|x\|_M^0 - \|x + x_n\|_M^0 \\ &\geq \frac{1}{k_n}(1 + \varrho_M(k_n x_n)) + \frac{1}{k}(1 + \varrho_M(kx)) \\ &\quad - \frac{k + k_n}{kk_n} \left(1 + \varrho_M\left(\frac{kk_n}{k + k_n}(x + x_n)\right) \right) \\ &= \frac{k + k_n}{kk_n} \sum_{j=1}^{\infty} \left(\frac{k}{k + k_n} M(k_n x_n(j)) + \frac{k}{k + k_n} M(kx(j)) \right) \\ &\quad - M\left(\frac{kk_n}{k + k_n}(x(j) + x_n(j))\right) \\ &\geq \frac{k + k_n}{kk_n} \left(\sum_{j > j_n} \left(\frac{k}{k + k_n} M(k_n x_n(j)) + \frac{k}{k + k_n} M(kx(j)) \right) \right) \\ &\quad - \sum_{j > j_n} M\left(\frac{kk_n}{k + k_n}(x(j) + x_n(j))\right) \end{aligned}$$

$$\begin{aligned}
 &\geq \frac{k+k_n}{kk_n} \left(\sum_{j>j_n} \frac{k}{k+k_n} M(k_n x_n(j)) \right. \\
 &\quad \left. - \left(\sum_{j>j_n} M\left(\frac{kk_n}{k+k_n} x_n(j)\right) + \frac{\delta\varepsilon}{2(1+\bar{k})} \right) \right) \\
 &\geq \frac{k+k_n}{kk_n} \left(\sum_{j>j_n} \frac{k}{k+k_n} M(k_n x_n(j)) \right. \\
 &\quad \left. - \sum_{j>j_n} (1-\delta) \frac{k}{k+k_n} M(k_n x_n(j)) - \frac{\delta\varepsilon}{2(1+\bar{k})} \right) \\
 &= \frac{k+k_n}{kk_n} \left(\delta \sum_{j>j_n} \frac{k}{k+k_n} M(k_n x_n(j)) - \frac{\delta\varepsilon}{2(1+\bar{k})} \right) \\
 &\geq \frac{2}{\bar{k}} \left(\frac{\delta\varepsilon}{1+\bar{k}} - \frac{\delta\varepsilon}{2(1+\bar{k})} \right) = \frac{\delta\varepsilon}{\bar{k}(1+\bar{k})},
 \end{aligned}$$

which is a contradiction. So (1) is true.

Since $k_n > 1$, we have

$$\lim_{j_0 \rightarrow \infty} \sup_n \sum_{j>j_0} M(x_n(j)) = 0.$$

Again from $M \in \Delta_2$ and [3], Th. 1.23,

$$(5) \quad \lim_{j_0 \rightarrow \infty} \sup_n \|x_n - [x_n]_{j_0}\|_M^0 = 0.$$

For arbitrary $\varepsilon > 0$, take j_0 such that

$$\sup_n \|x_n - [x_n]_{j_0}\|_M^0 < \varepsilon/3.$$

Since $\{[x_n]_{j_0}\}_{n=1}^\infty$ is a bounded set in a j_0 -dimensional space, there exist $\{\bar{x}_i\}_{i=1}^{i_0} \subset \{x_n\}$ such that for any n , there is i , $1 \leq i \leq i_0$, satisfying $\|[x_n]_{j_0} - [\bar{x}_i]_{j_0}\|_M^0 < \varepsilon/3$. Hence

$$\|x_n - \bar{x}_i\|_M^0 \leq \|[x_n]_{j_0} - [\bar{x}_i]_{j_0}\|_M^0 + \|x_n - [x_n]_{j_0}\|_M^0 + \|\bar{x}_i - [\bar{x}_i]_{j_0}\|_M^0 < \varepsilon,$$

that is, $\{\bar{x}_i\}_{i=1}^{i_0}$ is an ε -net of $\{x_n\}$. Thus $\{x_n\}$ is a compact set.

For Orlicz function spaces, criteria for CLUR points are more complicated.

THEOREM 2. *Let $x \in S(L_M^0)$. Then x is a CLUR point iff*

(a) $M \in \Delta_2$;

(b) $M \in \nabla_2$;

(c) $\mu\{t \in T : k|x(t)| \in (a, b)\} = 0$, where $k \in [k_x^*, k_x^{**}]$ and $[a, b]$ is an arbitrary SAI of M ;

(d) $\mu\{t \in T : k|x(t)| = a' \text{ or } k|x(t)| = b'\} = 0$, where a' (resp. b') is the left (resp. right) endpoint of an arbitrary SAI of M satisfying $p_-(a') = p(a')$ (resp. $p_-(b') = p(b')$);

(e) If $\mu\{t \in T : k|x(t)| = a\} > 0$, where a is the left endpoint of some SAI of M satisfying $p_-(a) < p(a)$, then $\varrho_N(p_-(k|x|)) = 1$;

(f) If $\mu\{t \in T : k|x(t)| = b\} > 0$, where b is the right endpoint of some SAI of M satisfying $p_-(b) < p(b)$, then $\varrho_N(p(k|x|)) = 1$.

Proof. Without loss of generality, assume $x(t) \geq 0$ ($t \in T$).

Necessity. The proof of (a), (b) is similar to (a), (b) of Theorem 1.

If (c) is not true then there is an SAI $[a, b]$ of M such that $\mu\{t \in T : kx(t) \in (a, b)\} > 0$. Take $\varepsilon > 0$ so small that the set $E = \{t \in T : kx(t) \in [a + \varepsilon, b - \varepsilon]\}$ has positive measure. Divide E into two disjoint measurable subsets E_1^1 and E_2^1 such that $\mu E_1^1 = \mu E_2^1$; divide E_1^1 and E_2^1 into disjoint measurable subsets E_1^2, E_2^2 and E_3^2, E_4^2 respectively such that $\mu E_1^2 = \mu E_2^2$ and $\mu E_3^2 = \mu E_4^2, \dots$; divide E_i^{n-1} into two disjoint measurable subsets E_{2i-1}^n, E_{2i}^n such that $\mu E_{2i-1}^n = \mu E_{2i}^n$ ($i = 1, \dots, 2^{n-1}$). Let

$$x_n = x|_{T \setminus E} + (x + \varepsilon/(2k))|_{\bigcup_{i=1}^{2^{n-1}} E_{2i-1}^n} + (x - \varepsilon/(2k))|_{\bigcup_{i=1}^{2^{n-1}} E_{2i}^n}.$$

It is easy to see $\varrho_N(p_-(kx_n)) = \varrho_N(p_-(kx))$ and $\varrho_N(p(kx_n)) = \varrho_N(p(kx))$. Hence $k \in [k_{x_n}^*, k_{x_n}^{**}]$ ($n = 1, 2, \dots$). Therefore

$$\begin{aligned} \|x_n\|_M^0 &= \frac{1}{k}(1 + \varrho_M(kx_n)) \\ &= \frac{1}{k} \left(1 + \varrho_M(kx|_{T \setminus E}) + \int_{\bigcup_{i=1}^{2^{n-1}} E_{2i-1}^n} M(kx(t) + \varepsilon/2) d\mu \right. \\ &\quad \left. + \int_{\bigcup_{i=1}^{2^{n-1}} E_{2i}^n} M(kx(t) + \varepsilon/2) d\mu \right) \\ &= \frac{1}{k} \left(1 + \varrho_M(kx) + p(a) \frac{\varepsilon}{2} \cdot \frac{\mu E}{2} - p(a) \frac{\varepsilon}{2} \cdot \frac{\mu E}{2} \right) = \|x\|_M^0 = 1. \end{aligned}$$

Obviously $k \in [k_{(x+x_n)/2}^*, k_{(x+x_n)/2}^{**}]$ ($n = 1, 2, \dots$). We have

$$\left\| \frac{x + x_n}{2} \right\|_M^0 = \frac{1}{k} \left(1 + \varrho_M \left(k \frac{x + x_n}{2} \right) \right) = \|x\|_M^0 = 1.$$

That is, $\|x + x_n\|_M^0 = 2$. But for $m \neq n$,

$$\|x_n - x_m\|_M^0 = \frac{\varepsilon}{2k} \cdot \frac{\mu E}{2} N^{-1} \left(\frac{2}{\mu E} \right)$$

(see [1], Ex. 1.22). Thus $\{x_n\}$ is not a compact set.

If (d) is not true then there is a SAI $[a', b']$ of M with $p_-(a') = p(a')$ and $E = \{t \in T : kx(t) = a'\}$ of positive measure. Take c' with $a' < c' < b'$. Divide E as above and put

$$x_n = x|_{T \setminus E} + \frac{a'}{k} \Big|_{\bigcup_{i=1}^{2^{n-1}} E_{2i-1}^n} + \frac{c'}{k} \Big|_{\bigcup_{i=1}^{2^{n-1}} E_{2i}^n} \quad (n = 1, 2, \dots).$$

We have $\varrho_N(p_-(kx_n)) = \varrho_N(p_-(kx))$ and $\varrho_N(p(kx_n)) = \varrho_N(p(kx))$. So $k \in [k_{x_n}^*, k_{x_n}^{**}]$ ($n = 1, 2, \dots$). Since $M \in \Delta_2$ there is $y \in S(I_N)$ such that $\langle x, y \rangle = \|x\|_M^0 = 1$. From [1], Th. 1.80, we have $\varrho_N(y) = 1$ and $p_-(kx(t)) \leq y(t) \leq p(kx(t))$ (a.e. $t \in T$). Now $p_-(kx_n(t)) \leq y(t) \leq p(kx_n(t))$ ($n = 1, 2, \dots$) too. Again by [1], Th. 1.80, y is a supporting functional of x_n ($n = 1, 2, \dots$). Thus

$$2 \geq \left\| x + \frac{x_n}{\|x_n\|_M^0} \right\|_M^0 \geq \int_T \left(x(t) + \frac{x_n(t)}{\|x_n\|_M^0} \right) y(t) d\mu = 2 \quad (n = 1, 2, \dots),$$

i.e. $\|x + x_n/\|x_n\|_M^0\|_M^0 = 2$. Clearly $\|x_n\|_M^0 = A < \infty$ ($n = 1, 2, \dots$) and for $m \neq n$,

$$\left\| \frac{x_n}{\|x_n\|_M^0} - \frac{x_m}{\|x_m\|_M^0} \right\|_M^0 = \frac{c' - a'}{A} \cdot \frac{\mu E}{2} N^{-1} \left(\frac{2}{\mu E} \right).$$

Hence $\{x_n/\|x_n\|_M^0\}$ is not a compact set.

Similarly we can prove that $\mu\{t \in T : kx(t) = b'\} = 0$.

If (e) is not true then there is a SAI $[a, b]$ of M with $p_-(a) < p(a)$, $E_0 = \{t \in T : kx(t) = a\}$ of positive measure and $\varrho_N(p_-(kx)) < 1$ (notice that we always have $\varrho_N(p_-(kx)) \leq 1$ by the definition of k). Take $E \subset E_0$ with $\mu E > 0$ such that

$$\begin{aligned} \int_{T \setminus E} N(p_-(kx(t))) d\mu + \int_E N(p(kx(t))) d\mu \\ = \int_{T \setminus E} N(p_-(kx(t))) d\mu + N(p(a))\mu E \leq 1. \end{aligned}$$

Divide E as above. Take c with $a < c < b$ and put

$$x_n = x|_{T \setminus E} + \frac{a}{k} \Big|_{\bigcup_{i=1}^{2^{n-1}} E_{2i-1}^n} + \frac{c}{k} \Big|_{\bigcup_{i=1}^{2^{n-1}} E_{2i}^n} \quad (n = 1, 2, \dots).$$

Then

$$\varrho_N(p_-(kx_n)) = \int_{T \setminus E} N(p_-(kx(t))) d\mu + N(p(a))\mu E \leq 1$$

and

$$\varrho_N(p(kx_n)) = \varrho_N(p(kx)) \geq 1.$$

So $k \in [k_{x_n}^*, k_{x_n}^{**}]$ ($n = 1, 2, \dots$). Since

$$\int_{T \setminus E} N(p_-(kx(t))) d\mu + N(p(a))\mu E \leq 1,$$

$$\int_{T \setminus E} N(p(kx(t))) d\mu + N(p(a))\mu E = \varrho_N(p(kx(t))) \geq 1,$$

there is $y \in L_N$ such that

$$y(t) = p(a) \quad (t \in E), \quad p_-(kx(t)) \leq y(t) \leq p(kx(t)) \quad (t \in T \setminus E)$$

and $\varrho_N(y) = 1$. By [1], Th. 1.80, y is a supporting functional of x . Obviously $p_-(kx_n(t)) \leq y(t) \leq p(kx_n(t))$ (a.e. $t \in T$, $n = 1, 2, \dots$), so y is a supporting functional of x_n ($n = 1, 2, \dots$) too. Thus

$$\left\| x + \frac{x_n}{\|x_n\|_M^0} \right\|_M^0 \geq \int_T \left(x(t) + \frac{x_n(t)}{\|x_n\|_M^0} \right) y(t) d\mu = 2 \quad (n = 1, 2, \dots).$$

Assume $\|x_n\|_M^0 = A$ ($n = 1, 2, \dots$). Then for $m \neq n$,

$$\left\| \frac{x_n}{\|x_n\|_M^0} - \frac{x_m}{\|x_m\|_M^0} \right\|_M^0 = \frac{c-a}{A} \cdot \frac{\mu E}{2} N^{-1} \left(\frac{2}{\mu E} \right),$$

which means $\{x_n/\|x_n\|_M^0\}$ is not a compact set.

The proof of (f) is similar to (e).

Sufficiency. By [6], x is a LUR point. Then of course it is a CLUR point.

Finally, we discuss a criterion for CLUR points in Orlicz function spaces with Luxemburg norm.

THEOREM 3. *Let $x \in S(L_M)$. Then x is a CLUR point iff*

- (a) $M \in \Delta_2$;
- (b) $\mu\{t \in T : |x(t)| \in (a, b)\} = 0$, where $[a, b]$ is an arbitrary SAI of M ;
- (c) $\mu\{t \in T : |x(t)| = b\} = 0$ for the right endpoint b of any SAI of M , or $\mu\{t \in T : |x(t)| = a\} = 0$ for the left endpoint a of any SAI of M , and $M \in \nabla_2$.

Proof. Still assume $x(t) \geq 0$ ($t \in T$).

Sufficiency. By [5], x is a LUR point. Then certainly it is a CLUR point.

Necessity. The proofs of (a) and (b) are similar to (a) and (c) in Theorem 2, respectively.

If (c) is not true, we consider the following two cases:

I. Both $E = \{t \in T : x(t) = b\}$ and $F = \{t \in T : x(t) = c\}$ have positive measure, where (a, b) and (c, d) are two SAI of M . Take $s \in (a, b)$ and $r \in (c, d)$ satisfying $(M(b) - M(s))\mu E = (M(r) - M(c))\mu F$. Divide E

and F as above and put

$$x_n = x|_{T \setminus (E \cup F)} + b|_{\bigcup_{i=1}^{2^{n-1}} E_{2i-1}^n} + s|_{\bigcup_{i=1}^{2^{n-1}} E_{2i}^n} \\ + r|_{\bigcup_{i=1}^{2^{n-1}} F_{2i-1}^n} + c|_{\bigcup_{i=1}^{2^{n-1}} F_{2i}^n} \quad (n = 1, 2, \dots).$$

We have

$$\varrho_M(x_n) = \varrho_M(x|_{T \setminus (E \cup F)}) + (M(b) + M(s))\frac{\mu E}{2} + (M(r) + M(c))\frac{\mu F}{2} \\ = \varrho_M(x|_{T \setminus (E \cup F)}) + M(b)\mu E + M(c)\mu F = \varrho_M(x) = 1.$$

So $\|x_n\|_M = 1$ ($n = 1, 2, \dots$). Moreover

$$\varrho_M\left(\frac{x_n + x}{2}\right) = \varrho_M(x|_{T \setminus (E \cup F)}) + M\left(\frac{b+s}{2}\right)\mu E + M\left(\frac{r+c}{2}\right)\mu F \\ = \varrho_M(x|_{T \setminus (E \cup F)}) + \frac{M(b) + M(s)}{2}\mu E + \frac{M(r) + M(c)}{2}\mu F = 1.$$

So $\|x + x_n\|_M = 2$ ($n = 1, 2, \dots$). But for $m \neq n$,

$$\varrho_M(x_n - x_m) = M(c - a)\frac{\mu E}{2} + M(b - d)\frac{\mu F}{2}.$$

Therefore $\{x_n\}$ is not a compact set.

II. $E = \{t \in T : x(t) = b\}$ has positive measure, where $[a, b]$ is a SAI of M and $M \notin \nabla_2$.

By [1], Th. 1.13, there exist $u_n \rightarrow \infty$ such that

$$M\left(\frac{u_n}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{M(u)}{2} \quad (n = 1, 2, \dots).$$

Since $\varrho_M(x|_{T \setminus E}) + M(b)\mu E = \varrho_M(x) = 1$, we can take $E_n \subset E$ such that

$$\varrho_M(x|_{T \setminus E}) + M(u_n - b)\mu E_n + M(a)\mu(E \setminus E_n) = 1 \quad (n = 1, 2, \dots).$$

It is easy to see that $\mu E_n \rightarrow 0$ ($n \rightarrow \infty$). Without loss of generality we can assume that $E_n \cap E_m = \emptyset$ ($m \neq n$). Put

$$x_n = x|_{T \setminus E} + a|_{E \setminus E_n} + (u_n - b)|_{E_n} \quad (n = 1, 2, \dots).$$

Then $\varrho_M(x_n) = 1$ and

$$\varrho_M\left(\frac{x_n + x}{2}\right) = \varrho_M(x|_{T \setminus E}) + M\left(\frac{b+a}{2}\right)\mu(E \setminus E_n) + M\left(\frac{u_n}{2}\right)\mu E_n \\ \geq \varrho_M(x|_{T \setminus E}) + \frac{M(b) + M(a)}{2}\mu(E \setminus E_n) + \left(1 - \frac{1}{n}\right)\frac{M(u_n)}{2}\mu E_n \\ = \frac{1}{2}\left[\varrho_M(x|_{T \setminus E}) + M(b)\mu(E \setminus E_n) \right. \\ \left. + \varrho_M(x|_{T \setminus E}) + M(a)\mu(E \setminus E_n) + \left(1 - \frac{1}{n}\right)M(u_n)\mu E_n\right] \\ \rightarrow 1 \quad (n \rightarrow \infty).$$

Hence $\|x_n\|_M = 1$ ($n = 1, 2, \dots$), $\|x_n + x\|_M \rightarrow 2$ ($n \rightarrow \infty$). But for any n, m , $m \neq n$,

$$\begin{aligned} \varrho_M(x_n - x_m) &\geq \varrho_M((x_n - x_m)|_{E_n}) \\ &= M(u_n - b - a)\mu E_n = M(u_n - b)\mu E_n - o(1/n) \\ &= (1 - \varrho_M(x|_{T \setminus E})) - M(a)\mu(E \setminus E_n) - o(1/n) \\ &= M(b)\mu E - M(a)\mu(E \setminus E_n) - o(1/n) \\ &\geq (M(b) - M(a))\mu E - o(1/n) \\ &\geq \frac{1}{2}(M(b) - M(a))\mu E \end{aligned}$$

for n large enough. This means $\{x_n\}$ is not a compact set.

Combining Theorems 1–3 of this paper and Theorem 6 of [2] we see that being a CLUR point is equivalent to being a LUR point in Orlicz function spaces, while in Orlicz sequence spaces a CLUR point does not even have to be an extreme point. Furthermore we can get

COROLLARY 1. L_M^0 (and L_M) is CLUR iff it is LUR.

COROLLARY 2. I_M^0 is CLUR iff it is reflexive.

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