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The Conley index for decompositions of isolated invariant sets

by

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Abstract. Let f be a continuous map of a locally compact metric space X into itself. Suppose that S is an isolated invariant set for f and a disjoint union of a fixed finite number of compact sets. We define an index of Conley type for isolated invariant sets admitting such a decomposition and prove some of its properties, which appear to be similar to that of the ordinary Conley index for maps. Our index takes into account the existence of the decomposition of S and therefore carries more information about the structure of the invariant set. In particular, it seems to be a more accurate tool for the detection of periodic trajectories and chaos of the Smale horseshoe type than the ordinary Conley index.

0. Introduction. The Conley index has become an important tool in the study of the qualitative behavior of dynamical systems, with both discrete and continuous time. The results concerning attractor-repeller decompositions ([1], [15], [18]), the connection matrix theory ([3], [4], [5]) as well as recent papers by Ch. McCord, K. Mischaikow and M. Mrozek [7] and the last two authors [9] (see also [20]) show that the Conley index reflects the structure of an isolated invariant set. In this paper we are mainly interested in the Conley index as a tool for the detection of chaos and periodic orbits. Comparing the results of [9] and [20] with the criterions for chaos based on the fixed point index in [19] or [23] shows that the ones based on the Conley index are, in some sense, weak. They only guarantee that some iteration of the map restricted to the isolated invariant set is semiconjugate to the shift map. Thus, they provide information about the dynamics of some iteration of the map rather than the map itself. The information about the number of periodic orbits is also not as accurate as that provided by the methods based on the fixed point index. The aim of this paper is to define an index of Conley type which fills this gap. Our index is defined for a decomposition

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of an isolated invariant set into a fixed number of disjoint compact sets. The knowledge of the decomposition allows us to equip the index with an additional structure, which carries more information than the ordinary Conley index. The main potential application of our index is for the detection of chaos. The Conley index for decompositions can also be used to state topological analogues of some results in the theory of smooth dynamical systems (e. g. the Poincaré–Birkhoff theorem). This topic will be discussed in a separate paper.

1. Preliminaries. We denote by $\mathbb{Z}^+$ and $\mathbb{R}$ the sets of nonnegative integer and real numbers (respectively). If X is a metric space and $Q = (Q_1, Q_0)$ is a pair of its compact subsets then by Q_1/Q_0 we denote the pointed space resulting from Q_1 when the points of Q_0 are identified to a single distinguished point, denoted by $[Q_0]$. $\mathcal{H}top$, $\mathcal{M}$ and $\mathcal{M}_{\mathrm{G}}$ stand for the homotopy category of pointed topological spaces, the category of modules and the category of graded modules over a fixed ring Ξ with unity. For a basepoint preserving map g its homotopy class is also denoted by g. This should not cause misunderstanding. For an object O in a category $\mathcal{K}$ we denote by $[O]$ the class of all objects in $\mathcal{K}$ isomorphic to O. A functor $F : \mathcal{K} \to \mathcal{L}$ induces the map sending an isomorphism class $[O]$ into $[F(O)]$ for each object $O \in \mathrm{Ob}(\mathcal{K})$. We denote this map by the same letter F.

Let us now recall the basic concepts of the Conley index theory. Our presentation is based mainly on [21] and [20] (see also [11], [13], [16], [17]). We begin with the definition of the category of objects equipped with a morphism over a given category $\mathcal{K}$, denoted by $\mathcal{K}_{\mathrm{m}}$. Put

$$\mathrm{Ob}(\mathcal{K}_{\mathrm{m}}) = \{(X, \alpha) : X \in \mathrm{Ob}(\mathcal{K}) \text{ and } \alpha \in \mathrm{Mor}_{\mathcal{K}}(X, X)\}$$

and

$$\mathrm{Mor}_{\mathcal{K}_{\mathrm{m}}}((X, \alpha), (X', \alpha')) = M((X, \alpha), (X', \alpha'))/\!\equiv,$$

where

$$M((X, \alpha), (X', \alpha')) = \{\beta \in \mathrm{Mor}_{\mathcal{K}}(X, X') : \beta \circ \alpha = \alpha' \circ \beta\} \times \mathbb{Z}^+$$

and $\equiv$ is the equivalence relation in the above set defined by

$$(\beta, n) \equiv (\overline{\beta}, \overline{n}) \Leftrightarrow \exists_{k \in \mathbb{Z}^+}\ \beta \circ \alpha^{\overline{n}+k} = \overline{\beta} \circ \alpha^{n+k}.$$

The morphism represented by $(\beta, n) \in M((X, \alpha), (X', \alpha'))$ will be denoted by $[\beta, n]$. The composition of morphisms in $\mathcal{K}_{\mathrm{m}}$ is defined by

$$[\beta', n'] \circ [\beta, n] = [\beta' \circ \beta, n' + n].$$

Given a functor $F : \mathcal{K} \to \mathcal{L}$ one can define the induced functor $F_{\mathrm{m}} : \mathcal{K}_{\mathrm{m}} \to \mathcal{L}_{\mathrm{m}}$ in the following way. For an object (X, α) and a morphism $[\beta, n]$ in $\mathcal{K}_{\mathrm{m}}$ we put

$$F_{\mathrm{m}}(X, \alpha) = (F(X), F(\alpha)), \qquad F_{\mathrm{m}}([\beta, n]) = [F(\beta), n].$$

We write $[X, \alpha]$ for the class of all objects in $\mathcal{K}_{\mathrm{m}}$ isomorphic to an object (X, α).

In each of the categories $\mathcal{H}top$, $\mathcal{M}$, $\mathcal{M}_{\mathrm{G}}$ for each object X there exists the zero morphism of X into itself (i.e. the homotopy class of the constant map or the zero homomorphism, according to the case). We denote this morphism by 0. In the same way we denote the trivial isomorphism classes in the categories of objects equipped with a morphism over each of the three categories, i.e. we put $0 = [C, 0]$ where C is any pointed space or (graded) $\varXi$-module. This class is independent of the choice of C. This notation is ambiguous, but it will always be clear from the context what is meant by 0. We note that $[X, \alpha] = 0$ if and only if $\alpha^n = 0$ for some $n \in \mathbb{Z}^+$.

For the rest of this section, fix a locally compact metric space X and a continuous map f of X into itself. Let S be an isolated invariant set with respect to f. A pair $Q = (Q_1, Q_0)$ of compact subsets of X is called an *index pair* for S with respect to f if and only if $S = \mathrm{Inv}_f \mathrm{cl}(Q_1 \setminus Q_0) \subset \mathrm{int}(Q_1 \setminus Q_0)$, Q_0 is *positively invariant* in Q_1 (i.e. $f(Q_0) \cap Q_1 \subset Q_0$) and Q_0 is an *exit set* for Q_1 (which means that $f(Q_1 \setminus Q_0) \subset Q_1$). For such Q, f induces a continuous map $f_Q : Q_1/Q_0 \to Q_1/Q_0$ which will be called the *index map*. The (*homotopy*) *Conley index* of S, denoted by $h(S, f, X)$, is defined as the class of all objects in $\mathcal{H}top_{\mathrm{m}}$ isomorphic to $(Q_1/Q_0, f_Q)$. We define the *cohomological* and the *q-dimensional cohomological Conley indices* by

$$h^*(S, f, X) = (H^*)_{\mathrm{m}}(h(S, f, X))$$

and

$$h^q(S, f, X) = (H^q)_{\mathrm{m}}(h(S, f, X)),$$

where $H^* : \mathcal{H}top \to \mathcal{M}_{\mathrm{G}}$ is a fixed cohomology functor with coefficients in $\varXi$.

Until the end of this section, let $\varXi$ be the field of rational numbers. Then $\mathcal{M}$ and $\mathcal{M}_{\mathrm{G}}$ are the categories of vector spaces and graded vector spaces over $\varXi$. An object (V, φ) in $\mathcal{M}_{\mathrm{m}}$ is called of *finite asymptotic dimension* (cf. [20], Definition 2.1 and Proposition 2.2) if there exists an object (W, ψ) isomorphic to (V, φ) with W finite-dimensional. In this case, we define the *trace* of (V, φ), denoted by $\mathrm{tr}(V, \varphi)$ as the ordinary trace of ψ. Using the methods of [20] (see Theorem 1.1, Definition 4.1 and Remark 4.1) one proves easily that it is independent of the choice of (W, ψ).

Now, let (V^*, φ^*) be an object in $(\mathcal{M}_{\mathrm{G}})_{\mathrm{m}}$. It is said to be of *finite type* if there exists an object (W^*, ψ^*) isomorphic to (V^*, φ^*) with W^* of finite type. In this case, we define the *Lefschetz number* of (V^*, φ^*), denoted by $\varLambda(V^*, \varphi^*)$, as the ordinary Lefschetz number of ψ^*. Clearly,

$$\varLambda(V^*, \varphi^*) = \sum_{q=-\infty}^{\infty} (-1)^q \, \mathrm{tr}(V^q, \varphi^q).$$

An isomorphism class I of objects in $(\mathcal{M}_{\mathrm{G}})_{\mathrm{m}}$ is said to be of *finite type* if it admits a representative of finite type. In this case, all its representatives are of finite type and they have the same Lefschetz number, which we call the *Lefschetz number of* I, and denote by $\Lambda(I)$. The following theorem is taken from [20] (see Lemma 5.2). For related results, see [10], [12], [14].

THEOREM 1.1. *If X is a Euclidean neighborhood retract* (*ENR*), *then the cohomological Conley index of any isolated invariant set for f is of finite type. If the Lefschetz number of the Conley index of an isolated invariant set S is nonzero then f has a fixed point in S.*

2. Categorical constructions. There are many Conley-type indices for isolated invariant sets (see [1], [11], [13], [17], [18], [21]), but all of them take the form of an isomorphism class of objects in a certain category. In the classical, continuous-time case, the homotopy category of pointed topological spaces is used and therefore the Conley index is simply the homotopy class of a pointed space. In the discrete-time case the situation is much more complicated: in order to give a good definition one has to use more sophisticated categorical constructions. The shape category ([13], [17]), the Leray functor ([11], [13]), the direct and inverse limit functors ([13]) and the category of objects equipped with a morphism ([21]) can serve as examples here. Below we define a generalization of the latter concept, which is suitable for the definition of the Conley index for decompositions of isolated invariant sets.

Let $\mathcal{K}$ be a category and A a finite set. In the sequel, we often deal with finite sequences of elements of A. We denote by $*$ the concatenation of such sequences, defined by

$$(Z_1,\ldots,Z_n)*(Z_1',\ldots,Z_m')=(Z_1,\ldots,Z_n,Z_1',\ldots,Z_m')\in A^{n+m}$$

for all $(Z_1,\ldots,Z_n)\in A^n$ and $(Z_1',\ldots,Z_m')\in A^m$. For $\overline{Z}$ being a sequence of members of A we denote by $\overline{Z}^k$ the concatenation of k copies of $\overline{Z}$. By $\iota(\overline{Z})$ we denote the sequence $\overline{Z}$ with entries in reverse order.

Let us now define the category $\mathcal{K}_{(A)}$. For $n\in\mathbb{Z}^+$ and $X,X'\in\mathrm{Ob}(\mathcal{K})$ put

$$\begin{gathered}\mathrm{Ob}(\mathcal{K}_{(A)})=\mathrm{Ob}(\mathcal{K}),\\ \mathrm{Mor}^n_{\mathcal{K}_{(A)}}(X,X')=(\mathrm{Mor}_{\mathcal{K}}(X,X'))^{A^n},\\ \mathrm{Mor}_{\mathcal{K}_{(A)}}(X,X')=\bigcup_{n\in\mathbb{Z}^+}\mathrm{Mor}^n_{\mathcal{K}_{(A)}}(X,X').\end{gathered}$$

The composition of morphisms

$$\alpha=\{\alpha^{\bar{Z}}\}_{\bar{Z}\in A^n}\in\mathrm{Mor}^n_{\mathcal{K}_{(A)}}(X,X')$$

and

$$\alpha' = \{\alpha'^{\bar{Z}}\}_{\bar{Z}\in A^m} \in \mathrm{Mor}^m_{\mathcal{K}_{(A)}}(X', X'')$$

is defined as follows:

$$\alpha' \circ \alpha = \{(\alpha' \circ \alpha)^{\bar{Z}}\}_{\bar{Z}\in A^{m+n}} \in \mathrm{Mor}^{m+n}_{\mathcal{K}_{(A)}}(X, X''),$$

where

$$(\alpha' \circ \alpha)^{\bar{Y}' * \bar{Y}} = \alpha'^{\bar{Y}'} \circ \alpha^{\bar{Y}}$$

for all $\overline{Y} \in A^n$ and $\overline{Y}' \in A^m$. Since A^0 consists of exactly one element, we identify $\mathrm{Mor}^0_{\mathcal{K}_{(A)}}(X, X')$ with $\mathrm{Mor}_{\mathcal{K}}(X, X')$ in the obvious way. Similarly, there is an obvious bijection of $\mathrm{Mor}^1_{\mathcal{K}}(X, X')$ onto $(\mathrm{Mor}_{\mathcal{K}}(X, X'))^A$. Therefore, we treat morphisms in the former set as families of morphisms of X into X' in $\mathcal{K}$, indexed by members of A. It is straightforward to verify that $\mathcal{K}_{(A)}$ is indeed a category. Notice that its identity morphism over an object X is $\mathrm{id}_X \in \mathrm{Mor}^0_{\mathcal{K}_{(A)}}(X, X)$.

The category $\mathcal{K}_{(A)}$ is only an intermediate step in the definition of the category $\mathcal{K}_{[A]}$, which we are going to use in the definition of the Conley index for decompositions of isolated invariant sets. Put

$$\mathrm{Ob}(\mathcal{K}_{[A]}) = \{(X, \alpha) : X \in \mathrm{Ob}(\mathcal{K}_{(A)}) = \mathrm{Ob}(\mathcal{K}),\ \alpha \in \mathrm{Mor}^1_{\mathcal{K}_{(A)}}(X, X)\}.$$

In order to define morphisms in $\mathcal{K}_{[A]}$, for objects (X, α) and (X', α') put

$$M((X, \alpha), (X', \alpha')) = \{(\beta, n) : \beta \in \mathrm{Mor}^n_{\mathcal{K}_{(A)}}(X, X'),\ n \in \mathbb{Z}^+,\ \beta\circ\alpha = \alpha'\circ\beta\}.$$

In this set we introduce an equivalence relation $\equiv$ in the following way.

$$(\beta, n) \equiv (\overline{\beta}, \overline{n}) \Leftrightarrow \exists_{k\in\mathbb{Z}^+}\ \beta \circ \alpha^{\bar{n}+k} = \overline{\beta} \circ \alpha^{n+k} \text{ (in } \mathcal{K}_{(A)}).$$

Now, define

$$\mathrm{Mor}_{\mathcal{K}_{[A]}}((X, \alpha), (X', \alpha')) = M((X, \alpha), (X', \alpha'))/\equiv.$$

The morphism represented by (β, n) will be denoted by $[\beta, n]$. The composition of morphisms $[\beta, n] : (X, \alpha) \to (X', \alpha')$ and $[\beta', n'] : (X', \alpha') \to (X'', \alpha'')$ is defined as follows:

$$[\beta', n'] \circ [\beta, n] = [\beta' \circ \beta, n' + n].$$

One can easily verify that this definition is correct, i.e. independent of the choice of the representatives for $[\beta, n]$ and $[\beta', n']$ and that $\mathcal{K}_{[A]}$ is indeed a category. Note that the identity morphism over (X, α) is $[\mathrm{id}_X, 0]$.

PROPOSITION 2.1. *For each* $[\beta, n] \in \mathrm{Mor}_{\mathcal{K}_{[A]}}((X, \alpha), (X', \alpha'))$ *and* $k \in \mathbb{Z}^+$,

$$[\beta, n] = [\alpha'^k \circ \beta, n + k] = [\beta \circ \alpha^k, n + k].$$

Proof. This follows immediately from the definition of $\equiv$. ∎

The Conley index for decompositions of isolated invariant sets will "contain" information about ordinary Conley indices of some sets which are important for understanding the dynamics of the map. Below we give the definition of functors which will enable us to extract this information.

Let k be a positive integer and $\overline{Y} = (Y_1, \ldots, Y_k) \in A^k$. The functor $\mathcal{P}_{\bar{Y}} : \mathcal{K}_{[A]} \to \mathcal{K}_{\mathrm{m}}$ is defined as follows. For an object (X, α) in $\mathcal{K}_{[A]}$ with $\alpha = \{\alpha^Z\}_{Z \in A}$ put

$$\mathcal{P}_{\bar{Y}}(X, \alpha) = (X, \alpha^{Y_1} \circ \ldots \circ \alpha^{Y_k}).$$

Now, let $[\beta, n]$ be a morphism of (X, α) into (X', α'). By Proposition 2.1, without loss of generality we can assume that $n = mk$ for some $m \in \mathbb{Z}^+$. Suppose that $\beta = \{\beta^{\bar{Z}}\}_{\bar{Z} \in A^n}$. Put

$$\mathcal{P}_{\bar{Y}}([\beta, n]) = [\beta^{\bar{Y}^m}, m].$$

A routine check that $\mathcal{P}_{\bar{Y}}$ is a well-defined functor is left to the reader.

Remark 2.1. An important property of the construction given above is the naturality with respect to functors. Let $F : \mathcal{K} \to \mathcal{L}$ be a functor. Then we have the induced functors $F_{(A)} : \mathcal{K}_{(A)} \to \mathcal{L}_{(A)}$ and $F_{[A]} : \mathcal{K}_{[A]} \to \mathcal{L}_{[A]}$ defined as follows:

$$F_{(A)}(X) = F(X),$$
$$F_{(A)}(\{\alpha^{\bar{Z}}\}_{\bar{Z} \in A^k}) = \begin{cases} \{F(\alpha^{\bar{Z}})\}_{\bar{Z} \in A^k} & \text{if } F \text{ is covariant,} \\ \{F(\alpha^{\iota(\bar{Z})})\}_{\bar{Z} \in A^k} & \text{if } F \text{ is contravariant,} \end{cases}$$

for all objects X and morphisms $\{\alpha^{\bar{Z}}\}_{\bar{Z} \in A^k}$ in $\mathcal{K}_{(A)}$ and

$$F_{[A]}(X, \alpha) = (F_{(A)}(X), F_{(A)}(\alpha)), \qquad F_{[A]}([\beta, n]) = [F_{(A)}(\beta), n]$$

for all objects (X, α) and morphisms $[\beta, n]$ in $\mathcal{K}_{[A]}$. Furthermore, the following diagram of categories and functors commutes for each $\overline{Y} \in A^k$:

$$\begin{tikzcd} \mathcal{K}_{[A]} \arrow[r, "F_{[A]}"] \arrow[d, "\mathcal{P}_{\bar{Y}}"] & \mathcal{L}_{[A]} \arrow[d, "\mathcal{P}_{\iota_F(\bar{Y})}"] \\ \mathcal{K}_{\mathrm{m}} \arrow[r, "F_{\mathrm{m}}"] & \mathcal{L}_{\mathrm{m}} \end{tikzcd}$$

where

$$\iota_F(\overline{Y}) = \begin{cases} \iota(\overline{Y}) & \text{if } F \text{ is contravariant,} \\ \overline{Y} & \text{if } F \text{ is covariant.} \end{cases}$$

As in the case of categories equipped with a morphism, we denote by 0 the isomorphism classes of the trivial (zero) objects in $\mathcal{H}top_{[A]}$, $(\mathcal{M}_{\mathrm{G}})_{[A]}$ and $\mathcal{M}_{[A]}$, defined in the obvious way.

3. The index. This section contains the basic definitions of the Conley index theory for decompositions of isolated invariant sets. In what follows, B, X and f will denote a fixed finite set, a locally compact metric space and a continuous map of X into itself. We shall make use of the categories of $\mathcal{K}_{[A]}$ type with $A = 2^B$. If $\{N_b\}$ is a family of sets indexed by members of B then for each set $Z \subset B$ we denote by N_Z the union of N_b over $b \in Z$.

DEFINITION 3.1. Let N be a compact subset of X. A family $\{N_b\} = \{N_b\}_{b\in B}$ of pairwise disjoint compact sets is called a *decomposition* of N if $N = \bigcup_{b\in B} N_b$.

Until the end of this section, we denote by S a fixed isolated invariant set for f and by $\{S_b\}$ its decomposition.

DEFINITION 3.2. An index pair $Q = (Q_1, Q_0)$ for S is said to be *compatible with the decomposition* $\{S_b\}$ of S if there exists a decomposition $\{D_b\}$ of $\mathrm{cl}(Q_1 \setminus Q_0)$ such that $S_b = S \cap D_b$ for each $b \in B$.

Let us emphasize that, in general, the decomposition $\{D_b\}$ is not uniquely determined by Q and $\{S_b\}$.

DEFINITION 3.3. Let $Q = (Q_1, Q_0)$ be an index pair for S compatible with the decomposition $\{S_b\}$ of S and $\{D_b\}$ be the corresponding decomposition of $\mathrm{cl}(Q_1 \setminus Q_0)$. Then for any $Z \in A$ we can define a continuous map

$$r^Z = r^Z_{(Q,\{D_b\})} : Q_1/Q_0 \to Q_1/Q_0$$

by the formula

$$r^Z([x]) = \begin{cases} [x] & \text{if } x \in D_Z, \\ [Q_0] & \text{otherwise.} \end{cases}$$

The *index object*, denoted by $I(Q, \{D_b\}, f)$, is the object in $\mathcal{H}top_{[A]}$ given by

$$I(Q, \{D_b\}, f) = (Q_1/Q_0, \{f^Z\}_{Z\in A}),$$

where $f^Z = f^Z_{(Q,\{D_b\})} = f_Q \circ r^Z$ (recall that f_Q is the index map).

In order to simplify the notation, we often write briefly $I(Q, \{D_b\})$ instead of $I(Q, \{D_b\}, f)$ whenever the map f is clear from the context. For each $Z \in A$ and $x \in Q_1$ we have the formula

$$f^Z([x]) = \begin{cases} [f(x)] & \text{if } x \in D_Z \cap (Q_1 \setminus Q_0), \\ [Q_0] & \text{otherwise.} \end{cases}$$

For further reference, let us note the following formula for compositions of the maps f^Z. Let $\overline{Z} = (Z_0, Z_1, \ldots, Z_{T-1}) \in A^T$. For all $x \in Q_1$,

$$f^{Z_{T-1}} \circ f^{Z_{T-2}} \circ \ldots \circ f^{Z_0}([x]) = \begin{cases} [f^T(x)] & \text{if } f^i(x) \in D_{Z_i} \cap (Q_1 \setminus Q_0) \text{ for each } i \in \{0, 1, \ldots, T-1\}, \\ [Q_0] & \text{otherwise.} \end{cases} \tag{3.1}$$

Let N be a compact neighborhood of S admitting a decomposition $\{N_b\}$ such that $S_b = N_b \cap S$ for each $b \in B$. By the existence theorems for index pairs (see [6], [10], [11], [13], [16]) there is an index pair $Q = (Q_1, Q_0)$ for S with Q_1 contained in N (Q may even be assumed regular in the sense of [10] or [20]). Obviously, such an index pair is compatible with the decomposition $\{S_b\}$. We have proved the following

Proposition 3.1. *There exist index pairs for S compatible with the decomposition $\{S_b\}$, arbitrarily close to S.*

The following theorem is of fundamental importance in our construction.

Theorem 3.1. *If $Q = (Q_1, Q_0)$ and $\overline{Q} = (\overline{Q}_1, \overline{Q}_0)$ are index pairs for S compatible with the decomposition $\{S_b\}$ of S, and $\{D_b\}$ and $\{\overline{D}_b\}$ are decompositions of $\mathrm{cl}(Q_1 \setminus Q_0)$ and $\mathrm{cl}(\overline{Q}_1 \setminus \overline{Q}_0)$ satisfying the conditions of Definition 3.2 then the index objects $I(Q, \{D_b\})$ and $I(\overline{Q}, \{\overline{D}_b\})$ are isomorphic in $\mathcal{H}top_{[A]}$.*

Proof. We proceed in several steps.

Step 1. There exists $T \in \mathbb{Z}^+$ such that for each sequence $(Z_0, Z_1, \ldots, Z_{2T-1})$ of members of A and $x \in X$,

$$(\forall_{i \in \{0,\ldots,2T-1\}}\ f^i(x) \in \overline{D}_{Z_i}) \Rightarrow f^T(x) \in D_{Z_T} \cap (Q_1 \setminus Q_0) \tag{3.2}$$

and

$$(\forall_{i \in \{0,\ldots,2T-1\}}\ f^i(x) \in D_{Z_i}) \Rightarrow f^T(x) \in \overline{D}_{Z_T} \cap (\overline{Q}_1 \setminus \overline{Q}_0). \tag{3.3}$$

For the proof, notice that the set U given by

$$U = \bigcup_{b \in B} (D_b \cap \overline{D}_b \cap (Q_1 \setminus Q_0) \cap (\overline{Q}_1 \setminus \overline{Q}_0))$$

is a neighborhood of S. As a consequence of Lemma 4.2 of [21] (cf. also Lemma 6.2 of [17]) we obtain the existence of a nonnegative integer T such that for each $x \in X$,

$$\begin{aligned} &(\forall_{i \in \{0,\ldots,2T-1\}}\ f^i(x) \in \mathrm{cl}(Q_1 \setminus Q_0)) \Rightarrow f^T(x) \in U, \\ &(\forall_{i \in \{0,\ldots,2T-1\}}\ f^i(x) \in \mathrm{cl}(\overline{Q}_1 \setminus \overline{Q}_0)) \Rightarrow f^T(x) \in U. \end{aligned} \tag{3.4}$$

Now, suppose that $\forall_{i \in \{0,\ldots,2T-1\}}\ f^i(x) \in D_{Z_i}$. Then, by (3.4), $f^T(x) \in U$. Since simultaneously $f^T(x) \in D_{Z_T}$,

$$f^T(x) \in U \cap D_{Z_T} \subset \overline{D}_{Z_T} \cap (\overline{Q}_1 \setminus \overline{Q}_0).$$

We have proved (3.3). In a similar way one proves (3.2).

Step 2. Let $T \in \mathbb{Z}^+$ be such that (3.2) and (3.3) hold. For a sequence $\overline{Z} = (Z_1, \ldots, Z_{3T})$ of members of A we define

$$f^{\bar{Z}} = f^{\bar{Z}}_{(Q,\{D_b\}),(\bar{Q},\{\bar{D}_b\})} : Q_1/Q_0 \to \overline{Q}_1/\overline{Q}_0$$

by

$$f^{\bar{Z}}([x]) = \begin{cases} [f^{3T}(x)] & \text{if } \forall_{i\in\{0,\ldots,2T-1\}}\ f^i(x) \in D_{Z_{3T-i}} \cap (Q_1 \setminus Q_0) \\ & \quad \text{and } f^{T+i}(x) \in \overline{D}_{Z_{2T-i}} \cap (\overline{Q}_1 \setminus \overline{Q}_0), \\ [\overline{Q}_0] & \text{otherwise.} \end{cases}$$

Our task is to prove the continuity of $f^{\bar{Z}}$.

The proof goes along the lines of other continuity proofs in the Conley index theory (cf. [17], [18], [21]). Let $f_0^{\bar{Z}} : Q_1 \to \overline{Q}_1/\overline{Q}_0$ be defined by the same formula as $f^{\bar{Z}}$. Clearly, it is enough to show that $f_0^{\bar{Z}}$ is continuous. Put

$$\begin{aligned} O_1 = \{x \in Q_1 : \forall_{i\in\{0,\ldots,2T-1\}}\ & f^i(x) \in D_{Z_{3T-i}} \cap (Q_1 \setminus Q_0) \\ & \text{and } f^{T+i}(x) \in \overline{D}_{Z_{2T-i}} \cap (\overline{Q}_1 \setminus \overline{Q}_0)\}, \\ O_2 = \{x \in Q_1 : \exists_{i\in\{0,\ldots,2T-1\}}\ & f^i(x) \notin D_{Z_{3T-i}} \ \text{ or } \ f^{T+i}(x) \notin \overline{D}_{Z_{2T-i}}\}. \end{aligned}$$

Clearly, O_2 is open in Q_1 and $f_0^{\bar{Z}}$ is constant on O_2 and therefore continuous at each point of this set. Since $f_0^{\bar{Z}}(x) = [f^{3T}(x)]$ for each $x \in O_1$, in order to prove the continuity of $f_0^{\bar{Z}}$ at each point of O_1 it is enough to show that this set is open in Q_1. Take $x \in O_1$. There exists an open neighborhood U of x in X such that

$$f^i(U) \cap (Q_0 \cup D_{B\setminus Z_{3T-i}}) = \emptyset = f^{T+i}(U) \cap (\overline{Q}_0 \cup \overline{D}_{B\setminus Z_{2T-i}}) \tag{3.5}$$

for each $i \in \{0, 1, \ldots, 2T-1\}$. Let us show that $U \cap Q_1 \subset O_1$. Let $y \in U \cap Q_1$. Our assumptions about U imply $y \in Q_1 \setminus Q_0$ and $f^i(y) \notin Q_0$ for each $i \in \{0, 1, \ldots, 2T-1\}$. Since Q_0 is an exit set for Q_1, $f^i(y) \in Q_1 \setminus Q_0$. By (3.5), $f^i(y) \in (Q_1 \setminus Q_0) \cap D_{Z_{3T-i}}$. Hence, by (3.3),

$$f^T(y) \in \overline{D}_{Z_{2T}} \cap (\overline{Q}_1 \setminus \overline{Q}_0) \subset \overline{Q}_1 \setminus \overline{Q}_0.$$

By the previous argument,

$$f^{T+i}(y) \in \overline{D}_{Z_{2T-i}} \cap (\overline{Q}_1 \setminus \overline{Q}_0) \quad \text{for all } i \in \{0, 1, \ldots 2T-1\}$$

so that $y \in O_1$.

We conclude that, in order to prove the continuity of $f_0^{\bar{Z}}$, it is enough to show that it is continuous at each point of $Q_1 \setminus (O_1 \cup O_2)$. Let x be in this set. Then, in particular,

$$\forall_{i\in\{0,\ldots,2T-1\}} \quad f^i(x) \in D_{Z_{3T-i}} \text{ and } f^{T+i}(x) \in \overline{D}_{Z_{2T-i}} \tag{3.6}$$

and $f_0^{\bar{Z}}(x) = [\overline{Q}_0]$. By (3.2) with x replaced with $f^T(x)$, $f^{2T}(x) \in D_{Z_T} \cap (Q_1 \setminus Q_0)$. Since $f^i(x) \in D_{Z_{3T-i}} \subset Q_1$ for each $i \in \{0, 1, \ldots, 2T-1\}$, positive invariance of Q_0 in Q_1 implies $f^i(x) \in D_{Z_{3T-i}} \cap (Q_1 \setminus Q_0)$. Thus, since $x \notin O_1$, for some $j \in \{0, 1, \ldots, 2T-1\}$ we must have $f^{T+j}(x) \notin \overline{D}_{Z_{2T-j}} \cap (\overline{Q}_1 \setminus \overline{Q}_0)$. By (3.6), $f^{T+j}(x) \in \overline{Q}_0$. By (3.6) and positive invariance of $\overline{Q}_0$ in $\overline{Q}_1$, $f^{3T-1}(x) \in \overline{Q}_0$. Let V' be an open neighborhood of $[\overline{Q}_0]$ in $\overline{Q}_1/\overline{Q}_0$. Denote by π the projection map of $\overline{Q}_1$ into $\overline{Q}_1/\overline{Q}_0$. Put $M = \pi^{-1}((\overline{Q}_1/\overline{Q}_0) \setminus V')$. Clearly, M is a compact subset of $\overline{Q}_1 \setminus \overline{Q}_0$. By positive invariance of $\overline{Q}_0$ in $\overline{Q}_1$, $f^{3T}(x) \notin M$. Let V be an open neighborhood of x in Q_1 such that $f^{3T}(V) \cap M = \emptyset$. Notice that for all $y \in V$, $f_0^{\bar{Z}}(y)$ is equal to either $[\overline{Q}_0]$ or $[f^{3T}(y)]$ (the second possibility can occur only if $f^{3T}(y) \in \overline{Q}_1$). Hence, $f_0^{\bar{Z}}(y) \in V'$. In this way we have proved that $f_0^{\bar{Z}}$ is continuous at x.

Step 3. If $(Z_1, \ldots, Z_{3T+1}) \in A^{3T+1}$ then

$$f^{(Z_1,\ldots,Z_{3T})} \circ f^{Z_{3T+1}}_{(Q,\{D_b\})} = f^{Z_1}_{(\bar{Q},\{\bar{D}_b\})} \circ f^{(Z_2,\ldots,Z_{3T+1})}.$$

Therefore, we have the following morphism in $\mathcal{H}top_{[A]}$:

$$f_{(Q,\{D_b\}),(\bar{Q},\{\bar{D}_b\})} = [\{f^{\bar{Z}}\}_{\bar{Z}\in A^{3T}}, 3T] : I(Q, \{D_b\}) \to I(\overline{Q}, \{\overline{D}_b\}).$$

To prove this, consider the following two conditions:

$$\text{(3.7)} \qquad \begin{aligned} &\forall_{i\in\{0,\ldots,2T\}} && f^i(x) \in D_{Z_{3T+1-i}} \cap (Q_1 \setminus Q_0) \quad \text{and} \\ &\forall_{i\in\{1,\ldots,2T\}} && f^{T+i}(x) \in \overline{D}_{Z_{2T+1-i}} \cap (\overline{Q}_1 \setminus \overline{Q}_0) \end{aligned}$$

and

$$\text{(3.8)} \qquad \begin{aligned} &\forall_{i\in\{0,\ldots,2T-1\}} && f^i(x) \in D_{Z_{3T+1-i}} \cap (Q_1 \setminus Q_0) \quad \text{and} \\ &\forall_{i\in\{0,\ldots,2T\}} && f^{T+i}(x) \in \overline{D}_{Z_{2T+1-i}} \cap (\overline{Q}_1 \setminus \overline{Q}_0). \end{aligned}$$

Notice that if (3.7) holds for some $x \in X$ then applying (3.3) gives $f^T(x) \in \overline{D}_{Z_{2T+1}} \cap (\overline{Q}_1 \setminus \overline{Q}_0)$. Therefore, (3.8) holds. We have proved that (3.7) $\Rightarrow$ (3.8). Since the reverse implication can be proved in a similar way using (3.2) with x replaced with $f^T(x)$, (3.7) and (3.8) are equivalent. To finish the proof apply the formulas for f^Z and $f^{\bar{Z}}$ to conclude that, for all $x \in Q_1$,

$$f^{(Z_1,\ldots,Z_{3T})} \circ f^{Z_{3T+1}}_{(Q,\{D_b\})}([x]) = \begin{cases} [f^{3T+1}(x)] & \text{if (3.7) holds,} \\ [\overline{Q}_0] & \text{otherwise,} \end{cases}$$

and

$$f^{Z_1}_{(\bar{Q},\{\bar{D}_b\})} \circ f^{(Z_2,\ldots,Z_{3T+1})} = \begin{cases} [f^{3T+1}(x)] & \text{if (3.8) holds,} \\ [\overline{Q}_0] & \text{otherwise.} \end{cases}$$

Step 4. The morphism $f_{(Q,\{D_b\}),(\bar{Q},\{\bar{D}_b\})}$ defined in Step 3 is an isomorphism in $\mathcal{H}top_{[A]}$.

Notice that Steps 1 through 3 remain valid if we replace Q by $\overline{Q}$ and vice versa. Hence we have the morphism

$$f_{(\bar{Q},\{\bar{D}_b\}),(Q,\{D_b\})}$$
$$= [\{f^{\bar{Z}}_{(\bar{Q},\{\bar{D}_b\}),(Q,\{D_b\})}\}_{\bar{Z}\in A^{3T}}, 3T] : I(\overline{Q},\{\overline{D}_b\}) \to I(Q,\{D_b\}).$$

Now, consider the composition

$$g = f^{\bar{Y}}_{(\bar{Q},\{\bar{D}_b\}),(Q,\{D_b\})} \circ f^{\bar{Z}}_{(Q,\{D_b\}),(\bar{Q},\{\bar{D}_b\})}$$

for given $\overline{Z} = (Z_1,\ldots,Z_{3T}) \in A^{3T}$ and $\overline{Y} = (Y_1,\ldots,Y_{3T}) \in A^{3T}$. Using the formula defining the maps in Step 2 we get

$$g([x]) = \begin{cases} [f^{6T}(x)] & \text{if the condition (3.9) below holds,} \\ [Q_0] & \text{otherwise,} \end{cases}$$

where

$$\forall_{i\in\{0,\ldots,2T-1\}} \quad \begin{aligned} &f^i(x) \in D_{Z_{3T-i}} \cap (Q_1 \setminus Q_0), \\ &f^{T+i}(x) \in \overline{D}_{Z_{2T-i}} \cap (\overline{Q}_1 \setminus \overline{Q}_0), \\ &f^{3T+i}(x) \in \overline{D}_{Y_{3T-i}} \cap (\overline{Q}_1 \setminus \overline{Q}_0), \\ &f^{4T+i}(x) \in D_{Y_{2T-i}} \cap (Q_1 \setminus Q_0). \end{aligned} \tag{3.9}$$

Using the implications (3.2) and (3.3) one can easily prove that (3.9) is equivalent to

$$\forall_{i\in\{0,\ldots,3T-1\}} \quad \begin{aligned} &f^i(x) \in D_{Z_{3T-i}} \cap (Q_1 \setminus Q_0), \\ &f^{3T+i}(x) \in D_{Y_{3T-i}} \cap (Q_1 \setminus Q_0). \end{aligned}$$

Hence, by (3.1),

$$g = f^{Y_1}_{(Q,\{D_b\})} \circ \ldots \circ f^{Y_{3T}}_{(Q,\{D_b\})} \circ f^{Z_1}_{(Q,\{D_b\})} \circ \ldots \circ f^{Z_{3T}}_{(Q,\{D_b\})}.$$

This means that

$$\{f^{\bar{Z}}_{(\bar{Q},\{\bar{D}_b\}),(Q,\{D_b\})}\}_{\bar{Z}\in A^{3T}} \circ \{f^{\bar{Z}}_{(Q,\{D_b\}),(\bar{Q},\{\bar{D}_b\})}\}_{\bar{Z}\in A^{3T}} = [(\{f^{Z}_{(Q,\{D_b\})}\}_{Z\in A})^{6T}]$$

in $\mathcal{H}top_{(A)}$ and therefore, by Proposition 2.1,

$$f_{(\bar{Q},\{\bar{D}_b\}),(Q,\{D_b\})} \circ f_{(Q,\{D_b\}),(\bar{Q},\{\bar{D}_b\})}$$
$$= [(\{f^{Z}_{(Q,\{D_b\})}\}_{Z\in A})^{6T}, 6T] = \mathrm{id}_{I(Q,\{D_b\})}.$$

In a similar way one proves that

$$f_{(Q,\{D_b\}),(\bar{Q},\{\bar{D}_b\})} \circ f_{(\bar{Q},\{\bar{D}_b\}),(Q,\{D_b\})} = \mathrm{id}_{I(\bar{Q},\{\bar{D}_b\})}. \ \blacksquare$$

The theorem just proved justifies the following definition.

Definition 3.4. Let $f : X \to X$ be a continuous map, S an isolated invariant set for f and $\{S_b\}$ a decomposition of S. The *Conley index* of $\{S_b\}$, denoted by $h(\{S_b\}, f, X)$, is defined as the class of all objects in $\mathcal{H}top_{[A]}$ isomorphic to the index object $I(Q,\{D_b\})$ for any index pair Q for S compatible with the decomposition $\{S_b\}$ and any decomposition $\{D_b\}$ of

$\mathrm{cl}(Q_1 \setminus Q_0)$ such that $D_b \cap S = S_b$ for each $b \in B$. The *cohomological* and the *q-dimensional cohomological indices* of $\{S_b\}$ are defined by

$$h^*(\{S_b\}, f, X) = (H^*)_{[A]}(h(\{S_b\}, f, X))$$

and

$$h^q(\{S_b\}, f, X) = (H^q)_{[A]}(h(\{S_b\}, f, X)),$$

respectively.

4. Properties. We begin this section with proving the continuation property of the Conley index for decompositions of isolated invariant sets.

THEOREM 4.1 (Continuation property). *Suppose that X is a locally compact metric space and a continuous map $f : X \times [0,1] \to X \times [0,1]$ is parameter-preserving, i.e. $f(X \times \lambda) \subset X \times \lambda$ for each $\lambda \in [0,1]$. For each set $A \subset X \times [0,1]$ and $\lambda \in [0,1]$ put $A_\lambda = \{x \in X : (x,\lambda) \in A\}$. Define $f_\lambda : X \to X$ by $f(x,\lambda) = (f_\lambda(x), \lambda)$. If S is an isolated invariant set for f and $\{S_b\}$ is a decomposition of S then, for each $\lambda \in [0,1]$, S_λ is an isolated invariant set for f_λ, $\{S_{b\lambda}\} = \{S_{b\lambda}\}_{b\in B}$ is a decomposition of S_λ and $h(\{S_{b\lambda}\}, f_\lambda, X)$ does not depend on λ.*

Proof. It is enough to show that each $\lambda \in [0,1]$ admits a neighborhood U such that $h(\{S_{b\mu}\}, f_\mu, X)$ is constant in $\mu \in U$. Let N be an isolating neighborhood for S_λ with respect to f_λ which admits a decomposition $\{N_b\}$ such that $N_b \cap S_\lambda = S_{b\lambda}$ for each $b \in B$. As a consequence of the existence theorem for index pairs for multivalued upper semicontinuous maps (see [6], Theorem 2.6) we obtain the existence of a *stable* index pair, i.e. a pair $Q = (Q_1, Q_0)$ which is an index pair for S_μ with respect to f_μ for each μ in an interval J which is a neighborhood of λ in $[0,1]$, such that $Q_1 \subset N$. Let $D_b = \mathrm{cl}(Q_1 \setminus Q_0) \cap N_b$ for each $b \in B$. Since $D_b \cap S_\lambda = S_{b\lambda}$, by making J smaller if necessary we may assume that $D_b \cap S_\mu = S_{b\mu}$ for each $\mu \in J$. This means that Q is compatible with the decomposition $\{S_{b\mu}\}$ of S_μ. Furthermore, since J is an interval, the homotopy class of the index map $(f_\mu)_Q : Q_1/Q_0 \to Q_1/Q_0$ does not depend on $\mu \in J$. By Definition 3.3, the index object $I(Q, \{D_b\}, f_\mu)$ is independent of $\mu \in J$. ∎

The next theorem shows that the Conley index for a decomposition $\{S_b\}$ carries information about ordinary Conley indices of some subsets of S.

THEOREM 4.2. *Let $\{S_b\}$ be a decomposition of an isolated invariant set S for a continuous map $f : X \to X$. For each sequence $\overline{Y} = (Y_0, Y_1, \ldots, Y_{n-1})$ of members of A put $\widetilde{S}_{\bar{Y}} = \bigcap_{i=0}^{n-1} f^{-i}(S_{Y_i})$ and $S_{\bar{Y}} = \mathrm{Inv}_{f^n}(\widetilde{S}_{\bar{Y}})$. Then $S_{\bar{Y}}$ is an isolated invariant set for f^n contained in S and*

$$h(S_{\bar{Y}}, f^n, X) = \mathcal{P}_{\iota(\bar{Y})}(h(\{S_b\}, f, X)).$$

P r o o f. Let N be an isolating neighborhood for S with respect to f admitting a decomposition $\{N_b\}$ such that $N_b \cap S = S_b$ for each $b \in B$. By Proposition 2.1 of [20], $N_{\bar{Y}} = \bigcap_{i=0}^{n-1} f^{-i}(N_{Y_i})$ is an isolating neighborhood with respect to f^n and its invariant part is contained in S. Therefore, $\mathrm{Inv}_{f^n}(N_{\bar{Y}}) = \mathrm{Inv}_{f^n}(\widetilde{S}_{\bar{Y}}) = S_{\bar{Y}}$, which means that $N_{\bar{Y}}$ is an isolating neighborhood for $S_{\bar{Y}}$ with respect to f^n. We have thus proved the first part of the theorem. In order to show the formula for the Conley index of $S_{\bar{Y}}$ we make use of the following fact (see [20], Lemma 3.1).

If $Q = (Q_1, Q_0)$ is a regular index pair for S such that $Q_1 \subset N$ then

$$h(S_{\bar{Y}}, f^n, X) = [Q_1/Q_0, f^{Y_{n-1}}_{(Q,\{D_b\})} \circ f^{Y_{n-2}}_{(Q,\{D_b\})} \circ \ldots \circ f^{Y_0}_{(Q,\{D_b\})}],$$

where $D_b = \mathrm{cl}(Q_1 \setminus Q_0) \cap N_b$ for each $b \in B$.

Since such index pairs exist, this formula together with the definition of $\mathcal{P}$-type functors proves the theorem. ■

Theorem 4.3 (Locality). *If $f : X \to X$ and $g : X \to X$ are continuous maps, S is an isolated invariant set for f, and f and g are equal on a neighborhood of S, then S is an isolated invariant set for g, and for any decomposition $\{S_b\}$ of S,*

$$h(\{S_b\}, f, X) = h(\{S_b\}, g, X).$$

P r o o f. By Proposition 3.1, there exists an index pair $Q = (Q_1, Q_0)$ for S compatible with the decomposition $\{S_b\}$ such that f and g restricted to Q_1 are equal. Since the corresponding index objects only depend on these restrictions, they are the same. ■

The rest of this section is devoted to the formulation and proof of the Ważewski property of the Conley index for decompositions of isolated invariant sets, and to giving a bound for the number of periodic points of f in terms of the Conley index for decompositions. Fix a locally compact metric space X, a continuous map f of X into itself, an isolated invariant set S for f and a decomposition $\{S_b\}$ of S. Let N be a fixed isolating neighborhood for S admitting a decomposition $\{N_b\}$ such that $N_b \cap S = S_b$. Let

$$S^+ = \mathrm{Inv}^+ N = \{x \in N : \forall_{i \in \mathbb{Z}^+}\ f^i(x) \in N\}.$$

The map $p : S^+ \to \Pi = \prod_{i \in \mathbb{Z}^+} B$ is defined by

$$p(x) = (\eta(f^i(x)))_{i=0}^{\infty}, \tag{4.1}$$

where $\eta : S^+ \to B$ is defined by

$$\eta(x) = b \quad \text{if and only if} \quad x \in N_b.$$

Clearly, both p and η are continuous if we endow B with the discrete topology. Furthermore, $p \circ f = \sigma \circ p$, where $\sigma : \Pi \to \Pi$ is the shift map. This means that p is a semiconjugacy onto its image. In what follows, we shall give lower bounds for the image of p and the image of the set of periodic points of f under p in terms of the Conley index for decompositions.

DEFINITION 4.1. Let $(Y,\{g^Z\}_{Z\in A})$ be an object in a category $\mathcal{K}_{[A]}$ ($\mathcal{K}=\mathcal{H}top$, $\mathcal{M}$ or $\mathcal{M}_{\mathrm{G}}$). We put

$$\begin{aligned}
\varPi_0(Y,\{g^Z\}_{Z\in A}) &= \{(b_i)_{i=0}^{\infty}\in\varPi:\forall_{n\in\mathbb{Z}^+}\ g^{\{b_n\}}\circ g^{\{b_{n-1}\}}\circ\ldots\circ g^{\{b_0\}}\neq 0\},\\
\varPi(Y,\{g^Z\}_{Z\in A}) &= \bigcap_{n\in\mathbb{Z}^+}\sigma^n(\varPi_0(Y,\{g^Z\}_{Z\in A})),\\
\varPi_0^*(Y,\{g^Z\}_{Z\in A}) &= \{(b_i)_{i=0}^{\infty}\in\varPi:\forall_{n\in\mathbb{Z}^+}\ g^{\{b_0\}}\circ g^{\{b_1\}}\circ\ldots\circ g^{\{b_n\}}\neq 0\},\\
\varPi^*(Y,\{g^Z\}_{Z\in A}) &= \bigcap_{n\in\mathbb{Z}^+}\sigma^n(\varPi_0^*(Y,\{g^Z\}_{Z\in A})).
\end{aligned}$$

PROPOSITION 4.1. *Let* $(Y,\{g^Z\}_{Z\in A})$ *and* $(\overline{Y},\{\overline{g}^Z\}_{Z\in A})$ *be objects in* $\mathcal{K}_{[A]}$.

(i) $\varPi_0(Y,\{g^Z\}_{Z\in A})$ *and* $\varPi_0^*(Y,\{g^Z\}_{Z\in A})$ *are compact.*

(ii) $\sigma(\varPi_0(Y,\{g^Z\}_{Z\in A}))\subset\varPi_0(Y,\{g^Z\}_{Z\in A})$ *and* $\sigma(\varPi_0^*(Y,\{g^Z\}_{Z\in A}))\subset\varPi_0^*(Y,\{g^Z\}_{Z\in A})$.

(iii) *If* $(Y,\{g^Z\}_{Z\in A})$ *and* $(\overline{Y},\{\overline{g}^Z\}_{Z\in A})$ *are isomorphic in* $\mathcal{K}_{[A]}$ *then*

$$\exists_{n\in\mathbb{Z}^+}\quad \sigma^n(\varPi_0(Y,\{g^Z\}_{Z\in A}))\subset\varPi_0(\overline{Y},\{\overline{g}^Z\}_{Z\in A})$$

and

$$\exists_{m\in\mathbb{Z}^+}\quad \sigma^m(\varPi_0^*(Y,\{g^Z\}_{Z\in A}))\subset\varPi_0^*(\overline{Y},\{\overline{g}^Z\}_{Z\in A}).$$

Therefore,

$$\varPi(Y,\{g^Z\}_{Z\in A})=\varPi(\overline{Y},\{\overline{g}^Z\}_{Z\in A})$$

and

$$\varPi^*(Y,\{g^Z\}_{Z\in A})=\varPi^*(\overline{Y},\{\overline{g}^Z\}_{Z\in A}).$$

In particular, *the sets* $\varPi(I)$ *and* $\varPi^*(I)$ *can be defined in the obvious way for isomorphism classes* I *in* $\mathcal{K}_{[A]}$.

(iv) *Suppose that* $\mathcal{K}$ *and* $\mathcal{L}$ *are categories*, *each of them equal to* $\mathcal{H}top$, $\mathcal{M}$ *or* $\mathcal{M}_{\mathrm{G}}$, *and* $F:\mathcal{K}\to\mathcal{L}$ *is a functor mapping zero morphisms into zero morphisms. If* F *is covariant then*

$$\varPi_0(F_{[A]}(Y,\{g^Z\}_{Z\in A}))\subset\varPi_0(Y,\{g^Z\}_{Z\in A})$$

and

$$\varPi_0^*(F_{[A]}(Y,\{g^Z\}_{Z\in A}))\subset\varPi_0^*(Y,\{g^Z\}_{Z\in A}).$$

If F *is contravariant then*

$$\varPi_0^*(F_{[A]}(Y,\{g^Z\}_{Z\in A}))\subset\varPi_0(Y,\{g^Z\}_{Z\in A})$$

and

$$\varPi_0(F_{[A]}(Y,\{g^Z\}_{Z\in A}))\subset\varPi_0^*(Y,\{g^Z\}_{Z\in A}).$$

The same inclusions hold for $\varPi_0$ *replaced with* $\varPi$.

P r o o f. We only prove the parts concerning $\Pi(Y,\{g^Z\}_{Z\in A})$ and $\Pi_0(Y, \{g^Z\}_{Z\in A})$. The proofs of the other statements are dual.

(i) If $(b_i)_{i=0}^{\infty} \notin \Pi_0(Y,\{g^Z\}_{Z\in A})$ then, for some $n\in\mathbb{Z}^+$,

$$g^{\{b_n\}}\circ g^{\{b_{n-1}\}}\circ\ldots\circ g^{\{b_0\}}=0.$$

Hence

$$\prod_{i=0}^{n}\{b_i\}\times\prod_{i=n+1}^{\infty}B\subset\Pi\setminus\Pi_0(Y,\{g^Z\}_{Z\in A}).$$

Thus, the complement of $\Pi_0(Y,\{g^Z\}_{Z\in A})$ is open in Π so that this set is compact.

(ii) If $(b_i)_{i=0}^{\infty}\in\Pi_0(Y,\{g^Z\}_{Z\in A})$ then, for each $n\in\mathbb{Z}^+$,

$$g^{\{b_{n+1}\}}\circ g^{\{b_n\}}\circ\ldots\circ g^{\{b_0\}}\neq 0.$$

Hence, $g^{\{b_{n+1}\}}\circ g^{\{b_n\}}\circ\ldots\circ g^{\{b_1\}}\neq 0$. It follows that $(b_{i+1})_{i=0}^{\infty}=\sigma((b_i)_{i=0}^{\infty})\in \Pi_0(Y,\{g^Z\}_{Z\in A})$.

(iii) Let

$$[\{h^{\bar Z}\}_{\bar Z\in A^n},n]:(Y,\{g^Z\}_{Z\in A})\to(\overline{Y},\{\overline{g}^Z\}_{Z\in A})$$

and

$$[\{l^{\bar Z}\}_{\bar Z\in A^m},m]:(\overline{Y},\{\overline{g}^Z\}_{Z\in A})\to(Y,\{g^Z\}_{Z\in A})$$

be reciprocal isomorphisms. Then, by the definition of the category $\mathcal{K}_{[A]}$ and Definition 4.1, for each $(b_i)_{i=0}^{\infty}\in\Pi_0(Y,\{g^Z\}_{Z\in A})$ and sufficiently large $k\in\mathbb{Z}^+$,

$$\begin{aligned}0&\neq g^{\{b_{n+m+k-1}\}}\circ\ldots\circ g^{\{b_0\}}\\&=l^{(b_{n+m+k-1},\ldots,b_{n+k})}\circ h^{(b_{n+k-1},\ldots,b_k)}\circ g^{\{b_{k-1}\}}\circ\ldots\circ g^{\{b_0\}}\\&=l^{(b_{n+m+k-1},\ldots,b_{n+k})}\circ\overline{g}^{\{b_{n+k-1}\}}\circ\ldots\circ\overline{g}^{\{b_n\}}\circ h^{(b_{n-1},\ldots,b_0)}.\end{aligned}$$

Hence, $g^{\{b_{n+k-1}\}}\circ\ldots\circ g^{\{b_n\}}\neq 0$ and therefore $\sigma^n((b_i)_{i=0}^{\infty})\in\Pi_0(\overline{Y}, \{\overline{g}^Z\}_{Z\in A})$.

The second part of (iii) follows immediately from the first, Proposition 4.1(ii) and Definition 4.1.

(vi) follows immediately from Definition 4.1. ∎

Theorem 4.4 (Ważewski property). *Let $(Y,\{g^Z\}_{Z\in A})$ be an object in $\mathcal{H}top_{[A]}$ such that $h(\{S_b\},f,X)=[Y,\{g^Z\}_{Z\in A}]$. Then*:

(i) *There exists $n\in\mathbb{Z}^+$ such that*

$$\sigma^n(\Pi_0(Y,\{g^Z\}_{Z\in A}))\subset p(S^+).$$

(ii) $\Pi(Y,\{g^Z\}_{Z\in A})\subset p(S)$.

P r o o f. Let $Q=(Q_1,Q_0)$ be an index pair for S such that $Q_1\subset N$. Let $D_b=\mathrm{cl}(Q_1\setminus Q_0)\cap N_b$. Clearly, $D_b\cap S=S_b$. The index object $I(Q,\{D_b\})$

is isomorphic to $(Y,\{g^Z\}_{Z\in A})$. Hence, by Proposition 4.1(iii),

$$\exists_{n\in\mathbb{Z}^+}\quad \sigma^n(\Pi_0(Y,\{g^Z\}_{Z\in A}))\subset \Pi_0(I(Q,\{D_b\})).$$

It follows that to prove (i) it is enough to show that the set on the right-hand side is contained in $p(S^+)$. Take $(b_i)_{i=0}^\infty\in\Pi_0(I(Q,\{D_b\}))$. Then for each $k\in\mathbb{Z}^+$ there exists an $x_k\in Q_1$ such that

$$f^{\{b_k\}}_{(Q,\{D_b\})}\circ\ldots\circ f^{\{b_0\}}_{(Q,\{D_b\})}([x_k])\neq[Q_0].$$

The formula (3.1) proves that

$$\forall_{i\in\{0,\ldots k\}}\qquad f^i(x_k)\in D_{b_i}\cap(Q_1\setminus Q_0).$$

Now, let x_* be an accumulation point of the sequence $\{x_k\}$. Clearly, $f^i(x_*)\in D_{b_i}\subset N$ for all $i\in\mathbb{Z}^+$. Thus, $x_*\in S^+$ and $p(x_*)=(b_i)_{i=0}^\infty$. The proof of (i) is complete.

To prove (ii), notice that

$$\sigma^{k+n}(\Pi_0(Y,\{g^Z\}_{Z\in A}))\subset\sigma^k(p(S^+))=p(f^k(S^+))$$

for each $k\in\mathbb{Z}^+$. Hence,

$$\Pi(Y,\{g^Z\}_{Z\in A})=\bigcap_{k\in\mathbb{Z}^+}\sigma^{k+n}(\Pi_0(Y,\{g^Z\}_{Z\in A}))\subset\bigcap_{k\in\mathbb{Z}^+}p(f^k(S^+)).$$

Since $\{f^k(S^+)\}$ is a decreasing sequence of compact sets intersecting in S, the set on the right-hand side is equal to $p(S)$. ∎

THEOREM 4.5 (Detection of periodic points). *Let* $(b_i)_{i=0}^\infty\in\Pi$ *be* T*-periodic. Put* $\overline{Y}=(\{b_0\},\{b_1\},\ldots,\{b_{T-1}\})\in A^T$. *If, for some positive integer* n,

$$\Lambda(\mathcal{P}_{\bar{Y}^n}(h^*(\{S_b\},f,X)))\neq 0\tag{4.2}$$

then there exists an nT*-periodic point* $x\in S$ *of* f *such that* $p(x)=(b_i)_{i=0}^\infty$. *If* (4.2) *holds for* $n=1$ *and* T *is the principal period of* $(b_i)_{i=0}^\infty$ *then* x *can be chosen in such a way that* T *is its principal period.*

Proof. By (4.2), Theorem 4.2 and Remark 2.1,

$$\Lambda(h^*(S_{\bar{Y}^n},f^{nT},X))\neq 0.$$

By Theorem 1.1, there exists an $x\in S_{\bar{Y}^n}$ such that $f^{nT}(x)=x$. Since $p(S_{\bar{Y}^n})=\{(b_i)_{i=0}^\infty\}$, this proves the first part of the theorem. Since $p\circ f=\sigma\circ p$, the principal period of $p(x)$ cannot exceed the principal period of x with respect to f. Thus, if $n=1$ then the principal period of x equals T. ∎

5. Horseshoes. In many chaotic dynamical systems arising from differential equations, behavior resembling that of Smale's horseshoe map is observed (see [8], [22]). In this section we provide an example of a criterion for chaos based on the index defined in Section 3 and compute the indices of decompositions of invariant sets of Smale's horseshoe (U-horseshoe) and

G-horseshoe maps. In the sequel, we only deal with decompositions of isolated invariant sets into two disjoint subsets, i.e. we set $B = \{1, 2\}$. By

$$\left[\begin{array}{cccc} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{k,1} & a_{k,2} & \dots & a_{k,n} \\ \hline a_{k+1,1} & a_{k+1,2} & \dots & a_{k+1,n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{array}\right]$$

we denote the class of all objects in $\mathcal{M}_{[A]}$ isomorphic to the object $(\Xi^n, \{\psi^Z\}_{Z\in A})$ such that the matrices of $\psi^{\{1\}}$ and $\psi^{\{2\}}$ in the standard basis are

$$\left[\begin{array}{cccc} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{k,1} & a_{k,2} & \dots & a_{k,n} \\ 0 & 0 & \dots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \dots & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cccc} 0 & 0 & \dots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \dots & 0 \\ a_{k+1,1} & a_{k+1,2} & \dots & a_{k+1,n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{array}\right]$$

(respectively) and $\psi^Z = \sum_{b\in Z} \psi^{\{b\}}$ for each $Z \in A$. Let us note the following

COROLLARY 5.1 *Let f be a continuous map of a locally compact metric space X into itself, S an isolated invariant set for f and $\{S_b\}$ a decomposition of S. Suppose that, for some $q \in \mathbb{Z}^+$,*

$$h^q(\{S_b\}, f, X) = \left[\begin{array}{cc} a_{1,1} & a_{1,2} \\ \hline a_{2,1} & a_{2,2} \end{array}\right].$$

where $a_{i,j} \in \Xi$ are nonzero. If Ξ is a domain then the map $p : S \to \Pi$ defined by (4.1) *is a surjection. If, in addition,*

$$h^r(\{S_b\}, f, X) = 0$$

for each $r \neq q$, Ξ is the field of rational numbers and X is an ENR then each periodic sequence in Π is the image (*under* p) *of a periodic point of f in S of the same principal period.*

Proof. Let $\psi_1, \psi_2 : \Xi^2 \to \Xi^2$ have matrices

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

in the standard basis. For each sequence $(b_0, b_1, \dots, b_{T-1})$ of members of B,

$$\psi_{b_{T-1}} \circ \psi_{b_{T-2}} \circ \dots \circ \psi_{b_0} \neq 0.$$

More precisely, the matrix of the composition on the left-hand side has one row of zeros and one row of nonzero members of Ξ (in particular, its trace is nonzero if Ξ is a field: we shall use this fact later). Thus,

$$\Pi^*\left(\left[\begin{array}{cc} a_{1,1} & a_{1,2} \\ \hline a_{2,1} & a_{2,2} \end{array}\right]\right) = \Pi.$$

Applying Proposition 4.1(iv) for F being the q-dimensional cohomology functor gives

$$\Pi(h(\{S_b\}, f, X)) = \Pi.$$

To finish the proof of the first part, apply Theorem 4.4(ii).

In order to prove the remaining part, take a sequence $(b_i)_{i=0}^{\infty} \in \Pi$ of principal period T. Let $\overline{Y} = (\{b_0\}, \{b_1\}, \ldots, \{b_{T-1}\})$. By assumptions, $\operatorname{tr}(\mathcal{P}_{\bar{Y}}(h^r(\{S_b\}, f, X))) = 0$ for all $r \neq q$. Since the trace is nonzero for $r = q$, the Lefschetz number of $\mathcal{P}_{\bar{Y}}(h^*(\{S_b\}, f, X))$ is nonzero. Theorem 4.5 implies that $(b_i)_{i=0}^{\infty}$ is the image of a periodic point of f of principal period T. ∎

In a moment we shall show that the assumptions of the above corollary are satisfied by the G-horseshoe and the U-horseshoe maps. We note that, by the continuation property of the Conley index for decompositions, Corollary 6.1 generalizes Theorem 2.4 of [9] and Theorem 2.1 of [23].

Example (cf. [11]). Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a G-horseshoe map, i.e. a map of $\mathbb{R}^2$ into itself which is affine on each of the rectangles $A_iB_iC_iD_i$ $(i = 1, 2)$. The rectangles and their images under G are depicted in Figure 1.

We are interested in the Conley index of the decomposition $\{S_b\}$ of the maximal invariant set S in $A_1B_1C_1D_1 \cup A_2B_2C_2D_2$, where $S_b = S \cap A_iB_iC_iD_i$ for each $b \in B = \{1, 2\}$. By Theorem 4.3 we can assume that G maps the rectangle $A_2B_2C_1D_1$ outside the rectangle $A_1B_1C_2D_2$. Let Q_0 be the set of all points in the rectangle $A_1B_1C_2D_2$ mapped by G outside its interior (it is the shadowed region, a disjoint union of three rectangles—see Fig. 1). One can easily see that if we put $Q_1 = A_1B_1C_2D_2$ then the pair $Q = (Q_1, Q_0)$ is an index pair for S, compatible with the decomposition $\{S_b\}$ of S. Clearly, the quotient space Q_1/Q_0 has the homotopy type of the wedge sum of two pointed circles. Thus,

$$H^q(Q_1/Q_0) = \begin{cases} \Xi^2 & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In $H^1(Q_1/Q_0)$ we can choose a basis consisting of vectors e_b $(b \in B)$ which are the images of the generators of $H^1(A_1B_1C_2D_2/(A_{1-b}B_{1-b}C_{1-b}D_{1-b} \cup Q_0))$ under the inclusion-induced homomorphism. Then $H^1(G_Q)$ maps e_b into $e_1 + e_2$ or $e_1 + e_2 - 2e_{1-b}$ according to the choice of the generators. Hence (notice that $H^1(r^Z)$ is the natural projection onto the submodule generated by $\{e_b : b \in Z\}$),

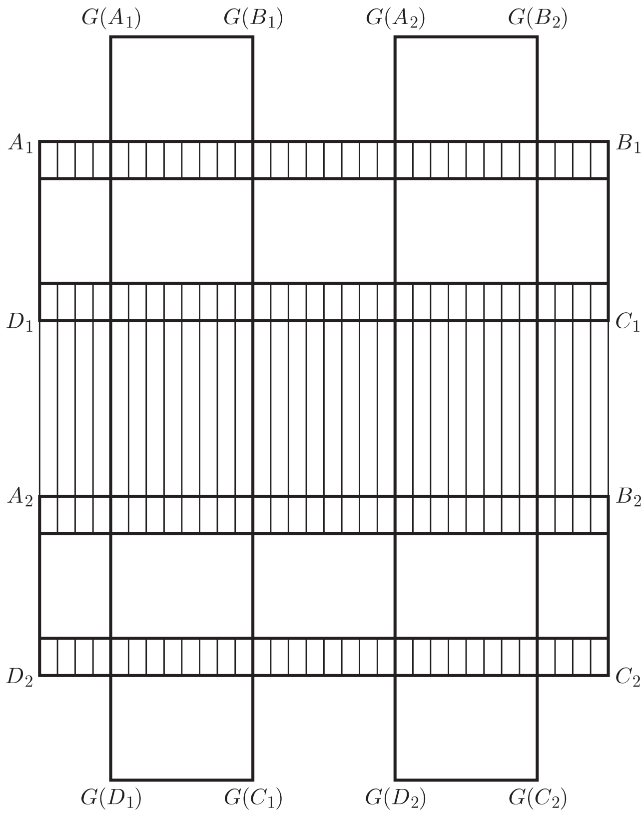

Fig. 1. G-horseshoe

$$h^1(\{S_b\}, G, \mathbb{R}^2) = \left[\begin{array}{cc} 1 & 1 \\ \hline 1 & 1 \end{array}\right] = \left[\begin{array}{cc} 1 & -1 \\ \hline -1 & 1 \end{array}\right].$$

Clearly, the cohomological indices in other dimensions are trivial. In the same way, one can prove that if U is the U-horseshoe map (cf. [11]) and $\{S_b\}$ is the decomposition of its invariant set obtained in the analogous way then

$$h^1(\{S_b\}, U, \mathbb{R}^2) = \left[\begin{array}{cc} 1 & -1 \\ \hline 1 & -1 \end{array}\right]$$

and the cohomological indices in all other dimensions are trivial.

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