FUNDAMENTA
MATHEMATICAE
147 (1995)

On finite-dimensional maps and other maps with "small" fibers

by

Yaki Sternfeld (Haifa)

Abstract. We prove that if f is a k-dimensional map on a compact metrizable space X then there exists a σ-compact $(k-1)$-dimensional subset A of X such that $f|X \setminus A$ is 1-dimensional. Equivalently, there exists a map g of X in I^k such that $\dim(f \times g) = 1$. These are extensions of theorems by Toruńczyk and Pasynkov obtained under the additional assumption that $f(X)$ is finite-dimensional.

These results are then extended to maps with fibers restricted to some classes of spaces other than the class of k-dimensional spaces. For example: if f has weakly infinite-dimensional fibers then $\dim(f|X \setminus A) \leq 1$ for some σ-compact weakly infinite-dimensional subset A of X.

The proof applies essentially the properties of hereditarily indecomposable continua.

1. Introduction. In this note we consider separable metric spaces and continuous functions (= maps). For a map f on X, $\dim f = \sup\{\dim f^{-1}(y) : y \in f(X)\}$. In [Pa] Pasynkov states the following

THEOREM 1. *Let f be a k-dimensional map on a compact space X with k and $\dim f(X)$ finite. Then there exists a map $g : X \to I^k$ such that $\dim(f \times g) = 0$.*

Toruńczyk ([T], Proposition 2) proves the following theorem (in a more general setting).

THEOREM 2. *Let X and f be as in Theorem 1. For each $0 \leq l \leq k-1$ there is a σ-compact subset A_l of X such that $\dim A_l \leq l$ and $\dim(f|X \setminus A_l) \leq k-l-1$.*

Theorems 1 and 2 can be easily derived from each other. To obtain Theorem 1 from Theorem 2 let $g_0 : X \to I$ be 1-1 on A_0. Then $\dim(f \times g_0) \leq k-1$, and proceed by induction on k. To derive Theorem 2 from Theorem 1 one needs the following lemma (to be proved in §3).

1991 *Mathematics Subject Classification*: 54F45, 54F15.

LEMMA 1. *Let Y and W be compact with* $\dim W \le k$. *There exists a* $(k-1)$-*dimensional* σ-*compact subset* B *of* $Y \times W$ *such that for all* y *in* Y, $\dim((\{y\} \times W) \setminus B) \le 0$. (*Equivalently*, $\dim(P|(Y \times W) \setminus B) \le 0$, *where* $P : Y \times W \to Y$ *is the projection.*)

Let now $g : X \to I^k$ be as in Theorem 1. Let $B \subset f(X) \times I^k$ be as in Lemma 1 (with $Y = f(X)$ and $W = I^k$). Set $A_{k-1} = (f \times g)^{-1}(B)$. Then $\dim A_{k-1} \le k-1$ and $\dim(f|X \setminus A_{k-1}) \le 0$, both by the Hurewicz theorem on closed maps which lower dimension.

The goal of this note is to prove theorems similar to Theorems 1 and 2 but without the finite-dimensionality assumption on $Y = f(X)$. The reduction in the dimension of f achieved in this case is merely to a 1-dimensional map and the question whether it can be further reduced to a 0-dimensional map as in Theorems 1 and 2 is left open.

In the following three theorems which we state here and prove in §3, as well as in the corollary and problem, f is assumed to be a k-dimensional map on a compact space X, with k finite.

THEOREM 3. *There exists a map* $g : X \to \mathbb{R}^k$ *such that* $\dim(f \times g) \le 1$. *Moreover*, $\dim(f \times g) \le 1$ *for almost all* g *in* $C(X, \mathbb{R}^k)$ (*where almost all* = *all but a set of first category*).

THEOREM 4. *There exists a* σ-*compact* $(k-1)$-*dimensional subset* A *of* X *such that* $\dim(f|X \setminus A) \le 1$.

THEOREM 5. *There exists a* k-*dimensional* (*not necessarily* σ-*compact though*) *subset* E *of* X *such that* $\dim(f|X \setminus E) = 0$.

The following corollary to Theorem 3 has been noticed by R. Pol.

COROLLARY 1. *f admits a representation as a composition of at most* $k+1$ 1-*dimensional maps.*

Proof. We have

$$X \xrightarrow{f \times g} f(X) \times I^k \xrightarrow{P_k} f(X) \times I^{k-1} \xrightarrow{P_{k-1}} f(X) \times I^{k-2} \longrightarrow \dots \\ \longrightarrow f(X) \times I \xrightarrow{P_1} f(X),$$

where $P_i : f(X) \times I^i \to f(X) \times I^{i-1}$ are the projections. ∎

PROBLEM 1. Does there exist a finite-dimensional σ-compact subset A of X such that $\dim(f|X \setminus A) = 0$? If so, does $\dim A$ depend on k only? (In view of Theorem 4 we may assume that $k = 1$.)

It turns out that only some very elementary properties of dimension are needed to prove theorems similar to Theorems 4 and 5.

Let Q be a topological property of separable metric spaces. Thus Q is a family of such spaces so that $X \in Q$ if and only if all homeomorphic copies of X are in Q. Assume also that Q satisfies the following three conditions:

Q_1. If $X \in Q$ and Y is a compact subset of X then $Y \in Q$.
Q_2. The union of countably many compact elements of Q is in Q.
Q_3. If X is compact and each component of X is in Q then $X \in Q$.

Examples of such properties are:

(i) $Q = \{X : \dim X \le k\}$ for some positive integer k.
(ii) Q = the family of weakly infinite-dimensional separable metric spaces.
(iii) $Q = \{X : \dim_G X \le k\}$, where k is a positive integer, G is an Abelian group and $\dim_G$ is the cohomological dimension.
(iv) More generally, $Q = \{X : D(X) \le k\}$, where D is a dimension function in the sense of Dobrowolski and Rubin [D-R].
(v) Let S be an ANE and let $Q = \{X : X\tau S\}$, where τ is the Kuratowski relation ([K], p. 332): $X\tau S$ if for every $H \subset X$ closed and every map $f : H \to S$, f is extendable over X.

Note that (v) includes (i), (iii) and (iv). We leave it to the reader to check that Q_1, Q_2 and Q_3 are satisfied.

Using almost the same arguments as in the proofs of Theorems 4 and 5 one can also prove the following result. (Q is assumed to satisfy Q_1, Q_2 and Q_3, and $f : X \to Y$ is a *Q-map* if for every $y \in Y$, $f^{-1}(y) \in Q$.)

THEOREM 6. *Let f be a Q-map on a compact space X. There exist a σ-compact Q-subset A of X and a 0-dimensional G_δ subset G of X such that* $\dim(f|X \setminus A) \le 1$ *and* $\dim(f|X \setminus (A \cup G)) = 0$.

Hence, for example, if each fiber of f is weakly infinite-dimensional, then $\dim(f|X \setminus A) \le 1$ for some σ-compact weakly infinite-dimensional subset A of X.

Remark. Theorem 6 is weaker than Theorems 4 and 5. Indeed, if $Q = \{X : \dim X \le k\}$ then Theorem 6 only states that $\dim A \le k$ while from Theorem 4 we obtain $\dim A \le k - 1$. (See Propositions 3 and 3^* where this difference originates.)

In the proofs of our theorems we apply the properties of hereditarily indecomposable continua in an essential manner. This is of interest as those continua do not appear explicitly in the statements of the results. See [Po] and [Le] for similar phenomena.

In §2 we study some properties of hereditarily indecomposable continua and of the closely related Bing spaces. In particular, we show in Proposition 3 that Problem 1 has an affirmative answer when X is a Bing space. These results are then applied in §3 to prove Theorems 3, 4 and 5. In §4, we state results which lead to the proof of Theorem 6. As their proofs are almost identical to the proofs of the results in §2 and §3, they are left to the reader.

2. Bing spaces. A continuum X (= a compact connected space) is *decomposable* if X is representable as $X = A \cup B$ with A, B proper subcontinua; otherwise X is called *indecomposable*. A compact space X is called a *Bing space* if every subcontinuum of X is indecomposable. Note that every continuum in a Bing space is hereditarily indecomposable, and that if A, B are continua in a Bing space with $A \cap B \neq \emptyset$ then either $A \subset B$ or $B \subset A$.

Bing [B] proved the following fundamental result.

BING'S THEOREM. *Let F and H be disjoint closed sets in a compact space X. Then there exists a Bing space $B \subset X$ which separates F from H in X.*

Bing's Theorem immediately implies the existence in every compact space of a basis whose elements' boundaries are Bing spaces; we call it a *Bing basis*.

THE BING BASIS THEOREM. *In every compact space there exists a countable basis B for the topology so that the boundary of each element of B is a Bing space.*

It also follows from Bing's Theorem that there exist Bing spaces of all (finite or infinite) dimensions.

In [B] Bing also proves the following result which he then applies to show that higher dimensional hereditarily indecomposable continua are not homogeneous. Since the proof is short and pretty, and as it demonstrates the special properties of Bing spaces and is essential for this note, we reproduce it here.

THE BING POINT THEOREM. *Let X be a k-dimensional Bing space (k finite or $k = \infty$). Then there exists a point p in X such that every nondegenerate continuum through p is k-dimensional.*

Proof. Let X_0 be a k-dimensional component of X. Let $X_1 \subset X_0$ be a continuum with $\operatorname{diam} X_1 \le \frac{1}{2} \operatorname{diam} X_0$. Such a continuum X_1 exists; indeed, one can cover X_0 by finitely many closed balls of diameter $\frac{1}{2} \operatorname{diam} X_0$; one of these balls must be k-dimensional and we may take X_1 to be a k-dimensional component of that ball.

Inductively we construct a decreasing sequence $X_0 \supset X_1 \supset X_2 \supset \ldots$ of k-dimensional continua with diameters tending to 0. Then $p = \bigcap X_i$ has the desired property. Indeed, let p belong to some nondegenerate subcontinuum A of X. Then either $X_i \subset A$ or $A \subset X_i$. But since $\operatorname{diam} X_i < \operatorname{diam} A$ for some i, A must contain X_i so $\dim A \ge \dim X_i = k$. ∎

Proposition 1. *Let W be a σ-compact subset of a Bing space. If W is the union of nondegenerate continua of dimension $\le k$ each then* $\dim W \le k$.

Proof. Let $W_0 \subset W$ be compact with $\dim W_0 = \dim W = n$. By Bing's Point Theorem there exists some point $p \in W$ such that each nondegenerate continuum through p is n-dimensional. But as $p \in W$, p belongs to some nondegenerate k-dimensional continuum. It follows that $n \le k$. ∎

Recall that a map is called *monotone* if its fibers are connected.

Proposition 2. *Let f be a k-dimensional monotone map on a Bing space. Let W denote the union of all the nondegenerate fibers of f (i.e., all those fibers which are not singletons). Then* $\dim W \le k$.

Proof. W is σ-compact and is the union of nondegenerate continua of dimension $\le k$ each. Thus $\dim W \le k$ by Proposition 1. ∎

Proposition 3. *Let f be a k-dimensional map on a Bing space X. Then* $\dim(f \times g) = 0$ *for almost all g in $C(X, \mathbb{R}^k)$, and there exists a σ-compact $(k-1)$-dimensional subset A of X such that* $\dim(f|X \setminus A) = 0$.

Proof. Let

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[dr, "f"'] & Z \arrow[d, "h"] \\ & f(X) \end{tikzcd}$$

denote the monotone-light decomposition of f with f_1 monotone and h light (= zero-dimensional; see [K], p. 84). Then $\dim f_1 = \dim f = k$. Let W denote the union of the nontrivial fibers of f_1. By Proposition 2, $\dim W \le k$. From a theorem of Hurewicz ([K], p. 125) and the σ-compactness of W it follows that $\dim(g|W) = 0$ for almost all g in $C(X, \mathbb{R}^k)$. For all such g, $\dim(f \times g) = 0$. Indeed, let L be a component of a fiber of $f \times g$. We must show that L is a singleton. If not, then as L is contained in some nondegenerate component of a fiber of f, it is also contained in a fiber of f_1 and hence $L \subset W$. But L is also contained in a fiber of g, and $\dim(g|W) = 0$, which implies that $\dim L = 0$. The existence of A follows from Lemma 1 as in the derivation of Theorem 2 from Theorem 1 in §1.

3. Proof of Theorems 3, 4 and 5. We prove the three theorems simultaneously. Let B be a (countable) Bing basis for X. For $U \in B$ there exists by Proposition 3 a σ-compact $(k-1)$-dimensional set A_U such that $\dim(f|\partial U \setminus A_U) = 0$. Then $A = \bigcup A_U$ satisfies the conclusion of Theorem 4 since each fiber L of $f|X \setminus A$ is covered by the two 0-dimensional sets $G = X \setminus \bigcup\{\partial U : U \in B\}$ and $L \cap \bigcup\{\partial U \setminus A_U : U \in B\}$.

To prove Theorem 5 set $E = A \cup G$, and to prove Theorem 3 note that for almost all $g : X \to I^k$ we have, by Proposition 3, $\dim(f \times g|\partial U) = 0$ for all U in B. Any such g satisfies $\dim(f \times g) \le 1$ by the same argument as above. ∎

Proof of Lemma 1. We first prove the case when $W = I = [0,1]$. Let $\Delta \subset I$ be a Cantor set and let $h : \Delta \to Y$ map Δ onto Y. Set

$$B_0 = \bigcup \{(h(t), t + r) : t \in \Delta,\ r \text{ rational}\} \subset Y \times I,$$

where the addition $t + r$ is taken in $\mathbb{R} \bmod 1$. Note that B_0 is a countable union of homeomorphic copies of the graph $\{(h(t), t) : t \in \Delta\}$ of h, which is a Cantor set. Hence B_0 is σ-compact and 0-dimensional. Let $y \in Y$. There exists some $t \in \Delta$ such that $h(t) = y$. Then $\{(y, t+r) : r \text{ rational}\} \subset \{y\} \times I$ and is dense there as $\{t + r : r \text{ rational}\}$ is dense in I. It follows that $\dim((\{y\} \times I) \setminus B_0) = 0$ and B_0 does the job for $Y \times I$.

Next, consider the case $W = I^k = I_1 \times \ldots \times I_k$. Let $P_i : Y \times I^k \to Y \times I_i$ denote the projection, let B_0^i denote the copy of B_0 in $Y \times I_i$ and set $A_i = P_i^{-1}(B_0^i)$. Then A_i as well as $A = \bigcup_{i=1}^k A_i$ are σ-compact $(k-1)$-dimensional subsets of $Y \times I^k$ and

$$\begin{aligned}(\{y\} \times I^k) \setminus A &= (\{y\} \times I^k) \setminus \bigcup_{i=1}^k A_i = \bigcap_{i=1}^k ((\{y\} \times I^k) \setminus A_i)\\ &= \bigcap_{i=1}^k (P_i^{-1}(\{y\} \times I_i) \setminus P_i^{-1}(B_0^i)) = \bigcap_{i=1}^k P_i^{-1}((\{y\} \times I_i) \setminus B_0^i)\\ &= \{y\} \times (I_1 \setminus B_0^1) \times (I_2 \setminus B_0^2) \times \ldots \times (I_k \setminus B_0^k).\end{aligned}$$

Hence $\dim((\{y\} \times I^k) \setminus A) = 0$ since $\dim(I_i \setminus B_0^i) = 0$ for all i.

Finally, let W be any k-dimensional compact space. Let $g : W \to I^k$ be 0-dimensional and set $B = (\mathrm{id} \times g)^{-1}(A) \subset Y \times W$. Then $\dim B \le k$ and since $(\{y\} \times W) \setminus B = (\mathrm{id} \times g)^{-1}((\{y\} \times W) \setminus A)$, $\dim((\{y\} \times W) \setminus B) = 0$, both by the Hurewicz Theorem. ∎

4. Proof of Theorem 6. Throughout this section we assume that Q is a property of separable metric spaces which satisfies Q_1, Q_2, and Q_3 of §1, and that P is the complementary property, i.e. $X \in P$ if and only if $X \notin Q$. The Q-versions of results in §2 and §3 are marked by $*$.

Bing's Point Theorem*. *Let X be a Bing space in P. There exists a point $p \in X$ such that each nondegenerate continuum through p in X is in P.* ∎

Note that Q_2 is needed here merely for a *finite* union of compacta.

Proposition 1*. *Let W be a σ-compact subset of a Bing space. If W is the union of nondegenerate continua in Q then $W \in Q$.* ∎

(Q_1 is applied here in its full generality.)

Proposition 2*. *Let f be a monotone Q-map on a Bing space. Let W denote the union of all nondegenerate fibers of f. Then $W \in Q$.*

Corollary. *Let f be a Q-map on a Bing space. Let V denote the union of all nondegenerate components of fibers of f. Then V is a σ-compact element of Q.* ∎

Proposition 3*. *Let f be a Q-map on a Bing space X. There exists a σ-compact Q-subset V of X (namely the set V of the above corollary) such that* $\dim(f|X \setminus V) = 0$.

Remark. The proof of Proposition 3* differs from that of Proposition 3, but is straightforward. The following fact is applied: Let H be compact and let $L = H \setminus \{$the union of all nondegenerate components of $H\}$. Then $\dim L = 0$.

Applying these propositions one proves Theorem 6 by constructing the sets A and G as in the proof of Theorem 4. ∎

References

[B] R. H. Bing, *Higher dimensional hereditarily indecomposable continua*, Trans. Amer. Math. Soc. 71 (1951), 653–663.

[D-R] T. Dobrowolski and L. Rubin, *The hyperspaces of infinite dimensional compacta for covering and cohomological dimension are homeomorphic*, Pacific J. Math. 104 (1994), 15–39.

[K] K. Kuratowski, *Topology II*, PWN, Warszawa, 1968.

[Le] M. Levin, *A short construction of hereditarily infinite dimensional compacta*, Topology Appl., to appear.

[Pa] B. A. Pasynkov, *On dimension and geometry of mappings*, Dokl. Akad. Nauk SSSR 221 (1975), 543–546 (in Russian).

[Po] R. Pol, *On light mappings without perfect fibers on compacta*, preprint.

[T] H. Toruńczyk, *Finite to one restrictions of continuous functions*, Fund. Math. 75 (1985), 237–249.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HAIFA
HAIFA 31905, ISRAEL
E-mail: YAKI@MATHCS2.HAIFA.AC.IL

Received 29 August 1994;
in revised form 12 December 1994