

Multifractal properties of the sets of zeroes of Brownian paths

by

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Abstract. We study Brownian zeroes in the neighborhood of which one can observe a non-typical growth rate of Brownian excursions. We interpret the multifractal curve for the Brownian zeroes calculated in [6] as the Hausdorff dimension of certain sets. This provides an example of the multifractal analysis of a statistically self-similar random fractal when both the spacing and the size of the corresponding nested sets are random.

1. Introduction

1.1. Notations. In this article we deal with the multifractal structure of zeroes of a Brownian path. A *Brownian path*, denoted by $\omega(t)$, is a point of the space $C[0, 1]$ equipped with the Wiener measure denoted by P . Recall that this measure is specified by the condition that for disjoint intervals $[t_1^1, t_2^1], [t_1^2, t_2^2], \dots, [t_1^n, t_2^n]$ the corresponding increments of the Brownian curve $\omega(t_2^1) - \omega(t_1^1), \omega(t_2^2) - \omega(t_1^2), \dots, \omega(t_2^n) - \omega(t_1^n)$ are independent normal variables with mean values 0 and variances $t_2^k - t_1^k$.

The set of zeroes $\{t : \omega(t) = 0\}$ is denoted by $Z[0, 1]$. It is random as long as ω is random. It is also well known that $Z[0, 1]$ is closed, nowhere dense and its Hausdorff dimension $h\text{-dim}(Z[0, 1])$ is $1/2$ for a.e. ω . The purpose of this paper is to study the fine structure of $Z[0, 1]$. Denote by $\text{Cm}[0, 1]$ the complement to $Z[0, 1]$. It is an open set consisting of a countable set of intervals. Take $\varepsilon > 0$ and delete from $[0, 1]$ all intervals of that set whose length is not less than ε . The connected components of the remaining set will be called ε -clusters. We denote them by $K_i(\varepsilon)$ (counting from left to right). Sometimes it will be convenient to consider ε -clusters on the whole halfline, assuming

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that the Wiener measure is considered on the space $C[0, \infty)$. We denote the ε -cluster containing $t \in [0, 1]$ by $K(\varepsilon, t)$. We also use the following notation:

- $L(t)$ is the local time on $Z[0, 1]$ (for the definition and basic properties of the local time see [5]); in Subsection 1.2 we discuss some properties of the local time connected with the fractal structure of $Z[0, 1]$;
- $l_i(\varepsilon), l(\varepsilon, t)$ are the increments of the local time on $K_i(\varepsilon)$ and $K(\varepsilon, t)$ respectively;
- $\delta_i(\varepsilon), \delta(\varepsilon, t)$ are the lengths of $K_i(\varepsilon)$ and $K(\varepsilon, t)$;
- $\Delta_i(\varepsilon)$ is the distance between $K_i(\varepsilon)$ and $K_{i+1}(\varepsilon)$;
- H^s is the s -dimensional Hausdorff measure; H_ε^s is the corresponding ε -measure ($H_\varepsilon^s(A) = \inf \sum_i |I_i|^s$, where the infimum is taken over all coverings of the set A with sets of diameter less than ε and $|\cdot|$ denotes the diameter);
- $\varepsilon_m = (1/2)^m$;
- $A_m(\gamma) = \left\{ t : \frac{\ln l(\varepsilon_m, t)}{\ln \delta(\varepsilon_m, t)} \geq \frac{1}{2} + \gamma \right\}$;
- $B_m(\gamma) = \{ t : \delta(\varepsilon_m, t) \leq \varepsilon_m^{1+\gamma} \}$;
- $C_m(\gamma) = \{ t : l(\varepsilon_m, t) \leq \varepsilon_m^{1/2+\gamma} \}$;
- $\nu_m([a, b])$ is the number of ε_m -clusters intersecting the segment $[a, b]$.

During the proofs we omit some indices if it does not lead to misunderstanding (for example we usually write $\delta(t)$ and A_m). All statements about $Z[0, 1]$ hold only for a subset of probability 1 even if we do not mention that explicitly.

1.2. Fractal geometry of $Z[0, 1]$ and local time. As was already mentioned, $h\text{-dim}(Z[0, 1]) = 1/2$. This fact follows from a stronger theorem. Define $\phi(s) = \sqrt{s \ln \ln s}$ and put

$$\phi\text{-}m(A) = \liminf_{\varepsilon \rightarrow 0} \sum_i \phi(|I_i|),$$

where the infimum is taken over all coverings of the set A by intervals of length less than ε . Then

$$(1.1) \quad \phi\text{-}m(Z[0, 1]) = \text{const} \cdot L(1) \quad (\text{see [9]}).$$

There is another curious property of the local time. According to the Frostman lemma (see [3]), for a given set A , for any $s < h\text{-dim}(A)$ one can find a measure $\mu(s)$ and a constant $c(s)$ such that $\mu([x, y]) < c(s) |x - y|^s$ for any x, y . For $A = Z[0, 1]$ we can describe this measure explicitly. Indeed, for sufficiently small ε the following inequality holds for a set of Wiener measure 1:

$$(1.2) \quad L(t + \varepsilon) - L(t) < \sqrt{3\varepsilon \ln(1/\varepsilon)} \quad (\text{see [5]}).$$

1.3. Main result. (1.2) implies that for all $t \in Z[0, 1]$,

$$\liminf_{\varepsilon \rightarrow 0} \frac{\ln(L(t + \varepsilon) - L(t))}{\ln \varepsilon} \geq \frac{1}{2}.$$

This can be reformulated in the following way: for all $t \in Z[0, 1]$,

$$\liminf_{m \rightarrow \infty} \frac{\ln l(\varepsilon_m, t)}{\ln \delta(\varepsilon_m, t)} \geq \frac{1}{2}.$$

The goal of this paper is to strengthen the last inequality.

THEOREM 1. For any γ with $0 < \gamma < 1/4$ and for a.e. ω ,

$$\begin{aligned} h\text{-dim} \left\{ t : \liminf_{m \rightarrow \infty} \frac{\ln l(\varepsilon_m, t)}{\ln \delta(\varepsilon_m, t)} \geq \frac{1}{2} + \gamma \right\} &= 0, \\ h\text{-dim} \left\{ t : \limsup_{m \rightarrow \infty} \frac{\ln l(\varepsilon_m, t)}{\ln \delta(\varepsilon_m, t)} \geq \frac{1}{2} + \gamma \right\} &= \frac{1}{2} - 2\gamma, \end{aligned}$$

or equivalently,

$$h\text{-dim} \liminf_{m \rightarrow \infty} A_m(\gamma) = 0, \quad h\text{-dim} \limsup_{m \rightarrow \infty} A_m(\gamma) = \frac{1}{2} - 2\gamma.$$

This theorem implies that

$$H^{1/2} \left\{ t : \lim_{m \rightarrow \infty} \frac{\ln l(\varepsilon_m, t)}{\ln \delta(\varepsilon_m, t)} \neq \frac{1}{2} \text{ or the limit of this ratio fails to exist} \right\} = 0,$$

while $H^{1/2}(Z[0, 1]) = \infty$ by (1.1).

1.4. Dimension of other singularities. The method used to prove Theorem 1 is also applicable to the investigation of $\liminf_{m \rightarrow \infty} B_m$, $\liminf_{m \rightarrow \infty} C_m$, $\limsup_{m \rightarrow \infty} B_m$, $\limsup_{m \rightarrow \infty} C_m$, that is, respectively,

$$\begin{aligned} &\left\{ t : \liminf_{m \rightarrow \infty} \frac{\ln \delta(\varepsilon_m, t)}{\ln \varepsilon_m} \geq 1 + \gamma \right\}, \\ &\left\{ t : \liminf_{m \rightarrow \infty} \frac{\ln l(\varepsilon_m, t)}{\ln \varepsilon_m} \geq \frac{1}{2} + \gamma \right\}, \\ &\left\{ t : \limsup_{m \rightarrow \infty} \frac{\ln \delta(\varepsilon_m, t)}{\ln \varepsilon_m} \geq 1 + \gamma \right\}, \\ &\left\{ t : \limsup_{m \rightarrow \infty} \frac{\ln l(\varepsilon_m, t)}{\ln \varepsilon_m} \geq \frac{1}{2} + \gamma \right\}. \end{aligned}$$

Roughly speaking, the structure of these sets is the following. Each X_m consists of about $(1/\varepsilon_m)^\varrho$ segments (where X is any of A , B or C), which are ‘‘almost equidistributed’’ on the segment $[0, 1]$, and most of these segments have length of order ε_m^θ . In this case $h\text{-dim}(\liminf X_m) = 0$ and $h\text{-dim}(\limsup X_m) = \varrho/\theta$.

Moreover, one can deduce from the proof that if we replace $\varepsilon_m = (1/2)^m$ by an arbitrary sequence $\tilde{\varepsilon}_m$, then the following statements hold with probability 1:

- if $\lim_{m \rightarrow \infty} \tilde{\varepsilon}_{m-1}/\tilde{\varepsilon}_m = \infty$, then $h\text{-dim}(\liminf X_m) = \varrho/\theta$;
- if $\tilde{\varepsilon}_{m-1}/\tilde{\varepsilon}_m$ remains bounded then $h\text{-dim}(\liminf X_m) = 0$. (Of course, the exceptional sets of measure 0 where neither of the statements above holds may be different for different sequences).

The plan of our paper is the following. In Subsection 1.5 we explain our results using the notion of multifractality applied to the set $Z[0, 1]$ (equipped with $L(t)$). In Section 2 we present some facts about the distribution of $l_i(\varepsilon)$, $\delta_i(\varepsilon)$, and $\Delta_i(\varepsilon)$. The proof of Theorem 1 is contained in Sections 3 and 4. In Section 3 we describe the set of Wiener measure 1 for which the statement of Theorem 1 is true. In Section 4 we give the proof of the main statement for this set. Essentially it does not differ too much from the one in the case when X_m is the union of $(1/\varepsilon_m)^\varrho$ equidistributed segments of length ε_m^θ . Finally, in Section 5 we calculate the above stated dimensions for B_m and C_m . Since the proof in this case almost completely coincides with the proof of Theorem 1, we restrict ourselves to the calculations of ϱ and θ . The answer is the following:

PROPOSITION 1. *With probability 1, for $0 < \gamma \leq 1$,*

$$h\text{-dim}(\liminf B_m) = 0, \quad h\text{-dim}(\limsup B_m) = \frac{1}{2} \cdot \frac{1 - \gamma}{1 + \gamma}.$$

PROPOSITION 2. *With probability 1, for $0 < \gamma \leq 1/2$,*

$$h\text{-dim}(\liminf C_m) = 0, \quad h\text{-dim}(\limsup C_m) = \frac{1}{2} \cdot \frac{1 - 2\gamma}{1 + 2\gamma}.$$

1.5. Singular points and the multifractal structure of $Z[0, 1]$. As one will see in Section 2, typical ε_m -clusters have size of order ε_m and for most of them

$$\frac{\ln l_i}{\ln \delta_i} \approx \frac{1}{2}.$$

At the same time there exist few ε_m -clusters for which

$$\alpha < \frac{\ln l_i}{\ln \delta_i} \leq \alpha + \Delta\alpha, \quad \text{where } \alpha \neq \frac{1}{2}.$$

For some α the share of such clusters varies polynomially with ε_m , i.e. it approximately equals $\varepsilon_m^{f(\alpha)}$ ($f(\alpha) = 3/2 - 2\alpha$, where $\alpha \in [1/2, 3/4]$ (cf. [6])). In this case one says that $f(\alpha)$ lies in the *multifractal spectrum* of $Z[0, 1]$. The multifractal structure of $Z[0, 1]$ is studied in a different way in [6].

In our paper we interpret $f(\alpha)$ as the Hausdorff dimension of $\limsup A_m$. It is quite clear why we use \limsup (instead of \liminf). Indeed, it reflects the very complicated behavior of

$$r_t(\varepsilon) = \frac{\ln l(\varepsilon, t)}{\ln \delta(\varepsilon, t)}$$

as a function of ε . Really, $r_t(\varepsilon)$ is piecewise constant, and it grows up at points whose coordinates are equal to the length of the intervals from $\text{Cm}[0, 1]$, lying close to t . The quite complicated structure of $\text{Cm}[0, 1]$ as compared with the complement to Cantor dust, for example, explains the chaotic behavior of $r_t(\varepsilon)$.

2. Basic distributions related to ε -clusters. Here we give some properties of the distributions related to ε -clusters which will be used in the following sections. The proofs can be found in [6].

PROPOSITION 3. (a) *The triples $(\Delta_i, \delta_i, l_i)$ are independent and identically distributed.*

(b) *The pair (δ_i, l_i) of random variables does not depend on Δ_i for any i .*

(c) *Introduce new random variables ξ_i^-, ξ_i^+ , and η_i , where $\delta_i(\varepsilon) = \varepsilon \xi_i^-$, $\Delta_i(\varepsilon) = \varepsilon \xi_i^+$, and $l_i(\varepsilon) = \sqrt{\pi \varepsilon / 2} \eta_i$. Then the distribution of ξ_i^+, ξ_i^- and η_i does not depend on ε and:*

(d) *η_i has exponential distribution with mean value 1, i.e. $F_{\eta_i}(x) = 1 - \exp\{-x\}$.*

(e) *The distribution function of ξ_i^+ is $F_{\xi_i^+}(x) = 1 - 1/\sqrt{x}$, $x > 1$.*

(f) *ξ_i^- all have positive moments and*

$$F_{\xi_i^-}(x) = \frac{1}{\sqrt{\pi} \sqrt{x}}(1 + O(1)), \quad x \rightarrow 0.$$

(g) *$P(\xi_i^- > s^\beta, \eta < s^\gamma) \sim \text{const}(\gamma, \beta) \cdot s^{2\gamma - \beta/2}$, $s \rightarrow 0$, $0 < \beta/2 < \gamma < 1/2$.*

(h) *Fix γ , $0 < \gamma < 1$. Call an ε_m -cluster poor in zeroes if it belongs to $A_m(\gamma)$. The probability of the event “ $K_i(\varepsilon)$ is poor in zeroes” has asymptotics $\text{const} \cdot \varepsilon^{2\gamma}$ as $\varepsilon \rightarrow 0$ when $\gamma \neq 1/2$, $\gamma < 1$ (we are only interested in a dense set of γ).*

(i) *A cluster which is poor in zeroes and satisfies the inequality $\varepsilon/2 < \delta_i < \varepsilon$ will be called a standard ε_m -cluster. Then the probability of a standard cluster has the same asymptotics as in (h), i.e. $\text{const} \cdot \varepsilon^{2\gamma}$.*

3. Description of the set of full measure where our results are valid

3.1. Number of ε -clusters. Recall that $\nu_m([a, b])$ is the number of ε_m -clusters intersecting the interval $[a, b]$.

LEMMA 1. For any given δ and a.e. ω , and for almost all m (i.e. all m except a finite set),

$$\left(\frac{1}{\varepsilon_m}\right)^{1/2-\delta} < \nu_m([0, 1]) < \left(\frac{1}{\varepsilon_m}\right)^{1/2+\delta}.$$

Proof. We have

$$\nu_m([0, 1]) = \min \left\{ n : \sum_{i=1}^n l_i(\varepsilon) \geq L(1) \right\} = \min \left\{ n : \sum_{i=1}^n \eta_i \geq \sqrt{\frac{2}{\pi\varepsilon}} L(1) \right\}.$$

We shall use Bernstein's inequality in the following form: let Z_i be independent and exponentially distributed random variables with mean value 1; then there are positive constants c_1, c_2, c_3, c_4 such that

$$(3.1) \quad P\left(c_1 n < \sum_{i=1}^n Z_i < c_2 n\right) > 1 - c_3 \exp\{-c_4 n\}.$$

In particular,

$$(3.2) \quad P\left(\sum_{i=1}^{(1/\varepsilon_m)^{1/2-\delta}} \eta_i(\varepsilon_m) > c_2 \left(\frac{1}{\varepsilon_m}\right)^{1/2-\delta}\right) < c_3 \exp\left\{-c_4 \left(\frac{1}{\varepsilon_m}\right)^{1/2-\delta}\right\}$$

and

$$(3.3) \quad P\left(\sum_{i=1}^{(1/\varepsilon_m)^{1/2+\delta}} \eta_i(\varepsilon_m) < c_1 \left(\frac{1}{\varepsilon_m}\right)^{1/2+\delta}\right) < c_3 \exp\left\{-c_4 \left(\frac{1}{\varepsilon_m}\right)^{1/2+\delta}\right\}.$$

By the Borel–Cantelli lemma, the inequalities in (3.2) and (3.3) with probability 1 take place only a finite number of times.

Since

$$c_2 \left(\frac{1}{\varepsilon_m}\right)^{1/2-\delta} < L(1) \sqrt{\frac{1}{\varepsilon_m}} < c_1 \left(\frac{1}{\varepsilon_m}\right)^{1/2+\delta}$$

if m is large enough, the lemma is proven. ■

LEMMA 2. For any given δ , with probability 1, for all m large enough the number of ε_m -clusters poor in zeroes is between $(1/\varepsilon_m)^{1/2-2\gamma-\delta}$ and $(1/\varepsilon_m)^{1/2-2\gamma+\delta}$.

Proof. Let us prove, for example, the lower estimate. In view of Lemma 1 it is sufficient to show that for all m except a finite number, no less than $(1/\varepsilon_m)^{1/2-2\gamma-\delta}$ ε_m -clusters of the first $(1/\varepsilon_m)^{(1-\delta)/2}$ of them are poor in zeroes. The probability of the complementary event is

$$\underline{P}(m) = \sum_{k=0}^{(1/\varepsilon_m)^{1/2-2\gamma-\delta}} b\left(\left(\frac{1}{\varepsilon_m}\right)^{(1-\delta)/2}, k, p_m\right),$$

where $b(n, k, p) = C_n^k p^k (1 - p)^{n-k}$ and $p_m = P\{\varepsilon_m\text{-cluster with a given number is poor in zeroes}\} \sim \text{const} \cdot \varepsilon^{2\gamma}$ (see Proposition 3(h)). The inequality

$$\sum_{k=0}^l b(n, k, p) < b(n, l, p) \frac{np - k}{np - kp},$$

which is valid for $k < np$, in our case gives

$$(3.4) \quad \underline{P}(m) < \frac{c(1/\varepsilon_m)^{1/2-2\gamma-\gamma/2} - (1/\varepsilon_m)^{1/2-2\gamma-\delta}}{c((1/\varepsilon_m)^{1/2-2\gamma-\delta/2} - (1/\varepsilon_m)^{1/2-4\gamma-\delta})} \\ \times \text{const} \cdot b\left(\left(\frac{1}{\varepsilon_m}\right)^{(1-\delta)/2}, \left(\frac{1}{\varepsilon_m}\right)^{1/2-2\gamma-\delta}, p_m\right).$$

It is easy to show (using Stirling's formula) that

$$\underline{P}(m) < c_5(\delta) \left(\frac{1}{\varepsilon_m}\right)^{c_6(\delta)} \exp\left\{-c_7\left(\frac{1}{\varepsilon_m}\right)^{1/2-\delta/2-2\gamma}\right\}.$$

Hence,

$$\sum_{m=1}^{\infty} \underline{P}(m) < \infty.$$

In the same way, if we define $\bar{P}(m)$ as

$$\bar{P}(m) = P\{\text{no less than } (1/\varepsilon_m)^{1/2-2\gamma+\delta} \text{ of the first } (1/\varepsilon_m)^{1/2+\delta/2} \\ \varepsilon_m\text{-clusters are poor in zeroes}\},$$

using again Stirling's formula and the inequality

$$\sum_{k=l}^n b(n, k, p) < b(n, l, p) \frac{k - np}{k - kp}$$

(which is valid for $k > np$), we obtain

$$\bar{P}(m) < c_8(\delta) \left(\frac{1}{\varepsilon_m}\right)^{c_9(\delta)} \exp\left\{-c_{10}\left(\frac{1}{\varepsilon_m}\right)^{1/2-\delta/2-2\gamma}\right\},$$

and therefore $\sum_{m=1}^{\infty} \bar{P}(m) < \infty$. Lemma 2 is proven. ■

PROPOSITION 4. *The statement of Lemma 2 also holds for standard ε_m -clusters.*

PROOF. $P\{\text{a certain } \varepsilon_m\text{-cluster is standard}\}$ has the same asymptotics as p_m . ■

3.2. *Decay of ε_m -clusters' size.* Denote by r_m the maximal size of ε_m -clusters in $[0,1]$.

LEMMA 3. *With probability 1,*

$$\limsup_{m \rightarrow \infty} \frac{\ln r_m}{\ln \varepsilon_m} \geq 1 \quad (\text{and, therefore, } \limsup_{m \rightarrow \infty} \frac{\ln r_m}{\ln \varepsilon_m} = 1).$$

Proof. Fix $\delta > 0$. It is sufficient to show that $r_m \leq \varepsilon^{1-\delta}$ for all m large enough. By Lemma 1 it is sufficient to prove that for all m large enough,

$$\max_{j \leq (1/\varepsilon_m)^{1/2+\delta}} \delta_j(\varepsilon_m) \leq \varepsilon_m^{1-\delta}, \quad \text{i.e.} \quad \max_{j \leq (1/\varepsilon_m)^{1/2+\delta}} \xi_j^- \leq \left(\frac{1}{\varepsilon_m}\right)^\delta.$$

Now,

$$\begin{aligned} P\left(\max_{j \leq (1/\varepsilon_m)^{1/2+\delta}} \xi_j^- \leq \left(\frac{1}{\varepsilon_m}\right)^\delta\right) &\leq \left(\frac{1}{\varepsilon_m}\right)^{1/2+\delta} P\left(\xi_j^- \geq \left(\frac{1}{\varepsilon_m}\right)^\delta\right) \\ &\leq \varepsilon^{3/2-\delta} E((\xi_j^-)^{2/\delta}). \end{aligned}$$

So,

$$\sum_{m=1}^{\infty} P\left(\max_{j \leq (1/\varepsilon_m)^{1/2+\delta}} \delta_j(\varepsilon_m) \leq \varepsilon_m^{1-\delta}\right) < \infty$$

and the lemma is proven. ■

Now we introduce two sequences of numbers:

$$k_n = \left(\frac{1/2 + \gamma + \delta}{1/2 - 2\gamma - \delta}\right)^n \quad \text{and} \quad \varepsilon(n) = 2^{-2^{n^2}}.$$

3.3. Equidistribution of ε -clusters

LEMMA 4. *Fix $\delta > 0$. Then the following statements hold a.e.:*

(a) *Fix a natural n . Then for almost all m and for any j ($1 \leq j \leq n$), any $\varepsilon_m^{k_j-1}$ -cluster poor in zeroes contains no more than*

$$\left(\frac{1}{\varepsilon_m}\right)^{[(k_j - k_{j-1})/2 - \gamma k_j - 2\gamma k_{j-1} + \delta(k_j + k_{j-1})]}$$

$\varepsilon_m^{k_j}$ -clusters which are poor in zeroes.

(b) *For almost all n the number of standard $\varepsilon(n)$ -clusters falling inside any standard $\varepsilon(n)$ -cluster lies between*

$$2^{[(2^{n^2} - 2^{(n-1)^2})/2 - \gamma 2^{(n-1)^2} - 2\gamma 2^{n^2} - \delta 2^{n^2}]}$$

and

$$2^{[(2^{n^2} - 2^{(n-1)^2})/2 - \gamma 2^{(n-1)^2} - 2\gamma 2^{n^2} + \delta 2^{n^2}]}.$$

Proof. We only prove the statement (a) (the proof of (b) is similar).

By Lemma 3, if m is large enough, the increment of the local time on any $\varepsilon_m^{k_j-1}$ -cluster poor in zeroes is less than $(\varepsilon_m^{k_j-1})^{1/2+\gamma-\delta}$, since otherwise

$$r_{mk_{j-1}} > \varepsilon_m^{k_j-1(1/2+\gamma-\delta)/(1/2+\gamma)}$$

and the number of these clusters is less than $(1/\varepsilon_m)^{k_j-1(1/2-2\gamma+\delta)}$. Let $l(K)$ be the number of $\varepsilon_m^{k_j}$ -clusters whose left end point coincides with the left end point of the $\varepsilon_m^{k_j-1}$ -cluster K . Then the number $n(K)$ of $\varepsilon_m^{k_j}$ -clusters lying inside K is less than

$$\begin{aligned} \min \left\{ n : \sum_{i=l}^{n+l} l_i(\varepsilon_m^{k_j}) \geq (\varepsilon_m^{k_j-1})^{1/2+\gamma-\delta} \right\} \\ = \min \left\{ \sum_{i=l}^{n+l} \eta_i \geq \left(\frac{1}{\varepsilon_m} \right)^{[(k_j-k_{j-1})/2-\gamma k_{j-1}+\delta k_{j-1}]} \right\}. \end{aligned}$$

By the estimate (3.2),

$$\sum_{m=1}^{\infty} \left(\frac{1}{\varepsilon_m} \right)^{k_j(1/2-2\gamma+\delta)} P \left\{ n(K) > \left(\frac{1}{\varepsilon_m} \right)^{[(k_j-k_{j-1})/2-\gamma k_{j-1}+\delta k_{j-1}+\delta k_j/2]} \right\} < \infty.$$

So, for all m large enough,

$$n(K) < \left(\frac{1}{\varepsilon_m} \right)^{[(k_j-k_{j-1})/2-\gamma k_{j-1}+\delta k_{j-1}+\delta k_j/2]}$$

and the estimate (3.2) implies that

$$\begin{aligned} \sum_{m=1}^{\infty} \left(\frac{1}{\varepsilon_m} \right)^{k_j(1/2-2\gamma+\delta)} P \{ \# \{ i : l(K) \leq i \leq \\ l(K) + (1/\varepsilon_m)^{[(k_j-k_{j-1})/2-\gamma k_{j-1}-\delta k_{j-1}-\delta k_j/2]} \\ \text{and } K_i(\varepsilon_m^{k_j}) \text{ is poor in zeroes} \} > \\ (1/\varepsilon_m)^{[(k_j-k_{j-1})/2-\gamma k_{j-1}-2\gamma k_j-\delta k_{j-1}-\delta k_j/2]} \} < \infty. \end{aligned}$$

The lemma is proven. ■

LEMMA 5. For any positive δ there is a constant $c(\delta)$ such that for a.e. ω and almost all n , any interval in $[0, 1]$ containing $c(\delta)$ standard $\varepsilon(n)$ -clusters contains no less than $(1/\varepsilon(n))^{2\gamma-\delta}$ $\varepsilon(n)$ -clusters.

Proof. By Lemma 2 it is sufficient to consider the case when the number of standard $\varepsilon(n)$ -clusters does not exceed $(1/\varepsilon(n))^{1/2-2\gamma+\delta}$. Then

$$\begin{aligned}
 & P\{\text{there exists an interval containing less than } (1/\varepsilon(n))^{1/2-2\gamma+\delta} \\
 & \quad \varepsilon(n)\text{-clusters among which } c(\delta) \text{ are standard}\} \\
 & \leq P\{\text{the interval beginning from a given standard } \varepsilon(n)\text{-cluster and} \\
 & \quad \text{containing } c(\delta) \text{ of them does not cover } (1/\varepsilon(n))^{2\gamma-\delta} \varepsilon(n)\text{-clusters}\} \\
 & \quad \times (1/\varepsilon(n))^{1/2-2\gamma+\delta} \\
 & \leq [P\{\text{there are less than } (1/\varepsilon(n))^{2\gamma-\delta} \varepsilon(n)\text{-clusters} \\
 & \quad \text{falling between two neighboring standard ones}\}]^{c(\delta)} \\
 & \quad \times (1/\varepsilon(n))^{1/2-2\gamma+\delta} \\
 & \leq (c_{11}\varepsilon(n)^{2\gamma})^{c(\delta)-1} (1/\varepsilon(n))^{1/2-2\gamma+\delta} \leq c_{11}^{c(\delta)-1} \varepsilon(n)^{2\gamma c(\delta)-1/2+\delta},
 \end{aligned}$$

i.e. for example, $1/\gamma + 1$ is a possible value for $c(\delta)$ and the lemma is proven. ■

LEMMA 6. Fix $\delta > 0$. For a.e. ω and almost all n , any interval I on the t -axis containing k standard $\varepsilon(n)$ -clusters has length exceeding

$$\varepsilon(n) \left\{ \left[\frac{k}{c(\delta)} \right] \left(\frac{1}{\varepsilon(n)} \right)^{2\gamma-\delta} \right\}^{2-\delta}.$$

Proof. By Lemma 5 it is sufficient to give the proof in the case when the number of $\varepsilon(n)$ -clusters in I is more than $[k/c(\delta)](1/\varepsilon(n))^{2\gamma-\delta}$.

Let us number the standard $\varepsilon(n)$ -clusters and denote by $p_{jk}(n)$ the probability of the event that the maximal distance between neighboring $\varepsilon(n)$ -clusters in the interval beginning from the j th standard $\varepsilon(n)$ -cluster and containing $[k/c(\delta)](1/\varepsilon(n))^{2\gamma-\delta} \varepsilon(n)$ -clusters, is less than

$$\varepsilon(n) \{ [k/c(\delta)] (1/\varepsilon(n))^{2\gamma-\delta} \}^{2-\delta}.$$

It is sufficient to check the convergence of the series

$$\begin{aligned}
 & \sum_{n=1}^{\infty} (1/\varepsilon(n))^{1/2-2\gamma+\delta} \sum_{j=1}^{(1/\varepsilon(n))^{1/2-2\gamma+\delta}} \sum_{k=1}^{[k/c(\delta)](1/\varepsilon(n))^{2\gamma-\delta}} p_{kj}(n) \\
 & \leq \sum_{n=1}^{\infty} \left(\sum_{k=1}^{(1/\varepsilon(n))^{1/2-2\gamma+\delta}} p_{1k}(n) \right) \left(\frac{1}{\varepsilon(n)} \right)^{1/2-2\gamma+\delta}.
 \end{aligned}$$

Now,

$$p_{1k}(n) = \left[P \left\{ \Delta < \varepsilon(n) \left\{ \left[\frac{k}{c(\delta)} \right] \left(\frac{1}{\varepsilon(n)} \right)^{2\gamma-\delta} \right\}^{2-\delta} \right\} \right]^{[k/c(\delta)](1/\varepsilon(n))^{2\gamma-\delta}}$$

$$\begin{aligned} &\sim \left(1 - \frac{1}{\sqrt{\{[k/c(\delta)](1/\varepsilon(n))^{2\gamma-\delta}\}^{2-\delta}}}\right)^{[k/c(\delta)](1/\varepsilon(n))^{2\gamma-\delta}} \\ &< \exp \left\{ -c_{12} \left(\left[\frac{k}{c(\delta)} \right] \left(\frac{1}{\varepsilon(n)} \right)^{2\gamma-\delta} \right)^{\delta/2-\delta/4} \right\} \\ &= \exp \left\{ -c_{12} \left(\left[\frac{k}{c(\delta)} \right] \left(\frac{1}{\varepsilon(n)} \right)^{2\gamma-\delta} \right)^{\delta/4} \right\}. \end{aligned}$$

The last inequality is valid when n is large enough (we have used the asymptotics $\ln p_{1k}(n) \sim ([k/c(\delta)](1/\varepsilon(n))^{2\gamma-\delta})^{\delta/4}$). ■

4. Geometrical considerations. In this part we consider those Brownian paths where the statements of Lemmas 1–6 and Proposition 4 are valid for all positive δ .

4.1. Dimension of the lower limit of A_m

LEMMA 7. $h\text{-dim}(\liminf_{m \rightarrow \infty} A_m) = 0$.

Proof. Denote by $A(m, n)$ the set $\{t : K(\varepsilon_m^{k_j}, t) \text{ is poor in zeroes for all } j \text{ with } 0 \leq j \leq n\}$. Denote by $N_m(n)$ the number of $\varepsilon_m^{k_n}$ -clusters included in $A(m, n)$. Then

$$\begin{aligned} N_m(n) &\leq (\text{the number of } \varepsilon_m\text{-clusters poor in zeroes}) \\ &\quad \times \prod_{j=1}^n (\text{the maximal number of } \varepsilon_m^{k_j}\text{-clusters} \\ &\quad \quad \quad \text{poor in zeroes inside an } \varepsilon_m^{k_{j-1}}\text{-cluster}) \\ &\leq \left(\frac{1}{\varepsilon_m}\right)^{1/2-2\gamma+\delta} \prod_{j=1}^n \left(\frac{1}{\varepsilon_m}\right)^{k_j(1/2-2\gamma+\delta)-k_{j-1}(1/2+\gamma-\delta)} = \left(\frac{1}{\varepsilon_m}\right)^\varphi, \end{aligned}$$

where

$$\varphi = \left(\frac{1}{2} - 2\gamma + \delta\right) + \frac{\delta \left[3 \frac{1/2 + \gamma + \delta}{1/2 - 2\gamma - \delta} + 2 \right] (k_n - 1)}{\frac{1/2 + \gamma + \delta}{1/2 - 2\gamma - \delta} - 1}.$$

By Lemma 3,

$$H_{(\varepsilon_m^{k_n})^{1-\delta}}^s(A(m, n)) \leq \varepsilon_m^{k_n(1-\delta)s-\varphi}.$$

Since δ is arbitrarily small, we conclude that $h\text{-dim}(\bigcap_{k=m}^\infty \bigcap_{n=1}^\infty A(k, n)) = 0$. But $\liminf A_m \subset \bigcup_{m=1}^\infty \bigcap_{k=m}^\infty \bigcap_{n=1}^\infty A(k, n)$, and the lemma is proven. ■

4.2. Upper estimate for the upper limit's dimension. To obtain an upper estimate of $h\text{-dim}(\limsup A_m)$ we need the following lemma.

LEMMA 8. Let sets X_n and a sequence $\varepsilon_n \rightarrow 0$ be such that

$$\sum_{n=1}^{\infty} H_{\varepsilon_n}^s(X_n) < \infty.$$

Then $h\text{-dim}(\limsup X_n) \leq s$.

Proof. We have

$$H_{\varepsilon_n}^s(\limsup X_n) \leq H_{\varepsilon_n}^s\left(\bigcup_{k=n}^{\infty} X_k\right) \leq \sum_{k=n}^{\infty} H_{\varepsilon_k}^s(X_k) \rightarrow 0. \blacksquare$$

COROLLARY 1. $h\text{-dim}(\limsup A_m) \leq 1/2 - 2\gamma$.

Proof. For any positive s ,

$$H_{\varepsilon_m}^s(A_m) \leq \text{const}(s) \left(\frac{1}{\varepsilon_m}\right)^{1/2-2\gamma+\delta} \varepsilon_m^{(1-\delta)s},$$

i.e.

$$\text{if } s > \frac{1/2 - 2\gamma + \delta}{1 - \delta}, \text{ then } h\text{-dim}(\limsup A_m) \leq s.$$

Since δ is arbitrarily small, the proof is complete. \blacksquare

4.3. Lower estimate for the upper limit's dimension. Let us now consider the n from which on the statement of Lemma 4(b) is true. Take an arbitrary standard $\varepsilon(n)$ -cluster. We introduce a probability measure μ on the set $\{t : \text{for any } k \geq n, K(\varepsilon(k), t) \text{ is a standard } \varepsilon(k)\text{-cluster and } K(\varepsilon(n), t) = K\}$ satisfying the following condition: all standard $\varepsilon(l+1)$ -clusters falling inside the same $\varepsilon(l)$ -cluster have equal measure.

By Lemma 4(b), for any standard $\varepsilon(l)$ -cluster ($l > n$),

$$\begin{aligned} \mu(K_l) &\leq \sum_{k=n+1}^l \left(\frac{1}{\varepsilon(n)}\right)^{[(1/2)(2^{k^2} - 2^{(k-1)^2}) - \gamma 2^{(k-1)^2} - 2\gamma 2^{k^2} - \delta 2^{k^2}]} \\ &\leq \text{const} \cdot \left(\frac{1}{2}\right)^{2^{l^2}(1/2-2\gamma-2\delta)}. \end{aligned}$$

LEMMA 9. For every interval I , we have $|I|^{1/2-2\gamma-2\delta} \geq \text{const} \cdot \mu(I)$.

Proof. Let j be the minimal natural number such that I covers an entire standard $\varepsilon(j)$ -cluster with positive measure, and let k be the number of $\varepsilon(j)$ -clusters inside I . There are two possibilities:

a) $k < c(\delta)$. Then

$$\mu(I) < (c(\delta) + 2) \text{const} \cdot \left(\frac{1}{2}\right)^{2^{j^2}(1/2-2\gamma-2\delta)}$$

($c(\delta) + 2$ takes account of more fine clusters as well) and

$$|I| > \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{2^{j^2}}.$$

b) $k > c(\delta)$. In this case the statement follows from Lemma 6 and the estimate

$$\mu(I) \leq (k + 2) \text{const} \cdot \left(\frac{1}{2}\right)^{2^{j^2} (1/2 - 2\gamma - 2\delta)}. \blacksquare$$

COROLLARY 2. $h\text{-dim}(\limsup A_m) \geq 1/2 - 2\gamma$.

PROOF. The implication (Lemma 9) \Rightarrow (Corollary 2) is well known in fractal geometry. We present the proof here, because it is short enough.

Let $\{I_j\}$ be an ε -cover of $(\bigcap_{k=n}^\infty A_k) \cap K$. Then

$$\sum_{j=1}^\infty |I_j|^{1/2 - 2\gamma - 2\delta} \geq \text{const} \cdot \sum_{j=1}^\infty \mu(I_j) \geq \text{const} \cdot \mu\left(\left(\bigcap_{k=n}^\infty A_k\right) \cap K\right) = \text{const},$$

i.e. for any $\varepsilon > 0$,

$$H_\varepsilon^{1/2 - 2\gamma - 2\delta} \left(\left(\bigcap_{k=n}^\infty A_k\right) \cap K\right) \geq \text{const}$$

and so

$$h\text{-dim} \left(\left(\bigcap_{k=n}^\infty A_k\right) \cap K\right) \geq \frac{1}{2} - 2\gamma - 2\delta. \blacksquare$$

This completes the proof of Theorem 1.

5. Dimension of other sets of singular points of the Brownian zeroes

5.1. Small size clusters. The proof of $h\text{-dim}(\liminf B_m) = 0$ is similar to the proof of Lemma 7.

The probability of small size clusters (belonging to B_m) has asymptotics $\text{const} \cdot \varepsilon^{\gamma/2}$. Hence, the number of those clusters has order $(1/\varepsilon)^{1/2 - \gamma/2}$ in the sense of Lemma 2. The length of clusters is bounded by $\varepsilon^{1+\gamma}$, therefore

$$h\text{-dim}(\limsup B_m) \leq \frac{1}{2} \cdot \frac{1 - \gamma}{1 + \gamma}.$$

To prove the reverse inequality one defines a standard small cluster to be one with length between $\frac{1}{2}\varepsilon^{1+\gamma}$ and $\varepsilon^{1+\gamma}$, and proceeds in a similar way to what we did with Lemmas 4–6, 9 and Corollary 2.

5.2. Small local time increment clusters. In the same way as in 4.1 and 5.1 we get $h\text{-dim}(\liminf C_m) = 0$.

To study the upper limit divide $[0, 2\gamma]$ into subintervals of length $1/n$. Define $\beta_i^{(n)} = i/n$ and

$$C_m(i, n) = \{t : \varepsilon_m^{1+\beta_{i+1}^{(n)}} < \delta(\varepsilon_m, t) \leq \varepsilon_m^{1+\beta_i^{(n)}} ; l_i(\varepsilon_m, t) < \varepsilon^{1/2+\gamma}\}.$$

The probability of clusters from $C_m(i, n)$ has asymptotics $\text{const} \cdot \varepsilon^{2\gamma-\beta_i^{(n)}/2}$.

Similarly to Lemma 3, with probability 1,

$$C_m \subset \bigcup_{i=1}^{\lfloor 2(1/2-\gamma)n \rfloor + 1} C_m(i, n)$$

if m is large enough. So

$$h\text{-dim}(\limsup C_m) = \max_i h\text{-dim}(\limsup C_m(i, n)).$$

Similarly to 4.2–4.3 and 5.1 we have the inequality

$$(5.1) \quad \frac{1/2 - 2\gamma - \beta_i^{(n)}/2}{1 + \beta_{i+1}^{(n)}} \leq h\text{-dim}(\limsup C_m(i, n)) \leq \frac{1/2 - 2\gamma - \beta_i^{(n)}/2}{1 + \beta_i^{(n)}}.$$

Since n is arbitrarily large, (5.1) implies that

$$h\text{-dim}(\limsup C_m) = \sup_{\beta \in [0, 2\gamma]} \frac{1/2 - 2\gamma - \beta/2}{1 + \beta} = \frac{1}{2} \cdot \frac{1 - 2\gamma}{1 + 2\gamma}.$$

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