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Linear orders and MA + ¬wKH

by

Zoran Spasojević (Madison, Wisc.)

Abstract. I prove that the statement that "every linear order of size 2^ω can be embedded in $(\omega^\omega, \ll)$" is consistent with MA + ¬wKH.

Let φ_κ denote the statement that every linear order of size κ can be embedded in $(\omega^\omega, \ll)$ for regular $\kappa \le 2^\omega = \mathfrak{c}$ where ω^ω denotes the set of all functions from ω to ω and $\ll$ is a partial order on ω^ω defined as follows: for $f, g \in \omega^\omega$ let $f \ll g$ if and only if $\exists n < \omega\, \forall i \ge n\, (f(i) \le g(i))$ and $f(i) < g(i)$ on an infinite set. Under CH, $\forall \kappa \le \mathfrak{c}\, (\varphi_\kappa)$, which basically follows from the fact that there are no (ω, ω)-gaps in $(\omega^\omega, \ll)$. If CH fails then $(\omega^\omega, \ll)$ may not even contain a well order of type ω_2 regardless of what $\mathfrak{c}$ is. On the other hand, MA + ¬CH $\rightarrow \forall \kappa < \mathfrak{c}\, (\varphi_\kappa)$. Kunen constructed a model for MA + ¬CH + $\neg\varphi_{\mathfrak{c}}$ and Laver [L] constructed a model for ¬CH + $\varphi_{\mathfrak{c}}$. For a while, the question was whether MA + ¬CH is strong enough to decide $\varphi_{\mathfrak{c}}$. Woodin [W] constructed a model for MA + $\mathfrak{c} = \omega_2$ + $\varphi_{\mathfrak{c}}$, therefore, together with Kunen's result, showing that $\varphi_{\mathfrak{c}}$ is independent of MA + ¬CH.

On the other hand, PFA → MA + ¬wKH → MA + ¬CH and neither of the implications is reversible. Therefore MA + ¬wKH is in strength somewhere between PFA and MA + ¬CH. But also PFA → $\mathfrak{c} = \omega_2$ + $\neg\varphi_{\mathfrak{c}}$. Therefore, it is reasonable to ask whether MA + ¬wKH is strong enough to decide $\varphi_{\mathfrak{c}}$. This question is the main consideration of this paper. The main result is Theorem 3.2 which states that if M is a countable transitive model (c.t.m.) for ZFC + V=L and κ is the first inaccessible cardinal in M then there is an extension N[J] of M which is a model for ZFC + MA + ¬wKH + $\mathfrak{c} = \omega_2$ + $\varphi_{\mathfrak{c}}$. The existence of an inaccessible cardinal is necessary to show the consistency of ¬wKH, as shown by Mitchell [M]. Todorčević [T] constructed a model for MA + ¬wKH + $\mathfrak{c} = \omega_2$, and I will use this result together with the result of Laver to construct the model N[J]. Therefore, when combined with PFA → MA + ¬wKH + $\mathfrak{c} = \omega_2$ + $\neg\varphi_{\mathfrak{c}}$, it shows that

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MA + ¬wKH is still not strong enough to decide $\varphi_{\mathfrak{c}}$. Woodin's construction cannot easily be modified to fit the additional arguments required in showing ¬wKH because his construction is completed in $\omega_2 \cdot \omega_2$ stages. In order to show that ¬wKH holds in the final model the construction here has to be finished in ω_2 steps. However, the treatment of stages of cofinality ω_1 resembles those in Woodin's construction. Consequently, the construction here can be regarded as an amalgamation of the constructions mentioned above.

To construct a model N[J], I start with a c.t.m. M for ZFC + V=L in which κ is the first inaccessible cardinal. Then, as in [M], extend M with a partial order to obtain a model N such that N $\models$ "$\neg\mathrm{wKH} + \mathfrak{c} = \kappa = \omega_2$". In N, I perform an iterated ccc forcing construction with finite supports of length ω_2. In the process I construct a $\mathfrak{c}$-saturated linearly ordered subset $(\mathbb{L}, \ll)$ of $(\omega^\omega, \ll)$. At the successor stages I alternate between ccc partial orders to make MA true and splitting partial orders for pregaps in $\mathbb{L}$. A difficulty occurs in splitting (ω_1, ω_1)-gaps. However, the construction is arranged in such a way that these gaps appear in $\mathbb{L}$ only at the limit stages of cofinality ω_1; at these stages I split all such gaps, all at once. The elements of ω^ω obtained at these stages will not be used directly, but they are needed to ensure that the splitting orders for all the pregaps in $\mathbb{L}$ continue to have the ccc until they are filled, one by one, at the later successor stages. The partial orders at these limit stages have cardinality ω_2, which causes some difficulty in the proof of ¬wKH. This difficulty is overcome by reducing the argument to suborders of size ω_1 of these partial orders.

Since trees and gaps play a central role in the construction, I begin with some notions and results on trees and gaps that are needed here. Many results included here are already known, however I present a different view point. Notation and terminology are adapted from [K], especially the part on iterated forcing.

1. Trees. A *tree* is a partial order in the strict sense, $\langle \mathbb{T}, \leq \rangle$, such that for each $x \in \mathbb{T}$, $\widehat{x} = \{y \in \mathbb{T} : y < x\}$ is well ordered by $<$. If $x \in \mathbb{T}$, the *height of* x *in* $\mathbb{T}$, $\mathrm{ht}(x, \mathbb{T})$, is the ordinal α which is the order type of $\widehat{x}$ and $\mathbb{T}_x = \{y \in \mathbb{T} : y \leq x \vee x < y\}$. For each ordinal α, the αth *level* of $\mathbb{T}$, $\mathrm{Lev}_\alpha(\mathbb{T})$, is the set $\{x \in \mathbb{T} : \mathrm{ht}(x, \mathbb{T}) = \alpha\}$. The *height of* $\mathbb{T}$, $\mathrm{ht}(\mathbb{T})$, is the least α such that $\mathrm{Lev}_\alpha(\mathbb{T}) = \emptyset$. A *chain* in $\mathbb{T}$ is a set $C \subseteq \mathbb{T}$ which is totally ordered by $<$. If C intersects every level of $\mathbb{T}$ then C is called a *path* through $\mathbb{T}$. $A \subseteq \mathbb{T}$ is an *antichain* iff $\forall x, y \in A\, (x \neq y \rightarrow (x \not\leq y \wedge y \not\leq x))$. I will only consider well pruned trees. A *well pruned tree* is a tree $\mathbb{T}$ such that

(i) $|\mathrm{Lev}_0(\mathbb{T})| = 1$,

(ii) $\forall \alpha < \beta < \mathrm{ht}(\mathbb{T})\ \forall x \in \mathrm{Lev}_\alpha(\mathbb{T})\ \exists y_1, y_2 \in \mathrm{Lev}_\beta(\mathbb{T})\, (y_1 \neq y_2 \wedge x \leq y_1, y_2)$,

(iii) $\forall\alpha < \mathrm{ht}(\mathbb{T})\ \forall x, y \in \mathrm{Lev}_\alpha(\mathbb{T})\,(\lim(\alpha) \to (x = y \leftrightarrow \widehat{x} = \widehat{y}))$.

From now on any mention of a tree $\mathbb{T}$ will automatically mean that $\mathbb{T}$ is a well pruned tree. An ω_1-*tree* is a tree $\mathbb{T}$ such that $\mathrm{ht}(\mathbb{T}) = |\mathbb{T}| = \omega_1$. An ω_1-tree is a *weak Kurepa tree* if it has at least ω_2 paths. The assertion that there is a weak Kurepa tree is denoted by wKH and ¬wKH denotes its negation. An *Aronszajn tree* is an ω_1-tree $\mathbb{T}$ without any paths such that $\forall\alpha < \omega_1\,(|\mathrm{Lev}_\alpha(\mathbb{T})| \le \omega)$. A *Suslin tree* is an Aronszajn tree with no uncountable antichains. If $\mathbb{T}$ is an ω_1-tree and $\exists(f : \mathbb{T} \to \omega)\,(\forall x, y \in \mathbb{T}\,(x < y \to f(x) \ne f(y)))$ then $\mathbb{T}$ is called a *special* ω_1-*tree* and f a *specializing function* for $\mathbb{T}$. It follows that if $\mathbb{T}$ is a special Aronszajn tree with a specializing function f then for some $n \in \omega$, $f^{-1}(n)$ is uncountable and as such an uncountable antichain in $\mathbb{T}$. Therefore neither $\mathbb{T}$ nor any subtree of $\mathbb{T}$ can be Suslin. Next, I define a partial order $\mathbb{S}_\mathbb{T}$, due to Baumgartner, which is intended to add a specializing function for $\mathbb{T}$.

Definition 1.1. Let $\mathbb{T}$ be an ω_1-tree. Then

$$\mathbb{S}_\mathbb{T} = \{p : \exists x \in [\mathbb{T}]^{<\omega}\,(p : x \to \omega) \wedge \forall s, t \in x\,(s < t \to p(s) \ne p(t))\}$$

with $p_1 \le p_2$ iff $p_1 \supseteq p_2$.

The symbol "$\perp$" denotes incompatibility in any partial order $\mathbb{P}$ and "$\veebar$" will be used to denote incompatibility in a tree $\mathbb{T}$, i.e.

$$\forall x, y \in \mathbb{T}\,(x \veebar y \leftrightarrow (x \not\le y \wedge y \not\le x)).$$

Then $\veebar$ extends to incompatibility in $[\mathbb{T}]^{<\omega}$ as follows:

$$\forall a, b \in [\mathbb{T}]^{<\omega}\,(a \veebar b \leftrightarrow (a \cap b = \emptyset \wedge \forall x \in a\ \forall y \in b\,(x \veebar y))).$$

Also note that if $p, q \in \mathbb{S}_\mathbb{T}$ and $\mathrm{dom}(p) \veebar \mathrm{dom}(q)$ then p and q are compatible in $\mathbb{S}_\mathbb{T}$.

Lemma 1.2. *If* $\mathbb{T}$ *is an Aronszajn tree then* $(\mathbb{S}_\mathbb{T}, \le)$ *has the ccc.*

Proof. By way of contradiction assume that $A = \{p_\alpha : \alpha < \omega_1\} \subseteq \mathbb{S}_\mathbb{T}$ is an uncountable antichain. Without loss of generality I may assume

(1) $\forall\alpha < \omega_1\,(|\mathrm{dom}(p_\alpha)| = n)$ for some $n < \omega$,
(2) $\forall\alpha, \beta < \omega_1\,(\alpha \ne \beta \to (\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = \emptyset))$.

To see that I may assume (2), first assume, by the Δ-system lemma, that $\{\mathrm{dom}(p_\alpha) : \alpha < \omega_1\}$ forms a Δ-system with root r. Then, since ω^r is countable, I may assume that $\forall\alpha, \beta < \omega_1\,(p_\alpha \restriction r = p_\beta \restriction r)$. Then (2) is implied at once by the claim below.

Claim. *If* $e_\alpha = \mathrm{dom}(p_\alpha) \setminus r$ *then* $(p_\alpha \restriction e_\alpha \perp p_\beta \restriction e_\beta) \leftrightarrow (p_\alpha \perp p_\beta)$.

P r o o f o f C l a i m. Let $p_\alpha \perp p_\beta$. Then

$$\exists x \in \mathrm{dom}(p_\alpha)\ \exists y \in \mathrm{dom}(p_\beta)$$
$$((x < y \wedge p_\alpha(x) = p_\beta(y)) \vee (y < x \wedge p_\beta(y) = p_\alpha(x))).$$

It cannot happen that $x, y \in \mathrm{dom}(p_\alpha)$ since $p_\alpha \in \mathbb{S}_\mathbb{T}$, and it cannot happen that $x, y \in \mathrm{dom}(p_\beta)$ for the same reason. Therefore $x, y \notin r$ so that $x \in e_\alpha$ and $y \in e_\beta$. This basically proves the claim since the implication in the other direction is trivial.

Now let $\mathrm{dom}(p_\alpha) = \{s_0^\alpha, s_1^\alpha, \ldots, s_{n-1}^\alpha\}$. Finally I may assume that if $\alpha < \beta < \omega_1$, $p_\alpha(s_i^\alpha) = p_\beta(s_j^\beta)$, and s_i^α and s_j^β are comparable (which must happen for some i and j since $p_\alpha \perp p_\beta$) then $s_i^\alpha < s_j^\beta$. Therefore for each α there must be $i(\alpha), j(\alpha) < n$ such that $\{\beta : s_{i(\alpha)}^\alpha < s_{j(\alpha)}^\beta\}$ is uncountable. Furthermore, there must be i and j such that $B = \{\alpha : i(\alpha) = i \wedge j(\alpha) = j\}$ is also uncountable. But now if $\alpha_1, \alpha_2 \in B$ there is $\beta > \alpha_1, \alpha_2$ such that $s_i^{\alpha_1}, s_i^{\alpha_2} < s_j^\beta$. And since $\mathbb{T}$ is a tree, $s_i^{\alpha_1}$ and $s_i^{\alpha_2}$ are comparable. Therefore $\{s_i^\alpha : \alpha \in B\}$ may be extended to a path through $\mathbb{T}$, contradicting the fact that $\mathbb{T}$ has no paths. Therefore A cannot be an uncountable antichain. ■

From the proof above immediately follow the two corollaries below.

COROLLARY 1.3. *Let* M *be a c.t.m. for* ZFC *and*, *in* M, *suppose that* $\mathbb{T}$ *is an Aronszajn tree and* $\mathbb{P}$ *a ccc partial order with* G $\mathbb{P}$*-generic over* M. *Then* $\mathbb{S}_\mathbb{T}$ *fails to have the ccc in* M[G] *iff a new path has been added through* $\mathbb{T}$ *in* M[G].

COROLLARY 1.4. *Let* M *be a c.t.m. for* ZFC, $\mathbb{T}$ *an Aronszajn tree in* M, *and* G $\mathbb{S}_\mathbb{T}$*-generic over* M. *Then*

$$\mathrm{M[G]} \models \text{“}\mathbb{T} \textit{ is a special Aronszajn tree”}.$$

DEFINITION 1.5. Let $\mathbb{P}$ be a partial order. Then $\mathbb{P}$ has the *property* K iff

$$\forall A \in [\mathbb{P}]^{\omega_1}\ \exists B \in [A]^{\omega_1}\ \forall x, y \in B\ (x \not\perp y).$$

LEMMA 1.6. *If* $\mathbb{T}$ *is a special Aronszajn tree then* $\mathbb{S}_\mathbb{T}$ *has the property* K.

P r o o f. Let $\{p_\alpha : \alpha < \omega_1\} \subseteq \mathbb{S}_\mathbb{T}$. Then, as in the proof of Lemma 1.2, I may assume

(1) $\forall \alpha < \omega_1\ (|\mathrm{dom}(p_\alpha)| = n)$ for some $n < \omega$,
(2) $\forall \alpha, \beta < \omega_1\ (\alpha \neq \beta \to \mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = \emptyset)$.

Let $\mathrm{dom}(p_\alpha) = e_\alpha$. To get p_α and p_β compatible it suffices to get $e_\alpha \veebar e_\beta$. Therefore the proof follows immediately from the following

CLAIM. $\exists A \in [\omega_1]^{\omega_1}\ \forall \alpha, \beta \in A\ (\alpha \neq \beta \to e_\alpha \veebar e_\beta)$.

P r o o f o f C l a i m. The proof is by induction on $|e_\alpha| = n$. Fix n and assume the result is true for all $m < n$.

Case 1: Suppose $\forall\gamma < \omega_1\ \exists x \in \mathrm{Lev}_\gamma(\mathbb{T})\ \exists\alpha < \omega_1\,(e_\alpha \subseteq \mathbb{T}_x)$. Then for $\mu < \omega_1$ choose $x_\mu \in \mathrm{Lev}_{\gamma_\mu}(\mathbb{T})$, α_μ and increasing γ_μ such that $e_{\alpha_\mu} \subseteq \mathbb{T}_{x_\mu}$ with $\gamma_\mu > \sup\{\mathrm{ht}(z) : z \in \bigcup_{\nu<\mu} e_{\alpha_\nu}\}$. Then, by the remarks before Definition 1.1, there is an $A \in [\omega_1]^{\omega_1}$ such that $\{x_\mu : \mu \in A\}$ is an uncountable antichain in $\{x_\mu : \mu < \omega_1\}$. But then $\forall\alpha,\beta \in A\,(\alpha \neq \beta \to e_\alpha \veebar e_\beta)$.

Case 2: This is just ¬Case 1. Fix γ such that $\forall x \in \mathrm{Lev}_\gamma(\mathbb{T})\ \forall\alpha < \omega_1\,(e_\alpha \not\subseteq \mathbb{T}_x)$. Then, since each level of $\mathbb{T}$ is countable and e_α are all pairwise disjoint, it follows that $n \geq 2$ and only countably many e_α meet $\mathrm{Lev}_\gamma(\mathbb{T})$ or below. Therefore without loss of generality I may throw those away and assume that $\forall\alpha < \omega_1\ \forall z \in e_\alpha\,(\mathrm{ht}(z) > \gamma)$. I may also assume that $\exists x \in \mathrm{Lev}_\gamma(\mathbb{T})\ \forall\alpha < \omega_1\,(e_\alpha \cap \mathbb{T}_x \neq \emptyset)$ since $e_\alpha \subseteq \bigcup\{\mathbb{T}_x : x \in \mathrm{Lev}_\gamma(\mathbb{T})\}$ and $|\mathrm{Lev}_\gamma(\mathbb{T})| \leq \omega$. So fix any such x. Then without loss of generality I may assume that

$$\forall\alpha < \omega_1\,((|e_\alpha \cap \mathbb{T}_x| = i > 0) \wedge (|e_\alpha \setminus \mathbb{T}_x| = j > 0))$$

since $e_\alpha \not\subseteq \mathbb{T}_x$. Then $0 < i, j < n$ and $i + j = n$. And by the induction hypothesis I may assume that

$$(*)\quad \forall\alpha,\beta < \omega_1\,(\alpha \neq \beta \to (((e_\alpha \cap \mathbb{T}_x) \veebar (e_\beta \cap \mathbb{T}_x)) \wedge ((e_\alpha \setminus \mathbb{T}_x) \veebar (e_\beta \setminus \mathbb{T}_x)))).$$

But then e_α are also pairwise incompatible in $[\mathbb{T}]^{<\omega}$. Here I claim that it is not possible to have $s \in e_\alpha$ and $t \in e_\beta$ with $s < t$ and $\alpha \neq \beta$. There are 4 cases to consider. If $s, t \in \mathbb{T}_x$ or $s, t \notin \mathbb{T}_x$ then I am done by $(*)$. The cases $s \in \mathbb{T}_x \wedge t \notin \mathbb{T}_x$ or $s \notin \mathbb{T}_x \wedge t \in \mathbb{T}_x$ cannot happen since $\mathbb{T}$ is a tree. This proves the claim and hence the lemma. ∎

LEMMA 1.7. *Let* M *be a c.t.m. for* ZFC *and suppose that* $\mathbb{U}$ *and* $\mathbb{T}$ *are Aronszajn trees in* M. *If* G *is* $\mathbb{S}_\mathbb{T}$*-generic over* M *then* M[G] $\models$ "$\mathbb{U}$ *is Aronszajn*".

Proof. It suffices to prove that no new paths through $\mathbb{U}$ are added in M[G]. So by way of contradiction let $p \in \mathbb{S}_\mathbb{T}$ and $\dot b \in \mathrm{M}^{\mathbb{S}_\mathbb{T}}$ with $p \Vdash$ "$\dot b$ is a new path through $\check{\mathbb{U}}$". Since $\mathbb{U}$ is Aronszajn in M, it follows that $\dot b_\mathrm{G} = b \notin \mathrm{M}$. Let

$$X = \{u \in \mathbb{U} : \exists p_u \leq p\,(p_u \Vdash \text{“}\check u \in \dot b\text{”})\}.$$

Let $u_\alpha \in \mathrm{Lev}_\alpha(\mathbb{U})$ and $p_\alpha \in \mathbb{S}_\mathbb{T}$ with $p_\alpha \leq p$ such that $p_\alpha \Vdash$ "$\check u_\alpha \in \dot b$". Now

$$\mathrm{M[G]} \models \text{“}\mathbb{S}_\mathbb{T} \text{ has the property K”}$$

so in M[G] let $B \in [\omega_1]^{\omega_1}$ such that $\{p_\alpha : \alpha \in B\}$ are pairwise compatible. Then there is a path, d, through $\mathbb{U}$ determined by B with $d \in \mathrm{M[G]}$ and $d \subseteq X$.

On the other hand, b is a new path through $\mathbb{U}$ so for each $u \in X$ there are $s,t\in X$ such that $u\le_{\mathbb U} s,t$ and t and s are incomparable. Let

$$Y=\{u\in X: u \text{ is } \le_{\mathbb U}\text{-minimal with } u\notin d\}.$$

Then $Y\in \mathrm{M[G]}$ and for each $u\in Y$ fix a $p_u\in\mathbb P$ such that $p_u\le p\wedge p_u\Vdash$ "$\check u\in\dot b$" and let $A=\{p_u\in\mathbb S_{\mathbb T}: u\in Y\}$. Then $A\in\mathrm{M[G]}$ and A is an uncountable subset of $\mathbb S_{\mathbb T}$ and any two elements of A are incompatible. Hence A is an uncountable antichain in $\mathbb S_{\mathbb T}$, which contradicts the fact that $\mathbb S_{\mathbb T}$ has the property K in M[G]. ■

Corollary 1.8. $\mathrm{M[G]}\models$ "$\mathbb S_{\mathbb U}$ *has the ccc*".

Lemma 1.9. *Let* M *be a c.t.m. for* ZFC *and*, *in* M, *suppose that* $\mathbb P$ *is a ccc partial order and* $\mathbb T$ *an* ω_1*-tree. If* G *is* $\mathbb P$*-generic over* M *with*

$$\mathrm{M[G]}\models \text{"}b \textit{ is a new path through } \mathbb T\text{"}$$

then there is a Suslin tree $\mathbb U\subseteq\mathbb T$ *with* $\mathbb U\in\mathrm M$ *such that*

$$\mathrm{M[G]}\models \text{"}b \textit{ is a new path through } \mathbb U\text{"}.$$

Proof. Let $p\in\mathbb P$ with $p\Vdash$ "$\dot b$ is a new path through $\check{\mathbb T}$". Let

$$\mathbb U=\{u\in\mathbb T:\exists q\le p\,(q\Vdash \text{"}\check u\in\dot b\text{"})\}.$$

Clearly $\mathbb U\in\mathrm M$, $\mathbb U\subseteq\mathbb T$, and $|\mathbb U|=\omega_1$. The fact that b is a new path through $\mathbb T$ also implies that $\mathrm{ht}(\mathbb U)=\omega_1$. If $\mathbb U$ is not Suslin in M then there is an $A\subseteq\mathbb U$ with $A\in\mathrm M$ and $|A|=\omega_1$ such that any two elements of A are incomparable. For each $u\in\mathbb U$ fix a $p_u\in\mathbb P$ such that $p_u\le p\wedge p_u\Vdash$ "$\check u\in\dot b$" and let

$$A_{\mathbb P}=\{p_u: u\in A\wedge p_u\le p\wedge\ (p_u\Vdash \text{"}\check u\in\dot b\text{"})\}.$$

Clearly $A_{\mathbb P}\in\mathrm M$. Then $A_{\mathbb P}$ is an antichain in $\mathbb P$. This follows since if $p_u,p_t\in A_{\mathbb P}$ for $u\ne t\in A$ and $q\in\mathbb P$ with $q\le p_u,p_t$ then $q\Vdash$ "$\check u\in\dot b\wedge\check t\in\dot b$" so that u and t are comparable, which is impossible by the choice of A. Furthermore, $A_{\mathbb P}$ is uncountable since A is. Hence $A_{\mathbb P}$ is an uncountable antichain in $\mathbb P$ contradicting the fact that $\mathbb P$ has the ccc in M. Therefore $\mathbb U$ is Suslin with $\mathrm{M[G]}\models$ "$b\subseteq\mathbb U$" so that

$$\mathrm{M[G]}\models \text{"}b \text{ is a new path through } \mathbb U\text{"}.$$

And this is precisely what I set out to show. ■

Let $\mathbb P$ be a partial order and $\dot{\mathbb Q}$ a $\mathbb P$-name for a partial order. Then $\mathbb P*\dot{\mathbb Q}$ denotes a two-step iteration. The following result is taken from [K] and is needed in the proof of Lemma 1.11.

Lemma 1.10. *Assume that in* M, $\mathbb P$ *is a ccc partial order and* $\dot{\mathbb Q}$ *a* $\mathbb P$*-name for a partial order such that* $\mathbf 1\Vdash_{\mathbb P}$ "$\dot{\mathbb Q}$ *has the ccc*". *Then* $\mathbb P*\dot{\mathbb Q}$ *has the ccc in* M.

LEMMA 1.11. *Suppose* M *is a c.t.m. for* ZFC *and* $\mathbb{P}$ *and* $\mathbb{Q}$ *two ccc partial orders in* M. *Then* $\mathbb{P}\times\mathbb{Q}$ *has the ccc iff* $\mathbf{1}\Vdash_{\mathbb{P}}$ "$\check{\mathbb{Q}}$ *has the ccc*".

Proof. If $\mathbf{1}\Vdash_{\mathbb{P}}$"$\check{\mathbb{Q}}$ has the ccc" then by Lemma 1.10, $\mathbb{P}*\check{\mathbb{Q}}$ has the ccc. Then since $\mathbb{P}*\check{\mathbb{Q}}$ and $\mathbb{P}\times\mathbb{Q}$ are isomorphic it follows that $\mathbb{P}\times\mathbb{Q}$ has the ccc.

Now suppose that $\mathbb{P}\times\mathbb{Q}$ has the ccc and by way of contradiction assume that

$$\mathbf{1}\Vdash_{\mathbb{P}} \text{“}\dot{A} \text{ is an uncountable antichain in } \check{\mathbb{Q}}\text{”}.$$

Let τ be a $\mathbb{P}$-name and $p'\in\mathbb{P}$ with

$$p'\Vdash_{\mathbb{P}} \text{“}\tau:\check{\omega}_1\to\dot{A} \text{ and } \tau \text{ is one-to-one and onto”}.$$

Also let $p_\xi\le p'$ and $q_\xi\in\mathbb{Q}$ with $p_\xi\Vdash_{\mathbb{P}}$ "$\tau(\xi)=\check{q}_\xi$". Then $B=\{\langle p_\xi,\check{q}_\xi\rangle:\xi<\omega_1\}$ is an uncountable antichain in $\mathbb{P}*\check{\mathbb{Q}}$. To see this suppose that $\langle p_\alpha,\check{q}_\alpha\rangle$ and $\langle p_\beta,\check{q}_\beta\rangle$ are compatible for some $\alpha\ne\beta$. Let $\langle p,\check{q}\rangle\le\langle p_\alpha,\check{q}_\alpha\rangle,\langle p_\beta,\check{q}_\beta\rangle$. Then $p\le p_\alpha,p_\beta$ and $p\Vdash_{\mathbb{P}}$ "$\check{q}\le\check{q}_\alpha,\check{q}_\beta$". But this leads to a contradiction since also $p\le p'$ so that

$$p\Vdash_{\mathbb{P}} \text{“}\dot{A} \text{ is an antichain in } \check{\mathbb{Q}} \text{ and } \check{q}_\alpha,\check{q}_\beta\in\dot{A}\text{”}.$$

Therefore $\mathbf{1}\Vdash_{\mathbb{P}}$ "$\check{\mathbb{Q}}$ has the ccc". ∎

LEMMA 1.12. *Let* M *be a c.t.m. for* ZFC *and*, *in* M, $\mathbb{P}$ *a ccc partial order and* $\langle\mathbb{P}_\xi:\xi\le\alpha\rangle$ *an iterated ccc forcing construction with finite supports where* α *is a limit ordinal. If* $\forall\xi<\alpha\,(\mathbf{1}\Vdash_{\mathbb{P}_\xi}$ "$\check{\mathbb{P}}$ *has the ccc*"$)$ *then* $\mathbf{1}\Vdash_{\mathbb{P}_\alpha}$ "$\check{\mathbb{P}}$ *has the ccc*".

Proof. If $\mathrm{cf}(\alpha)=\omega$ and

$$\mathbf{1}\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{A} \text{ is an uncountable antichain in } \check{\mathbb{P}}\text{”}$$

then since $\langle\mathbb{P}_\xi:\xi\le\alpha\rangle$ has finite supports it follows that some uncountable subset of A is constructed at some earlier stage. But any subset of A is also an antichain in $\mathbb{P}$. Therefore

$$\exists\beta<\alpha\,(\mathbf{1}\Vdash_{\mathbb{P}_\beta} \text{“}\check{\mathbb{P}} \text{ fails to have the ccc”}),$$

contradicting the hypothesis.

If $\mathrm{cf}(\alpha)>\omega_1$ then any subset of $\mathbb{P}$ of size ω_1 is constructed by some stage $\beta<\alpha$. Therefore if

$$\mathbf{1}\Vdash_{\mathbb{P}_\alpha} \text{“}\check{\mathbb{P}} \text{ fails to have the ccc”}$$

then

$$\exists\beta<\alpha\,(\mathbf{1}\Vdash_{\mathbb{P}_\beta} \text{“}\check{\mathbb{P}} \text{ fails to have the ccc”}),$$

again contradicting the hypothesis.

Finally, let $\mathrm{cf}(\alpha) = \omega_1$ and suppose that the conclusion of the lemma fails. Therefore

$$\forall\beta < \alpha\,(\mathbf{1} \Vdash_{\mathbb{P}_\beta} \text{“}\check{\mathbb{P}} \text{ has the ccc”})$$

but

$$\mathbf{1} \Vdash_{\mathbb{P}_\alpha} \text{“}\check{\mathbb{P}} \text{ fails to have the ccc”}.$$

Then according to Lemma 1.11, $\mathbb{P} \times \mathbb{P}_\beta$ has the ccc for each $\beta < \alpha$, but $\mathbb{P} \times \mathbb{P}_\alpha$ does not have the ccc. Then again by Lemma 1.11,

$$\forall\beta < \alpha\,(\mathbf{1} \Vdash_{\mathbb{P}} \text{“}\check{\mathbb{P}}_\beta \text{ has the ccc”})$$

but

$$\mathbf{1} \Vdash_{\mathbb{P}} \text{“}\check{\mathbb{P}}_\alpha \text{ fails to have the ccc”}.$$

Let G be $\mathbb{P}$-generic over M and, working in M[G], let $A = \{p_\xi : \xi < \omega_1\}$ be an uncountable antichain in $\mathbb{P}_\alpha$. Then by the Δ-system lemma I may assume that $\{\mathrm{supp}(p_\xi) : \xi < \omega_1\}$ forms a Δ-system with root r. Let $\beta < \alpha$ with $r \subseteq \beta$. Then since $\mathbb{P}_\beta$ has the ccc let $\xi, \eta < \omega_1$ and $p \in \mathbb{P}_\beta$ be such that $p \le p_\xi{\restriction}\beta, p_\eta{\restriction}\beta$. Now define p^* as follows:

$$p^*(\theta) = \begin{cases} p(\theta) & \text{if } \theta < \beta, \\ p_\xi(\theta) & \text{if } \theta \in \mathrm{supp}(p_\xi) \cap (\alpha \setminus \beta), \\ p_\eta(\theta) & \text{if } \theta \in \mathrm{supp}(p_\eta) \cap (\alpha \setminus \beta), \\ \mathbf{1}(\theta) & \text{otherwise.} \end{cases}$$

Then $p^* \in \mathbb{P}_\alpha$ and $p^* \le p_\xi, p_\eta$, which contradicts the assumption that A is an uncountable antichain in $\mathbb{P}_\alpha$. Therefore $\mathbf{1} \Vdash_{\mathbb{P}}$ “$\check{\mathbb{P}}_\alpha$ has the ccc” and hence by Lemma 1.11, $\mathbf{1} \Vdash_{\mathbb{P}_\alpha}$ “$\check{\mathbb{P}}$ has the ccc”. ■

According to the lemma just proved if $\mathbb{T}$ is Aronszajn in the ground model and $\mathbb{S}_{\mathbb{T}}$ fails to have the ccc then this cannot happen at a limit stage. Equivalently, if any new paths are added through $\mathbb{T}$ then it can only happen at a successor stage.

This concludes the work on trees required for the final model.

2. Gaps. In the construction of a $\mathfrak{c}$-saturated linear order in $(\omega^\omega, \ll)$ gaps occur naturally. This section deals with gaps and their properties that are necessary for the construction in Section 3.

For convenience I choose to work with $(\mathbb{Z}^\omega, \ll)$ rather than $(\omega^\omega, \ll)$ and construct a $\mathfrak{c}$-saturated linear order in $(\mathbb{Z}^\omega, \ll)$ instead of $(\omega^\omega, \ll)$. This will imply the result for $(\omega^\omega, \ll)$ since $(\mathbb{Z}^\omega, \ll)$ can easily be embedded in $(\omega^\omega, \ll)$. Recall that $\mathbb{Z}^\omega$ is the set of all functions that map ω into $\mathbb{Z}$, the set of integers. This set has a natural partial order , “$\ll$”, which is defined as follows: If $f, g \in \mathbb{Z}^\omega$ then $f \ll g$ iff $\exists n < \omega\ \forall i \ge n\,(f(i) \le g(i))$ and $f(i) < g(i)$ on an infinite set.

Definition 2.1. Let I, J be two linearly ordered sets and $\langle f, g\rangle = \langle f_\xi, g_\eta : \xi \in I,\ \eta \in J\rangle \subseteq \mathbb{Z}^\omega$ such that $\forall \xi, \eta \in I\,(\xi \leq \eta \to f_\xi \ll f_\eta)$ and $\forall \zeta, \theta \in J\,(\zeta \leq \theta \to g_\theta \ll g_\zeta)$ and $\forall \xi \in I\ \forall \eta \in J\,(f_\xi \ll g_\eta)$. Then $\langle f, g\rangle$ is called an (I, J)-*pregap* in $\mathbb{Z}^\omega$. If $\exists h \in \mathbb{Z}^\omega\ \forall \xi \in I\ \forall \eta \in J\,(f_\xi \ll h \ll g_\eta)$ then h splits $\langle f, g\rangle$. If no such h exists then $\langle f, g\rangle$ is called an (I, J)-*gap*.

Definition 2.2. Let I, J, I', J' be linearly ordered sets and $\langle f, g\rangle$ an (I, J)-pregap and $\langle f', g'\rangle$ an (I', J')-pregap. Then $\langle f, g\rangle$ and $\langle f', g'\rangle$ are *equivalent* iff $\forall \xi \in I\ \exists \zeta \in I'\ \forall \eta \in J\ \exists \theta \in J'\,(f_\xi \ll f'_\zeta \wedge g'_\theta \ll g_\eta)$ and $\forall \xi \in I'\ \exists \zeta \in I\ \forall \eta \in J'\ \exists \theta \in J\,(f'_\xi \ll f_\zeta \wedge g_\theta \ll g'_\eta)$.

Let $\langle f, g\rangle$ and $\langle f', g'\rangle$ be two equivalent gaps. Then $h \in \mathbb{Z}^\omega$ splits $\langle f, g\rangle$ if and only if h splits $\langle f', g'\rangle$. From this fact it easily follows that there is a ccc partial order that splits $\langle f, g\rangle$ if and only if there is a ccc partial order that splits $\langle f', g'\rangle$. Therefore considering splitting orders for an (I, J)-pregap is equivalent to considering splitting orders for an (I', J')-pregap where I' is a cofinal well ordered subset of I and J' is a cofinal well ordered subset of J. Thus in considering splitting orders for pregaps I can use ordinals for indexing sets and an (I, J)-pregap will also be called a (λ, κ)-*pregap* if $\mathrm{cf}(I) = \lambda$ and $\mathrm{cf}(J) = \kappa$. One such splitting order is given by the following

Definition 2.3. Let $\langle f, g\rangle = \langle f_\xi, g_\eta : \xi < \lambda, \eta < \kappa\rangle$ be a (λ, κ)-pregap where λ, κ are ordinals. Set

$$\mathbb{S}_{\langle f,g\rangle} = \{\langle x, y, n, s\rangle : x \in [\lambda]^{<\omega} \wedge y \in [\kappa]^{<\omega} \wedge n < \omega \wedge (s : n \to \mathbb{Z}) \wedge \forall \xi \in x\ \forall \eta \in y\ \forall i \geq n\,(f_\xi(i) \leq g_\eta(i))\}$$

with $\langle x_2, y_2, n_2, s_2\rangle \leq \langle x_1, y_1, n_1, s_1\rangle$ iff

(1) $x_1 \subseteq x_2,\ y_1 \subseteq y_2,\ n_1 \leq n_2,\ s_1 = s_2{\restriction}n_1$,
(2) $\forall \xi \in x_1\ \forall \eta \in y_1\ \forall i < \omega\,(n_1 \leq i < n_2 \to (f_\xi(i) \leq s_2(i) \leq g_\eta(i)))$.

The splitting function h for $\langle f, g\rangle$ is given by

$$h = \bigcup\{s : \exists x, y, n\,(\langle x, y, n, s\rangle \in \mathrm{G})\}$$

where G is $\mathbb{S}_{\langle f,g\rangle}$-generic. Note that if $\lambda = \kappa = 0$ then $\mathbb{S}_{\langle f,g\rangle}$ is isomorphic to the partial order that adds a generic element to $\mathbb{Z}^\omega$.

Definition 2.4. Let $\langle f, g\rangle = \langle f_\xi, g_\eta : \xi < \lambda,\ \eta < \kappa\rangle$ be a (λ, κ)-pregap where λ, κ are ordinals. Then the function h is $\mathbb{S}_{\langle f,g\rangle}$-*generic* if the filter

$$\mathrm{G} = \{\langle x, y, n, s\rangle \in \mathbb{S}_{\langle f,g\rangle} : (s = h{\restriction}n) \wedge \forall \xi \in x\ \forall \eta \in y\ \forall i \geq n\ (f_\xi(i) \leq h(i) \leq g_\eta(i))\}$$

is $\mathbb{S}_{\langle f,g\rangle}$-generic.

Note that h is $\mathbb{S}_{\langle f,g\rangle}$-generic if and only if $-h$ is $\mathbb{S}_{\langle -g,-f\rangle}$-generic where $\langle -g, -f\rangle = \langle -g_\eta, -f_\xi : \eta < \kappa, \xi < \lambda\rangle$. This fact will be used later and this is precisely the reason why I chose to work with $(\mathbb{Z}^\omega, \ll)$ rather than $(\omega^\omega, \ll)$.

The partial order in Definition 2.3 is due to Kunen as is the following

LEMMA 2.5. *Let $\langle f,g\rangle$ be a (λ,κ)-pregap.*

(1) *If the pregap is split then $\mathbb{S}_{\langle f,g\rangle}$ has the property* K.
(2) *If* $\mathrm{cf}(\lambda)\neq\omega_1$ *or* $\mathrm{cf}(\kappa)\neq\omega_1$ *then $\mathbb{S}_{\langle f,g\rangle}$ has the property* K.
(3) *If $\lambda=\kappa=\omega_1$ and $\mathbb{S}_{\langle f,g\rangle}$ fails to have the ccc then there is an $m<\omega$ and there are $X,Y\in[\omega_1]^{\omega_1}$ with $X=\{\xi_\alpha:\alpha<\omega_1\}$ and $Y=\{\eta_\alpha:\alpha<\omega_1\}$ such that*

(i) $\forall\alpha<\omega_1\ \forall i\geq m\,(f_{\xi_\alpha}(i)\leq g_{\eta_\alpha}(i))$ *and*
(ii) $\forall\alpha,\beta<\omega_1\,(\alpha\neq\beta\rightarrow\exists i\geq m\,(f_{\xi_\alpha}(i)\not\leq g_{\eta_\beta}(i)\vee f_{\xi_\beta}(i)\not\leq g_{\eta_\alpha}(i)))$.

Proof. (1) Let $\{p_\alpha:\alpha<\omega_1\}\subseteq\mathbb{S}_{\langle f,g\rangle}$ where $p_\alpha=\langle x_\alpha,y_\alpha,n_\alpha,s_\alpha\rangle$. Suppose h splits $\langle f,g\rangle$. For each $\alpha<\omega_1$ fix $k_\alpha<\omega$ such that

$$\forall\xi\in x_\alpha\ \forall\eta\in y_\alpha\ \forall i\geq k_\alpha\,(f_\xi(i)\leq h(i)\leq g_\eta(i)).$$

By extending each p_α if necessary I may assume that $\forall\alpha<\omega_1\,(k_\alpha\leq n_\alpha)$. Then it is easily seen that

$$\exists A\in[\omega_1]^{\omega_1}\ \exists n<\omega\ \exists\,(s:n\rightarrow\mathbb{Z})\ \forall\alpha\in A\,(n_\alpha=n\wedge s_\alpha=s).$$

Now it clearly follows that $\forall\alpha,\beta\in A\,(p_\alpha\not\perp p_\beta)$ so that $\mathbb{S}_{\langle f,g\rangle}$ has the property K.

(2) Let $\{p_\alpha:\alpha<\omega_1\}\subseteq\mathbb{S}_{\langle f,g\rangle}$ where $p_\alpha=\langle x_\alpha,y_\alpha,n_\alpha,s_\alpha\rangle$. First assume that $\mathrm{cf}(\lambda)>\omega_1$. Then there exists $\mu<\lambda$ such that $\forall\alpha<\omega_1\,(x_\alpha\subseteq\mu)$. Therefore $\{p_\alpha:\alpha<\omega_1\}\subseteq\mathbb{S}_{\langle f_\xi,g_\eta:\xi<\mu,\eta<\kappa\rangle}$. Then the result follows from (1) since f_μ splits $\langle f_\xi,g_\eta:\xi<\mu,\ \eta<\kappa\rangle$. If $\mathrm{cf}(\lambda)<\omega_1$ then $\exists\mu<\lambda$ such that $x_\alpha\subseteq\mu$ for uncountably many α and this is sufficient to obtain the result as above. The case $\mathrm{cf}(\kappa)\neq\omega_1$ is handled in the same way.

(3) Let $A=\{p_\alpha=\langle x_\alpha,y_\alpha,n_\alpha,s_\alpha\rangle:\alpha<\omega_1\}$ be an uncountable antichain in $\mathbb{S}_{\langle f,g\rangle}$. For each $\alpha<\omega_1$ fix k_α such that

$$\forall\xi,\zeta\in x_\alpha\ \forall i\geq k_\alpha\,(\xi\leq\zeta\rightarrow f_\xi(i)\leq f_\zeta(i))$$

and

$$\forall\eta,\theta\in y_\alpha\ \forall i\geq k_\alpha\,(\eta\leq\theta\rightarrow g_\theta(i)\leq g_\eta(i)).$$

Then without loss of generality I may make the following assumptions:

(a) $\forall\alpha<\omega_1\,(k_\alpha=k\wedge n_\alpha=n\wedge s_\alpha=s)$,
(b) $n\geq k$ (by extending each p_α if necessary),
(c) $\forall\alpha,\beta<\omega_1\,(\alpha<\beta\rightarrow(\max(x_\alpha)<\max(x_\beta)))$,
(d) $\forall\alpha,\beta<\omega_1\,(\alpha<\beta\rightarrow(\max(y_\alpha)<\max(y_\beta)))$.

Let $m=n$, $\xi_\alpha=\max(x_\alpha)$ and $\eta_\alpha=\max(y_\alpha)$. Now it easily follows from the fact that A is an uncountable antichain that if $X=\{\xi_\alpha:\alpha<\omega_1\}$ and $Y=\{\eta_\alpha:\alpha<\omega_1\}$ then both (i) and (ii) hold. ∎

In the discussion that follows I will usually work with equivalent gaps. Therefore when referring to the lemma above I may without loss of generality assume that $X = Y = \omega_1$ and $m = 0$.

LEMMA 2.6. *Let* M *be a c.t.m. for* ZFC *and assume that*, *in* M, $\langle f, g\rangle$ *is a* (λ,κ)*-pregap*, *for regular* λ,κ, *such that* $\mathbb{S}_{\langle f,g\rangle}$ *has the ccc*, *and* $\mathbb{T}$ *is an Aronszajn tree. If* G *is* $\mathbb{S}_{\langle f,g\rangle}$*-generic over* M *then* $\mathrm{M[G]} \models$ "$\mathbb{S}_{\mathbb{T}}$ *has the ccc*".

Proof. According to Corollary 1.3 it is sufficient to show that no new paths are added through $\mathbb{T}$ in M[G]. So by way of contradiction assume that $\dot{b}$ is an $\mathbb{S}_{\langle f,g\rangle}$-name for a new path through $\mathbb{T}$ and $p \in \mathbb{S}_{\langle f,g\rangle}$ such that

$$p \Vdash \text{"}\dot{b} \text{ is a new path through } \check{\mathbb{T}}\text{"}.$$

Let

$$X = \{t \in \mathbb{T} : \exists p_t \le p\, (p_t \Vdash \text{"}\check{t} \in \dot{b}\text{"})\}.$$

Since b is a new path through $\mathbb{T}$, for each $s \in X$ there are $t, u \in X$ such that $s \le_{\mathbb{T}} t, u$ and t and u are incomparable in $\mathbb{T}$. Working in M[G], let

$$Y = \{t \in X : t \text{ is } \le_{\mathbb{T}}\text{-minimal with } t \notin b\}.$$

Then $Y \in \mathrm{M[G]}$ and for each $t \in Y$ fix a $p_t \le p$ with $p_t \Vdash$ "$\check{t} \in \dot{b}$" and let $A = \{p_t \in \mathbb{S}_{\langle f,g\rangle} : t \in Y\}$. Then A is an uncountable subset of $\mathbb{S}_{\langle f,g\rangle}$ in M[G] and any two elements of A are incompatible. Hence A is an uncountable antichain in $\mathbb{S}_{\langle f,g\rangle}$ which contradicts the fact that $\mathbb{S}_{\langle f,g\rangle}$ has the property K in M[G]. Therefore $\mathrm{M[G]} \models$ "$\mathbb{S}_{\mathbb{T}}$ has the ccc". ∎

Let $\mathbb{L} \subseteq \mathbb{Z}^\omega$ such that $(\mathbb{L}, \ll)$ is a linear order. Then $I \subseteq \mathbb{L}$ is an interval in $\mathbb{L}$ iff

$$\forall x, y \in I\ \forall z \in \mathbb{L}\, (x \ll z \ll y \to z \in I).$$

If $\langle f_\xi, g_\eta : \xi < \lambda, \eta < \kappa\rangle \subseteq \mathbb{L}$ is a (λ,κ)-pregap and I is an interval in $\mathbb{L}$ then $\langle f_\xi, g_\eta : \xi < \lambda,\ \eta < \kappa\rangle \subseteq I$ will mean that

$$\exists \alpha < \lambda\ \exists \beta < \kappa\, (\langle f_\xi, g_\eta : \alpha \le \xi < \lambda,\ \beta \le \eta < \kappa\rangle \subseteq I).$$

LEMMA 2.7. *Let* M *be a c.t.m. for* ZFC *and suppose that*, *in* M, $\mathbb{P}$ *is a ccc partial order and* $(\mathbb{L}, \ll)$ *a linear order in* $(\mathbb{Z}^\omega, \ll)$. *If* G *is* $\mathbb{P}$*-generic over* M *with*

$$\mathrm{M[G]} \models \text{"}\langle f, g\rangle \textit{ is a new } (\omega_1,\omega_1)\textit{-gap in } \mathbb{L}\text{"}$$

then, *in* M, *there is a Suslin tree* $\mathbb{T}$ *and a* $\mathbb{P}$*-name* $\dot{b}$ *such that*

$$\mathrm{M[G]} \models \text{"}b \textit{ is a new path through } \mathbb{T}\text{"}.$$

Proof. Let $p_0 \in \mathrm{G}$ with

$$(\diamond) \qquad p_0 \Vdash \text{"}\langle \dot{f}, \dot{g}\rangle \text{ is a new } (\check{\omega}_1, \check{\omega}_1)\text{-gap in } \check{\mathbb{L}}\text{"}.$$

By recursion on $\alpha < \omega_1$ I construct sequences $\langle S_\alpha : \alpha < \omega_1\rangle$ and $\langle A_I^\alpha : \alpha < \omega_1,\ I \in S_\alpha\rangle$ where for each $\alpha < \omega_1$ each element of S_α is a non-empty interval in $\mathbb{L}$ such that

(1) $\forall I, J \in S_\alpha\,(I \neq J \to I \cap J = \emptyset)$,
(2) $\bigcup\{A_I^\alpha : I \in S_\alpha\}$ is a maximal antichain in $\mathbb{P}$ below p_0,
(3) $\forall p \in A_I^\alpha\,(p \le p_0 \wedge p \Vdash \text{“}\langle \dot f, \dot g\rangle \subseteq \check I\text{”})$,
(4) $\forall I \in S_\alpha\ \forall \beta \ge \alpha\ \exists I_1, I_2 \in S_{\beta+1}\,(I_1 \cap I_2 = \emptyset \wedge I_1 \cup I_2 \subseteq I)$,
(5) $\forall \beta > \alpha\ \forall I \in S_\beta\ \exists J \in S_\alpha\,(I \subseteq J)$.

Let $S_0 = \{\mathbb{L}\}$ and $A_{\mathbb{L}}^0 = \{p_0\}$. Fix $\alpha < \omega_1$ and assume that $\forall \xi < \alpha$, S_ξ is constructed together with A_I^ξ, for each $I \in S_\xi$, such that (1)–(5) are satisfied.

First assume $\alpha = \beta + 1$. Note that (1)–(3) and the fact that $\mathbb{P}$ has the ccc imply that $|S_\beta| \le \omega$. Choose $I \in S_\beta$ and $q \in A_I^\beta$. Then since

$$q \Vdash \text{“}\langle \dot f, \dot g\rangle \text{ is a new } (\check\omega_1, \check\omega_1)\text{-gap in } \check{\mathbb{L}}\text{”}$$

there are $r_1, r_2 \le q$ and disjoint intervals $I_0, I_1 \subseteq I$ with $r_i \Vdash \text{“}\langle \dot f, \dot g\rangle \subseteq \check I_i\text{”}$, for $i < 2$, and $I_0 \cup I_1 = I$. Let B_{I_i} be a maximal antichain below p_0 such that

$$r_i \in B_{I_i} \wedge \forall r \in B_{I_i}\ \exists q \in A_I^\beta\,(r \le q \wedge r \Vdash \text{“}\langle \dot f, \dot g\rangle \subseteq \check I_i\text{”}).$$

Now repeat this construction for each $I \in S_\beta$. Then $S_\alpha = \{I_i : I \in S_\beta \wedge i < 2\}$ and for each $i < 2$ and $I \in S_\beta$ let $A_{I_i}^\alpha = B_{I_i}$. Note that $\langle S_\xi : \xi \le \alpha\rangle$ and $\langle A_I^\xi : \xi \le \alpha, I \in S_\xi\rangle$ satisfy (1)–(5). This finishes the construction for successor stages.

Now suppose $\mathrm{cf}(\alpha) = \omega$. Let S be the set of all intervals in $\mathbb{L}$ such that for each $I \in S$ there is a $p \le p_0$ and an increasing sequence $\langle \alpha_n : n < \omega\rangle$ with $\sup\{\alpha_n : n < \omega\} = \alpha$ and for each $n < \omega$ an $I_n \in S_{\alpha_n}$ such that $(m < n \to I_n \subseteq I_m)$ and $I = \bigcap_{n<\omega} I_n$ with $p \Vdash \text{“}\langle \dot f, \dot g\rangle \subseteq \check I\text{”}$. Note that $\forall I, J \in S\,(I \neq J \to I \cap J = \emptyset)$ and $((\diamond) \to S \neq \emptyset)$. Furthermore, S is countable since $\mathbb{P}$ has the ccc. Let $S_\alpha = S$ and for each $I \in S$ let A_I^α be a maximal antichain below p_0 such that $\forall p \in A_I^\alpha\,(p \le p_0 \wedge p \Vdash \text{“}\langle \dot f, \dot g\rangle \subseteq \check I\text{”})$. Then by the definition of S, each A_I^α is non-empty and by maximality of S, $\bigcup\{A_I^\alpha : I \in S\}$ is a maximal antichain in $\mathbb{P}$ below p_0. This finishes the construction.

It is easy to see now that $\langle S_\alpha : \alpha < \omega_1\rangle$ and $\langle A_I^\alpha : \alpha < \omega_1,\ I \in S_\alpha\rangle$ satisfy (1)–(5). Furthermore, $(\diamond)$ implies that $\mathbb{T} = \langle \bigcup_{\alpha<\omega_1} S_\alpha, \supseteq\rangle$ is a Suslin tree in M. However, in M[G], $\langle f, g\rangle$ is a new (ω_1, ω_1)-gap in $\mathbb{L}$ so that $\langle f, g\rangle$ determines a path, b, through $\mathbb{T}$. ∎

The results so far are all that is necessary for treatment of successor stages in the construction of the final model. Now I present several results that will enable me to go beyond the limit stages. Lemmas 1.9 and 1.12

are used to show, as indicated earlier, that no new paths can be added through existing ω_1-trees at limit stages. For gaps the situation is slightly different. In the construction of a $\mathfrak{c}$-saturated linear order $\mathbb{L}$ in $(\mathbb{Z}^\omega, \ll)$, new gaps can appear at limit stages in the portion of $\mathbb{L}$ constructed by that stage. According to Lemma 2.5 there is no problem with non-(ω_1, ω_1)-gaps. But (ω_1, ω_1)-gaps can be somewhat problematic. However, with the aid of Lemma 2.7 the construction will be arranged in such a way that such gaps can only occur at stages of cofinality ω_1 and the splitting orders for such gaps will have the ccc. The next sequence of results is a formalization of the facts just stated. But first some terminology.

In the discussion that follows nice names play an important role. Let M be a c.t.m. for ZFC and $\mathbb{P} \in \mathrm{M}$ a partial order. If $\sigma \in \mathrm{M}^{\mathbb{P}}$, a *nice* $\mathbb{P}$*-name* for a subset of σ is $\tau \in \mathrm{M}^{\mathbb{P}}$ of the form $\bigcup\{\{\pi\} \times A_\pi : \pi \in \mathrm{dom}(\sigma)\}$, where each A_π is an antichain in $\mathbb{P}$. It is shown in [K] that if $\sigma, \mu \in \mathrm{M}^{\mathbb{P}}$ then there is a nice $\mathbb{P}$-name $\tau \in \mathrm{M}^{\mathbb{P}}$ for a subset of σ such that $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\mu \subseteq \sigma \to \mu = \tau$". Since isomorphic partial orders lead to the same generic extensions, it is then justified to use cardinals κ for base sets of partial orders and subsets of $\kappa \times \kappa$ for ordering relations. Therefore, the phrase "let $\dot{\mathbb{Q}}$ be a nice $\mathbb{P}$-name ..." will mean that $\dot{\mathbb{Q}}$ is of the form $(\check{\kappa}, \sigma)$, where κ is some cardinal and σ is a nice $\mathbb{P}$-name for a subset of $(\kappa \times \kappa)\check{}$. Now, in M, let α be a limit ordinal and $\langle \mathbb{P}_\xi : \xi \le \alpha \rangle$ an iterated forcing construction with finite supports where the limit stages are handled in the usual way and the successor stages are obtained as follows: Let $\Lambda = \{\xi : \xi < \alpha \wedge \xi \text{ is even} \wedge \mathrm{cf}(\xi) \ne \omega_1\}$ and let $\mathbb{P}_0$ be the trivial partial order. Let $\gamma + 1 = \beta < \alpha$ and assume that $\langle \mathbb{P}_\xi : \xi < \beta \rangle$ has been constructed together with the sequence $\langle f_\xi : \xi \in \Lambda \cap \beta \rangle$ of functions in $\mathbb{Z}^\omega$ linearly ordered by $\ll$. For the simplicity of notation denote "$\Vdash_{\mathbb{P}_\xi}$" by "$\Vdash_\xi$".

If γ is an odd ordinal, let $\dot{\mathbb{Q}}_\gamma$ be a nice $\mathbb{P}_\gamma$-name for a partial order such that $\mathbf{1} \Vdash_\gamma$ "$\dot{\mathbb{Q}}_\gamma$ has the ccc" and let $\mathbb{P}_\beta = \mathbb{P}_\gamma * \dot{\mathbb{Q}}_\gamma$. At this point it is not important how $\dot{\mathbb{Q}}_\gamma$ are selected, but in the final construction $\dot{\mathbb{Q}}_\gamma$ will be chosen in a way that will ensure Martin's Axiom holds in the final model.

If γ is an even ordinal and not of cofinality ω_1 (i.e. $\gamma \in \Lambda$), then choose a pregap C_γ in $\langle f_\xi : \xi \in \Lambda \cap \beta \rangle$ and let $\mathbb{P}_\beta = \mathbb{P}_\gamma * \dot{\mathbb{S}}_\gamma$ where $\dot{\mathbb{S}}_\gamma$ is a nice $\mathbb{P}_\gamma$-name for the partial order that splits C_γ and let f_γ be an element of $\mathbb{Z}^\omega$ obtained in such a way. The function f_γ will be a part of $\mathbb{L}$ and only at these stages new elements are added to $\mathbb{L}$. At this point also assume that $\mathbf{1} \Vdash_\gamma$ "$\dot{\mathbb{S}}_\gamma$ has the ccc". Once again, at this point it is not important how C_γ are selected, but in the final construction, C_γ will be chosen in a way that will ensure $\mathbb{L} = \langle f_\xi : \xi \in \Lambda \rangle$ is a $\mathfrak{c}$-saturated linear order. However, the description of stages γ, where γ is a limit ordinal of cofinality ω_1 (which follows next), will imply at once that $\mathbf{1} \Vdash_\gamma$ "$\dot{\mathbb{S}}_\gamma$ has the ccc".

Finally, let γ be a limit ordinal of cofinality ω_1. Let $\dot{\mathbb{R}}_\gamma$ be a nice $\mathbb{P}_\xi$-name for the partial order obtained by taking the product of all the splitting orders for (ω_1, ω_1)-pregaps in $\langle f_\xi : \xi \in \Lambda \cap \beta \rangle$ which are also gaps in $(\mathbb{Z}^\omega, \ll)$, and let $\mathbb{P}_\beta = \mathbb{P}_\gamma * \dot{\mathbb{R}}_\gamma$. The rest of this section is devoted to precisely defining this product and showing that $\mathbf{1} \Vdash_\gamma$"$\dot{\mathbb{R}}_\gamma$ has the ccc" so that at the end $\mathbb{P}_\alpha$ will have the countable chain condition. No element of $\mathbb{Z}^\omega$ obtained at this stage will be a part of $\mathbb{L}$. Their existence only ensures that each (ω_1, ω_1)-pregap in the portion of $\mathbb{L}$ constructed by this stage can be split by a ccc partial order at some later stage. Now let G be $\mathbb{P}_\alpha$-generic over M, with $\mathrm{G}_\xi = \mathrm{G} \cap \mathbb{P}_\xi$ and $\mathrm{M}_\xi = \mathrm{M}[\mathrm{G}_\xi]$. Let $\theta < \beta < \alpha$ with $\beta \in \Lambda$ and $A \subseteq \theta \cap \Lambda$ with $A \in \mathrm{M}_\theta$. Then C_β also defines a pregap in $\langle f_\xi : \xi \in A \rangle$. For $p \in \mathbb{S}_\beta$ let $p \restriction A = \langle x_p \cap A, y_p \cap A, n_p, s_p \rangle$ and $\mathbb{S}_{\beta,A} = \{q : \exists p \in \mathbb{S}_\beta \, (q = p \restriction A)\}$ and assume that $\mathbb{S}_{\beta,A} \in \mathrm{M}_\theta$.

LEMMA 2.8. *Let* M, α, $\langle \mathbb{P}_\xi : \xi \le \alpha \rangle$, *and* G *be as above with* $\mathrm{cf}(\alpha) \ne \omega_1$ *and* $\mathbb{L} = \langle f_\xi : \xi \in \Lambda \rangle$ *the linear order in* M[G] *obtained by the construction. If* $\langle f, g \rangle$ *is an* (ω_1, ω_1)*-pregap in* $\mathbb{L}$, *in* M[G], *then there is a* $\beta < \alpha$ *and an equivalent* (ω_1, ω_1)*-pregap* $\langle f', g' \rangle$ *such that* $\langle f', g' \rangle$ *is added to* $\mathbb{L}$ *at stage* β.

Proof. If $\mathrm{cf}(\alpha) = \omega$ then the result follows from the fact that if A is a set of size ω_1 constructed at stage α then there is a $B \in [A]^{\omega_1}$ and $\beta < \alpha$ such that B is constructed at stage β.

If $\mathrm{cf}(\alpha) > \omega_1$ then the result follows from the fact that all sets of size ω_1 constructed at stage α are in fact constructed at some earlier stage. ∎

PROPOSITION 2.9. *In* M, *let* $l < \omega$ *and let* $\langle \mathbb{P}_\xi : \xi \le \alpha \rangle$ *and* G *be as before with* $\mathrm{cf}(\alpha) = \omega_1$. *In* M[G], *let* $A_i, B_i \subseteq \alpha$, *with* $A_i \cup B_i$ *cofinal in* α, *and let* $\langle f_{a^i}, f_{b^i} : a^i \in A_i, b^i \in B_i \rangle$ *be* (ω_1, ω_1)*-gaps with the corresponding splitting orders* $\mathbb{S}^i$, *for* $i < l$. *Then* $\mathbb{S}^0 \times \ldots \times \mathbb{S}^{l-1}$ *has the countable chain condition in* M[G].

The next two lemmas are needed in the proof of this proposition.

LEMMA 2.10. f_β *is* $\mathbb{S}_{\beta,A}$*-generic over* M_θ.

Proof. It suffices to show that the filter obtained from f_β, in $\mathbb{S}_{\beta,A}$, intersects each dense subset of $\mathbb{S}_{\beta,A}$ in M_θ. So let D be a dense subset of $\mathbb{S}_{\beta,A}$, in M_θ. By recursion I define a sequence of sets $\langle D_\xi : \xi \le \beta \rangle$ in M_β as follows: Let $\mathbb{S}_{\beta,\xi}$ be the partial order that fills the pregap in $\langle f_\zeta : \zeta \in \xi \cap \Lambda \rangle$ determined by C_β. Then

$$D_0 = \{q \in \mathbb{S}_{\beta,0} : \exists p \in D \, (q \le p \restriction 0)\}.$$

Fix $\xi < \beta$ and assume D_ζ has been defined for each $\zeta < \xi$. If $\xi = \zeta + 1$ then

$$D_\xi = \{q \in \mathbb{S}_{\beta,\xi} : \exists q_1 \in D_\zeta \ \exists p \in D \, (q \le q_1, p \restriction \xi)\}.$$

And if ξ is a limit then $D_\xi = \bigcup_{\zeta < \xi} D_\zeta$.

Then by induction I show that each D_ξ is dense in $\mathbb{S}_{\beta,\xi}$. Since D is dense it follows that D_0 is also dense. If ξ is a limit ordinal then the result also follows easily from the definition of D_ξ and the induction hypothesis. Now assume that D_ξ is dense and show that $D_{\xi+1}$ is also dense. If $\xi \notin \Lambda$ then $D_{\xi+1} = D_\xi$ and the result follows from the induction hypothesis. So assume $\xi \in \Lambda$.

Case 1: C_ξ and C_β define the same pregap in $\langle f_\zeta : \zeta \in \xi \cap \Lambda\rangle$. In this case $\mathbb{S}_{\beta,\xi} = \mathbb{S}_\xi$ so that f_ξ is $\mathbb{S}_{\beta,\xi}$-generic over M_ξ. Let $p \in \mathbb{S}_{\beta,\xi+1}$ and by extending p if necessary I may assume that $\xi \in x_p \cup y_p$, say $\xi \in y_p$. Let $q = \langle x_p, y_p \setminus \{\xi\}, n_p, s_p\rangle$ and note that $q \in \mathbb{S}_\xi$. Now D_ξ is dense in $\mathbb{S}_\xi$, so let $q_1 \in D_\xi$ and $p_1 \in D$ from which q_1 is defined ($q_1 \le p_1 \restriction \xi$) such that $q_1 \le q$. Note that D_ξ may not be in M_ξ, but q_1 is. Now f_ξ is $\mathbb{S}_\xi$-generic over M_ξ so that f'_ξ is also $\mathbb{S}_\xi$-generic over M_ξ where f'_ξ is just f_ξ modified by s_{q_1}. So let q_2 be an element in the $\mathbb{S}_\xi$-generic filter over M_ξ determined by f'_ξ with $q_2 \le q_1$. Then it is easily seen that $q_3 = \langle x_{q_2}, y_{q_2} \cup \{\xi\}, n_{q_2}, s_{q_2}\rangle \in \mathbb{S}_{\beta,\xi+1}$ with $q_3 \le q_1, p$. But also $q_3 \le p_1 \restriction (\xi+1)$ so that $q_3 \in D_{\xi+1}$, showing that $D_{\xi+1}$ is dense in $\mathbb{S}_{\beta,\xi+1}$.

Case 2: C_ξ and C_β do not define the same pregap in $\langle f_\zeta : \zeta \in \xi \cap \Lambda\rangle$. Then there is a $\zeta_0 < \xi$ such that f_{ζ_0} is between the pregaps C_ξ and C_β in $\langle f_\zeta : \zeta \in \xi \cap \Lambda\rangle$. I may assume C_ξ is to the right of C_β. Let $p \in \mathbb{S}_{\beta,\xi+1}$ and by extending p if necessary I may assume that $\xi \in y_p$. In addition I may assume that $\zeta_0 \in y_p$ and that $n_0 < \omega$ is such that $\forall i \ge n_0\,(f_{\zeta_0}(i) \le f_\xi(i))$ with $n_0 \le n_p$. Let $q = \langle x_p, y_p \setminus \{\xi\}, n_p, s_p\rangle$ and choose $q_1 \in D_\xi$ and $p_1 \in D$ from which q_1 is defined ($q_1 \le p_1 \restriction \xi$) such that $q_1 \le q$. Let $q_2 = \langle x_{q_1}, y_{q_1} \cup \{\xi\}, n_{q_1}, s_{q_1}\rangle$. Then it is clear that $q_2 \in \mathbb{S}_{\beta,\xi+1}$ with $q_2 \le q_1, p, p_1 \restriction (\xi+1)$ so that $q_2 \in D_{\xi+1}$, showing that $D_{\xi+1}$ is dense in $\mathbb{S}_{\beta,\xi+1}$.

And now I conclude that D_β is dense in $\mathbb{S}_\beta$. Therefore let q be an element in the intersection of D_β and the $\mathbb{S}_\beta$-generic filter determined by f_β. By definition of D_β, let $p \in D$ with $q \le p$. Then p is also in the filter obtained from f_β in $\mathbb{S}_{\beta,A}$. ■

LEMMA 2.11. *Let* M, $\langle \mathbb{P}_\xi : \xi \le \alpha\rangle$, *and* G *be as before with* $\mathrm{cf}(\alpha) = \omega_1$ *in* M. *In addition, assume that* $\mathbb{P}_\alpha$ *has the ccc in* M. *Let* $A, B \subseteq \alpha \cap \Lambda$ *with each* A *and* B *of order type* ω_1, $A \cup B$ *cofinal in* α *and* $\langle f_a, f_b : a \in A,\ b \in B\rangle$ *an* (ω_1, ω_1)*-gap in* $\langle f_\xi : \xi \in \alpha \cap \Lambda\rangle$, *in* M[G], *with its splitting order* $\mathbb{S}_{A,B}$. *Then* $\mathbb{S}_{A,B}$ *has the ccc in* M[G].

Proof. Working in M[G], let $A = \langle a_\xi : \xi < \omega_1\rangle$ and $B = \langle b_\xi : \xi < \omega_1\rangle$ be increasing enumerations of A and B. By way of contradiction assume that the conclusion of the lemma is false. Then by restricting the discussion to an equivalent gap or to $\langle -f_b, -f_a : b \in B,\ a \in A\rangle$ I may assume, according to Lemma 2.5, that for some $m < \omega$,

$$(\dagger)\qquad \forall\xi < \omega_1\,\forall i \ge m\,(f_{a_\xi}(i) \le f_{b_\xi}(i)) \quad\text{and}\quad \forall\xi < \eta < \omega_1\;\exists i \ge m\,(f_{a_\xi}(i) \not\le f_{b_\eta}(i) \vee f_{a_\eta}(i) \not\le f_{b_\xi}(i)).$$

The rest of the argument involves only integers greater than or equal to m, therefore, for the sake of simplicity I will assume that $m = 0$, which completely eliminates any reference to m in the rest of the argument. I may also assume that for each $\eta < \omega_1$ each element of $\{f_{a_\xi}, f_{b_\zeta} : \xi \le \eta,\ \zeta < \eta\}$ is constructed before f_{b_η}. This makes B cofinal in α. In M, let τ be a nice $\mathbb{P}_\alpha$-name for B and for each $\beta < \alpha$ let $\pi(\beta) = \min\{\xi : \mathbf{1} \Vdash_\alpha (\exists\check{p} \in \dot{G}_\xi\,(\check{p} \Vdash_\xi \check{\beta} \in \tau))\}$. Then since $\mathbb{P}_\alpha$ has the ccc and τ is a nice name there is a closed and unbounded $C \subseteq \alpha$ such that $\mathbf{1} \Vdash_\alpha$ "$\check{\beta}$ is a limit point of τ" for $\beta \in C$ and $\pi(\xi) < \beta$ for each $\xi < \beta$. Note that also $C \subseteq \text{closure}(B)$ and $\forall\xi \in C\,(B \cap \xi \in \mathrm{M}_\xi)$. By performing a similar construction for A, if necessary, I may also assume that $\{a_\zeta \in A : b_\zeta \in B \cap \xi\} \in \mathrm{M}_\xi$ for each $\xi \in C$. Now for each $\gamma \in C$ let $\beta_\gamma = \min(B \setminus \gamma)$. Therefore $\beta_\gamma = b_\xi$ for some $\xi < \omega_1$, in which case let $\alpha_\gamma = a_\xi$. Let $\mathbb{S}_{\alpha_\gamma,\beta_\gamma}$ be the splitting order for $\langle f_a, f_b : a \le \alpha_\gamma,\ b < \beta_\gamma\rangle$. Then by Lemma 2.10 and (†) it follows that for each $\gamma \in C$ there is a p_γ, in the $\mathbb{S}_{\alpha_\gamma,\beta_\gamma}$-generic filter over $\mathrm{M}_{\beta_\gamma}$ determined by f_{β_γ}, such that

$$(\ddagger)\qquad p_\gamma \Vdash_{\beta_\gamma} \text{“}\dot{f}_{\alpha_\gamma} \le \dot{f}_{\beta_\gamma} \wedge \forall\dot{b}_\xi \in \dot{B} \cap \check{\gamma}\,(\dot{f}_{a_\xi} \not\le \dot{f}_{\beta_\gamma} \vee \dot{f}_{\alpha_\gamma} \not\le \dot{f}_{b_\xi})\text{”},$$

where $f \le g$ iff $\forall i < \omega\,(f(i) \le g(i))$. By extending p_γ if necessary I may assume that $\alpha_\gamma \in x_{p_\gamma}$. Now define ψ on C by

$$\psi(\gamma) = \max(x_{p_\gamma} \cup y_{p_\gamma} \setminus \{\alpha_\gamma\}).$$

Then $\psi(\gamma) < \gamma$ for each $\gamma \in C$ so that there is a $D \subseteq C$, cofinal in C, hence α, and a θ such that $\forall\gamma \in D\,(\psi(\gamma) = \theta)$. Let $\gamma_0 = \min(C \setminus \theta)$ and, by shrinking D if necessary, assume that

$$\forall\gamma \in D\,(\beta_\gamma > \beta_{\gamma_0}) \wedge \forall\gamma,\delta \in D\,(x_{p_\gamma} \setminus \{\alpha_\gamma\} = x_{p_\delta} \setminus \{\alpha_\delta\} \wedge y_{p_\gamma} = y_{p_\delta} \wedge n_{p_\gamma} = n_{p_\delta} \wedge s_{p_\gamma} = s_{p_\delta}).$$

For $\delta \in D$ let $G_{\alpha_\delta,\beta_\delta}$ be the $\mathbb{S}_{\alpha_\delta,\beta_\delta}$-generic filter determined by f_{β_δ}. Then for each $\gamma < \delta \in D$ there is a $q' \in G_{\alpha_\delta,\beta_\delta}$ such that $q' \le p_\delta$ with $\alpha_\gamma \in x_{q'}$ and $\beta_\gamma \in y_{q'}$. Hence, it follows that if $q = \langle x_{p_\delta} \cup \{\alpha_\gamma\}, y_{p_\delta} \cup \{\beta_\gamma\}, n_{q'}, s_{q'}\rangle$ then $q \in G_{\alpha_\delta,\beta_\delta}$ and $q \le p_\delta$. Therefore, since $|D| = \omega_1$ and $|\mathbb{Z}^{<\omega}| = \omega$ there are $\gamma < \delta \in D$ and $k < \omega$, with $n_{p_\delta} \le k$, such that $f_{\beta_\gamma}\restriction k = f_{\beta_\delta}\restriction k$ and $q = \langle x_{p_\delta} \cup \{\alpha_\gamma\}, y_{p_\delta} \cup \{\beta_\gamma\}, k, f_{\beta_\gamma}\restriction k\rangle \in G_{\alpha_\delta,\beta_\delta}$ with $q \le p_\delta$ in $\mathbb{S}_{\alpha_\delta,\beta_\delta}$. But now, since $q \in \mathbb{S}_{\alpha_\delta,\beta_\delta} \cap G_{\alpha_\delta,\beta_\delta}$ it follows that

$$(*)\qquad q \Vdash_{\beta_\delta} \text{“}\forall\check{i} \ge \check{k}\,(\dot{f}_{\alpha_\gamma}(\check{i}), \dot{f}_{\alpha_\delta}(\check{i}) \le \dot{f}_{\beta_\delta}(\check{i}) \le \dot{f}_{\beta_\gamma}(\check{i}))\text{”}.$$

Also, by (‡) it follows that $p_\gamma \Vdash_{\beta_\gamma}$ "$\dot{f}_{\alpha_\gamma} \le \dot{f}_{\beta_\gamma}$" and $p_\delta \Vdash_{\beta_\delta}$ "$\dot{f}_{\alpha_\delta} \le \dot{f}_{\beta_\delta}$".

Therefore, since $f_{\beta_\gamma}\restriction k = f_{\beta_\delta}\restriction k$ it follows that

$$(\circ)\qquad q \Vdash_{\beta_\delta} \text{“}\forall \check{i} < \check{k}\,(\dot{f}_{\alpha_\gamma}(\check{i}), \dot{f}_{\alpha_\delta}(\check{i}) \le \dot{f}_{\beta_\delta}(\check{i}))\text{”}.$$

Now, from (∗) and (∘) it follows that

$$q \Vdash_{\beta_\delta} \text{“}\forall \check{i} < \check{\omega}\,(\dot{f}_{\alpha_\gamma}(\check{i}) \le \dot{f}_{\beta_\delta}(\check{i}) \wedge \dot{f}_{\alpha_\delta}(\check{i}) \le \dot{f}_{\beta_\gamma}(\check{i}))\text{”}.$$

But this clearly contradicts the part of (‡) which states that $p_\delta \Vdash_{\beta_\delta}$ “$\dot{f}_{\alpha_\gamma} \not\le \dot{f}_{\beta_\delta} \vee \dot{f}_{\alpha_\delta} \not\le \dot{f}_{\beta_\gamma}$”, since $q \le p_\delta$. Therefore $\mathbb{S}_{A,B}$ has the ccc in M[G]. ∎

Proof of Proposition 2.9. For the sake of notational simplicity I will present the proof of the proposition for the case when $l = 2$. The proof presented below can easily be modified to prove the general case when l is an arbitrary integer. Let $A_i = \{a^i_\xi : \xi < \omega_1\}$ and $B_i = \{b^i_\xi : \xi < \omega_1\}$ be the enumerations in the increasing order of A_i and B_i for $i < 2$. Define $\langle f_a, f_b\rangle = \langle f^a_\xi, f^b_\xi : \xi < \omega_1\rangle$ as follows:

$$f^a_\xi(n) = \begin{cases} f_{a^0_\xi}(k) & \text{if } n = 2k,\\ f_{a^1_\xi}(k) & \text{if } n = 2k+1,\end{cases}\qquad f^b_\xi(n) = \begin{cases} f_{b^0_\xi}(k) & \text{if } n = 2k,\\ f_{b^1_\xi}(k) & \text{if } n = 2k+1.\end{cases}$$

Then $\langle f_a, f_b\rangle$ is an (ω_1,ω_1)-pregap and since $\mathbb{S}_{\langle f_a,f_b\rangle}$ can be densely embedded in $\mathbb{S}^0 \times \mathbb{S}^1$ it suffices to show that $\mathbb{S}_{\langle f_a,f_b\rangle}$ has the ccc in M[G]. By way of contradiction suppose not. Then as in the previous lemma I may assume that

$$\forall \xi < \eta < \omega_1\,(f^a_\eta \le f^b_\eta \wedge (f^a_\xi \not\le f^b_\eta \vee f^a_\eta \not\le f^b_\xi))$$

and also that for each $\eta < \omega_1$, each element of $\{f_{a^i_\xi}, f_{b^i_\zeta} : \xi \le \eta,\ \zeta < \eta,\ i < 2\}$ is constructed before $f_{b^i_\eta}$ for $i < 2$ and that $f_{b^0_\eta}$ is constructed before $f_{b^1_\eta}$. Let C_0 and C_1 be the corresponding closed and unbounded subsets of α as in the previous lemma. Then $C = C_0 \cap C_1$ is also closed and unbounded in α. For $\gamma \in C$ and $i < 2$ let $\beta^i_\gamma = \min(B_i \setminus \gamma)$. Then $\beta^i_\gamma = b^i_\xi$ for some $\xi < \omega_1$, in which case let $\alpha^i_\gamma = a^i_\xi$. Let $\mathbb{S}_{\alpha^i_\gamma,\beta^i_\gamma}$ be the splitting order for $\langle f_{a^i}, f_{b^i} : a^i \le \alpha^i_\gamma,\ b^i < \beta^i_\gamma\rangle$ and $\mathbb{S}_{\alpha_\gamma,\beta_\gamma}$ the splitting order for $\langle f^a_\xi, f^b_\eta : \xi \le \alpha_\gamma,\ \eta < \beta_\gamma\rangle$. Then by Lemma 2.10, for each $\gamma \in C$ and $i < 2$, $f_{\beta^i_\gamma}$ is $\mathbb{S}_{\alpha^i_\gamma,\beta^i_\gamma}$-generic over $\mathrm{M}_{\beta^0_\gamma}$. Therefore $f^b_{\beta_\gamma}$ is $\mathbb{S}_{\alpha_\gamma,\beta_\gamma}$-generic over $\mathrm{M}_{\beta^0_\gamma}$. Now the rest of the proof continues as in the previous lemma in order to get a contradiction. Therefore $\mathbb{S}_{\langle f_a,f_b\rangle}$ has the ccc in M[G]. ∎

Finally, I explain what is meant by the product of all splitting orders for (ω_1,ω_1)-gaps and present some of its properties.

Definition 2.12. Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. An $i : \mathbb{P} \to \mathbb{Q}$ is a *complete embedding* if

(1) $\forall p, p' \in \mathbb{P}\,(p' \le p \to i(p') \le i(p))$,
(2) $\forall p, p' \in \mathbb{P}\,(p' \perp p \leftrightarrow i(p') \perp i(p))$,

(3) $\forall q \in \mathbb{Q}\ \exists p \in \mathbb{P}\ \forall p' \in \mathbb{P}\,(p' \le p \to i(p') \not\le q)$.

The following lemma is taken from [K].

Lemma 2.13. *Suppose $i, \mathbb{P}, \mathbb{Q}$ are in* M, $i : \mathbb{P} \to \mathbb{Q}$ *and i is a complete embedding. Let* H *be $\mathbb{Q}$-generic over* M. *Then $i^{-1}(\mathrm{H})$ is $\mathbb{P}$-generic over* M *and* $\mathrm{M}[i^{-1}(\mathrm{H})] \subseteq \mathrm{M}[\mathrm{H}]$.

Definition 2.14. Let A be a set and $\langle \mathbb{P}_a : a \in A\rangle$ a sequence of partial orders. Then $\prod_{a\in A} \mathbb{P}_a$ denotes the set of all sequences $\langle p_a : a \in A\rangle$ such that $p_a \in \mathbb{P}_a$ and $p_a = \mathbf{1}_a$ for all but finitely many $a \in A$. If $B \subseteq A$ then

$$\prod_{a\in A}^{B} \mathbb{P}_a = \Big\{p \in \prod_{a\in A} \mathbb{P}_a : \forall a \in A \setminus B\,(p_a = \mathbf{1}_a)\Big\}.$$

And let $i : \prod_{a\in A}^{B} \mathbb{P}_a \to \prod_{a\in A} \mathbb{P}_a$ be the inclusion map $i(p) = p$.

In the final construction each $\mathbb{P}_a$ will be a splitting order for some (ω_1, ω_1)-gap. Proposition 2.9, in conjunction with the next lemma, whose proof is standard, is used to show that such products have the countable chain condition.

Lemma 2.15. *Let A be a set and $\langle \mathbb{P}_a : a \in A\rangle$ a sequence of partial orders.*

(1) *If $B \subseteq A$ then the inclusion $i : \prod_{a\in A}^{B} \mathbb{P}_a \to \prod_{a\in A} \mathbb{P}_a$ is a complete embedding.*

(2) *$\prod_{a\in A} \mathbb{P}_a$ has the ccc iff for every finite $B \subseteq A$, $\prod_{a\in A}^{B} \mathbb{P}_a$ has the ccc.*

This essentially finishes the treatment of gaps. Now I am ready for the final construction.

3. Final model. In this section I combine the work of Todorčević and Laver to obtain the final model. In his construction, Todorčević starts with Mitchell's model, in [M], for ¬wKH. Therefore I begin with a brief discussion of that model.

Let M be a c.t.m. for ZFC + V=L and, in M, let κ be the first strongly inaccessible cardinal. From now on inaccessible will mean strongly inaccessible. If A and B are sets and μ a cardinal then

$$\mathbb{F}\mathrm{n}(A, B, \mu) = \{p : (|p| < \mu) \wedge (p \text{ is a function}) \wedge (\mathrm{dom}(p) \subseteq A) \wedge (\mathrm{ran}(p) \subseteq B)\}.$$

Let $\mathbb{C} = \mathbb{F}\mathrm{n}(\kappa, 2, \omega)$ and partially order $\mathbb{C}$ by $p \le_{\mathbb{C}} q$ iff $p \supseteq q$. $\mathbb{C}$ is the standard partial order for adding κ generic subsets of ω. Then $\mathbb{C}$ has the ccc in M and as such preserves cardinals. For $\gamma < \kappa$ let

$$\mathbb{C}_\gamma = \{p \in \mathbb{C} : \mathrm{dom}(p) \subseteq \gamma\} \quad \text{and} \quad \mathbb{C}^\gamma = \{p \in \mathbb{C} : \mathrm{dom}(p) \cap \gamma = \emptyset\}.$$

Then $\mathbb{C} \cong \mathbb{C}_\gamma \times \mathbb{C}^\gamma$ and if G is $\mathbb{C}$-generic over M then $\mathrm{G}_\gamma = \mathrm{G} \cap \mathbb{C}_\gamma$ is $\mathbb{C}_\gamma$-generic over M and $\mathrm{G}^\gamma = \mathrm{G} \cap \mathbb{C}^\gamma$ is $\mathbb{C}^\gamma$-generic over $\mathrm{M}[\mathrm{G}_\gamma]$ with $\mathrm{M}[\mathrm{G}] = \mathrm{M}[\mathrm{G}_\gamma][\mathrm{G}^\gamma]$. Let $\mathbb{B}$ be the complete Boolean algebra of regular open subsets of $\mathbb{C}$. Then $\mathbb{C}$ is dense in $\mathbb{B}$. For $\gamma < \kappa$, let $\mathbb{B}_\gamma$ be the complete Boolean algebra of regular open subsets of $\mathbb{C}_\gamma$ and identify each $\mathbb{B}_\gamma$ with its image in $\mathbb{B}$ under the normal complete embedding. Then for each $\gamma < \delta < \kappa$ it follows that $\mathbb{B}_\gamma$ is a complete subalgebra of $\mathbb{B}_\delta$, which in turn is a complete subalgebra of $\mathbb{B}$.

In M, let

$$\mathbb{D} = \{f : f \in \mathbb{F}\mathrm{n}(\kappa, \mathbb{B}, \omega_1) \wedge \forall \gamma \in \mathrm{dom}(f)\,(f(\gamma) \in \mathbb{B}_{\gamma+\omega})\}.$$

For $f \in \mathbb{D}$ define $\bar{f} : \mathrm{dom}(f) \to 2$, in M[G], by $\bar{f}(\gamma) = 1$ iff $\exists p \in \mathrm{G}\,(p \le_{\mathbb{B}} f(\gamma))$. Also in M[G], let $\mathbb{E} = \{\bar{f} : f \in \mathbb{D}\}$ partially ordered by $\bar{f} \le_{\mathbb{E}} \bar{g}$ iff $\bar{f} \supseteq \bar{g}$. Also partially order $\mathbb{D}$, in M, by $f \le_{\mathbb{D}} g$ iff $\mathbf{1} \Vdash_{\mathbb{C}}$ "$\bar{f} \le_{\mathbb{E}} \bar{g}$". In M, let $\mathbb{F}$ be a partial order with domain $\mathbb{C} \times \mathbb{D}$ partially ordered by

$$(p, f) \le_{\mathbb{F}} (q, g) \quad \text{iff} \quad p \le_{\mathbb{C}} q \wedge (p \Vdash_{\mathbb{C}} \text{“} \bar{f} \le_{\mathbb{E}} \bar{g} \text{”}).$$

Now I list a few properties of the partial orders defined above and refer the reader to [M] for proofs and further details. Let K be $\mathbb{F}$-generic over M. Then $\mathrm{G} = \{p \in \mathbb{C} : (p, \emptyset) \in \mathrm{K}\}$ is $\mathbb{C}$-generic over M and $\mathrm{H} = \{\bar{f} \in \mathbb{E} : (\emptyset, f) \in \mathrm{K}\}$ is $\mathbb{E}$-generic over M[G] and M[K]=M[G][H]. Also $\omega_1^{\mathrm{M}} = \omega_1^{\mathrm{M[G]}} = \omega_1^{\mathrm{M[K]}}$, and $\mathbb{F}$ has the κ-cc so that κ is a cardinal in M[K] with $\kappa = \omega_2^{\mathrm{M[K]}}$.

In M, let $\mathbb{D}_\gamma = \{f \restriction \gamma : f \in \mathbb{D}\}$, $\mathbb{D}^\gamma = \{f \setminus (f \restriction \gamma) : f \in \mathbb{D}\}$, $\mathbb{F}_\gamma = \mathbb{C}_\gamma \times \mathbb{D}_\gamma$ and $\mathbb{F}^\gamma = \mathbb{C}^\gamma \times \mathbb{D}^\gamma$ for $\gamma < \kappa$. Then $\mathrm{K}_\gamma = \mathrm{K} \cap \mathbb{F}_\gamma$ and $\mathrm{K}^\gamma = \mathrm{K} \cap \mathbb{F}^\gamma$. In M[G], let $\mathbb{E}_\gamma = \{f \restriction \gamma : f \in \mathbb{E}\}$ and $\mathbb{E}^\gamma = \{f \setminus (f \restriction \gamma) : f \in \mathbb{E}\}$ for $\gamma < \kappa$. Partially order $\mathbb{F}^\gamma$, in $\mathrm{M}[\mathrm{G}_\lambda]$, by

$$(p, f) \le_{\mathbb{F}^\gamma} (q, g) \quad \text{iff} \quad p \le_{\mathbb{C}} q \wedge \exists p' \in \mathrm{G}_\lambda\,(p \cup p' \Vdash_{\mathbb{C}} \text{“} \bar{f} \le_{\mathbb{E}} \bar{g} \text{”}).$$

Then for each γ such that $\forall \gamma' < \gamma\,(\gamma' + \omega < \gamma)$, K_γ is $\mathbb{F}_\gamma$-generic over M and K^γ is $\mathbb{F}^\gamma$-generic over $\mathrm{M}[\mathrm{K}_\gamma]$ with $\mathrm{M}[\mathrm{K}] = \mathrm{M}[\mathrm{K}_\gamma][\mathrm{K}^\gamma]$. Also, since $|\mathbb{F}_\gamma| < \kappa$, it follows that κ is still inaccessible in $\mathrm{M}[\mathrm{K}_\gamma]$. If λ is an uncountable cardinal in $\mathrm{M}[\mathrm{K}_\gamma]$ with $\lambda < \kappa$, then λ is collapsed onto ω_1 in M[K]. In addition, in M[K], $2^\omega = 2^{\omega_1} = \omega_2$. Furthermore, if $\mathbb{R}$ is a ccc partial order in $\mathrm{M}[\mathrm{K}_\gamma]$ and I is $\mathbb{R}$-generic over M[K] then I is also $\mathbb{R}$-generic over $\mathrm{M}[\mathrm{K}_\gamma]$ with

$$\mathrm{M}[\mathrm{K}_\gamma][\mathrm{I}][\mathrm{K}^\gamma] = \mathrm{M}[\mathrm{K}_\gamma][\mathrm{K}^\gamma][\mathrm{I}] = \mathrm{M}[\mathrm{K}][\mathrm{I}].$$

The following lemma and its proof are due to Todorčević [T].

Lemma 3.1. *Let $\nu > \omega_1^{\mathrm{M}}$ be a regular cardinal in* M *and $\mathbb{R}$ a ccc partial order in* $\mathrm{M}[\mathrm{K}_\nu]$. *Let* I *be $\mathbb{R}$-generic over* $\mathrm{M}[\mathrm{K}_\nu]$ *and $\mathbb{T}$ an ω_1-tree in* $\mathrm{M}[\mathrm{K}_\nu][\mathrm{I}]$. *If b is a path through $\mathbb{T}$ in* M[K][I] *then* $b \in \mathrm{M}[\mathrm{K}_\nu][\mathrm{I}]$.

Now I am ready for the construction of the main model.

THEOREM 3.2. *Let* M *be a c.t.m. for* ZFC + V=L *and* κ *the first inaccessible cardinal in* M. *Then there is an extension of* M *which is a model for*

$$\mathrm{ZFC} + \mathrm{MA} + \neg\mathrm{wKH} + \mathfrak{c} = \omega_2 + \varphi_{\mathfrak{c}}.$$

Proof. Let $\mathbb{F}$ be the partial order described above and K $\mathbb{F}$-generic over M with N=M[K]. In N I construct a finite support ccc iteration

$$\langle\langle \mathbb{P}_\xi : \xi \le \omega_2\rangle, \langle \mathbb{Q}_\xi : \xi < \omega_2\rangle\rangle$$

of length ω_2. In the process I construct a $\mathfrak{c}$-saturated linear order $(\mathbb{L}, \ll)$ in $(\mathbb{Z}^\omega, \ll)$. At the successor stages I alternate between ccc partial orders to make MA true and partial orders which are the splitting orders for some pregap in $\mathbb{L}$. According to Lemma 2.5 only the splitting orders for (ω_1, ω_1)-gaps may fail to have the ccc. However, the construction is arranged in such a way that such gaps occur in $\mathbb{L}$ only at stages of cofinality ω_1; at these stages the splitting orders for all these gaps will have the ccc and at these stages I split these gaps, all at once. The splitting functions added at these stages will not be a part of $\mathbb{L}$, but they are needed to ensure that the splitting orders for all pregaps in $\mathbb{L}$ remain ccc until the pregaps are filled, one by one, at the later successor stages. The partial orders that are used at these limit stages of the iteration have cardinality ω_2, which causes some difficulty in the proof of $\neg$wKH. This difficulty is overcome by reducing the argument to suborders of size ω_1 of these partial orders. If γ is a limit ordinal then $\mathbb{P}_\gamma$ is obtained in the usual way.

In N, let

$$\Lambda = \{\xi < \omega_2 : \xi \text{ is an even ordinal and } \mathrm{cf}(\xi) \ne \omega_1\}$$

and let $g : \omega_2 \to \omega_2 \times \omega_2$ such that g maps both Λ and $\omega_2 \setminus \Lambda$ onto $\omega_2 \times \omega_2$ with the property that

$$\forall \xi, \eta, \gamma < \omega_2\, (g(\xi) = \langle \eta, \gamma\rangle \to \eta \le \xi).$$

The function g will be used in deciding how to choose each $\mathbb{Q}_\xi$.

Let $\mathbb{P}_0$ be the trivial partial order. Suppose $\xi < \omega_2$ and that $\mathbb{P}_\xi$ has been constructed and let $\mathbb{L}_\xi = \{f_\zeta : \zeta \in \xi \cap \Lambda\}$ be the portion of $\mathbb{L}$ constructed by stage ξ and N_ξ the extension of N by $\mathbb{P}_\xi$. First consider the case when ξ is an odd ordinal. At these stages no new elements are added to $\mathbb{L}$ so that $\mathbb{L}_{\xi+1} = \mathbb{L}_\xi$. In N, let $\langle\langle \lambda^\xi_\gamma, \sigma^\xi_\gamma\rangle : \gamma < \omega_2\rangle$ be an enumeration of all pairs $\langle \lambda, \sigma\rangle$ such that $\lambda < \omega_2$, λ is a cardinal and σ is a nice $\mathbb{P}_\xi$-name for a subset of $(\lambda \times \lambda)\check{}$. Let $g(\xi) = \langle \eta, \gamma\rangle$. Since $\eta \le \xi$, the $\mathbb{P}_\eta$-name, σ^η_γ, has been defined. Let σ be the corresponding $\mathbb{P}_\xi$-name and $\lambda = \lambda^\eta_\gamma$. There are three cases to consider.

Case 1. If it is not the case that $\mathbf{1} \Vdash_\xi$ "$\langle \check{\lambda}, \sigma\rangle$ has the ccc" then let $\mathbb{Q}_\xi$ be a nice $\mathbb{P}_\xi$-name for the trivial partial order and $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \mathbb{Q}_\xi$.

C a s e 2. If $\mathbf{1} \Vdash_\xi$ "$\langle\check{\lambda}, \sigma\rangle$ has the ccc" and it is not the case that extending by $\langle\check{\lambda}, \sigma\rangle$ adds any new paths through an ω_1-tree, in N_ξ, then by Lemmas 1.9 and 2.7 it is not the case that a new (ω_1, ω_1)-gap is added in $\mathbb{L}_\xi$. Then let $\mathbb{Q}_\xi$ be $\langle\check{\lambda}, \sigma\rangle$ and $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \mathbb{Q}_\xi$.

C a s e 3. If $\mathbf{1} \Vdash_\xi$ "$\langle\check{\lambda}, \sigma\rangle$ has the ccc" and extending by $\langle\check{\lambda}, \sigma\rangle$ adds a new path through an ω_1-tree $\mathbb{T}$ in N_ξ, or extending by $\langle\check{\lambda}, \sigma\rangle$ adds a new (ω_1, ω_1)-gap in $\mathbb{L}_\xi$ then by Lemma 1.9 or Lemma 2.7, respectively, there is a Suslin tree $\mathbb{U}$, in N_ξ, such that a new path is added through $\mathbb{U}$. Therefore in the extension by $\langle\check{\lambda}, \sigma\rangle$ the specializing partial order $\mathbb{S}_\mathbb{U}$ for $\mathbb{U}$ fails to have the ccc. Then by Lemma 1.11 there is an element p in $\mathbb{S}_\mathbb{U}$ such that, in N_ξ,

$$p \Vdash_{\mathbb{S}_\mathbb{U}} \text{“}\langle\check{\lambda}, \sigma\rangle \text{ fails to have the ccc”}.$$

Let $\mathbb{Q}_\xi$ be a nice $\mathbb{P}_\xi$-name for the suborder of $\mathbb{S}_\mathbb{U}$ below p. Then $\mathbf{1} \Vdash_\xi$ "$\mathbb{Q}_\xi$ has the ccc" and let $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \mathbb{Q}_\xi$. Note that

$$\mathbf{1} \Vdash_{\xi+1} \text{“}\langle\check{\lambda}, \sigma\rangle \text{ fails to have the ccc”},$$

and once a partial order fails to have the ccc it fails to have the ccc in all further extensions. Also note that no new paths are added through ω_1-trees and hence no new (ω_1, ω_1)-gaps in $\mathbb{L}_\xi$ in the extension by $\mathbb{Q}_\xi$.

This finishes the treatment of odd successor stages. Now assume ξ is an even ordinal with $\mathrm{cf}(\xi) = \omega_1$. In N_ξ, let $\langle C^\xi_\zeta : \zeta < \omega_2\rangle$ be an enumeration of all pregaps in $\mathbb{L}_\xi$ represented by (ω_1, ω_1)-gaps constructed in $\mathbb{L}_\xi$ at stage ξ, with the corresponding splitting orders $\mathbb{S}^\xi_\zeta$. Then by Proposition 2.9 and Lemma 2.15, $\prod_{\zeta<\omega_2} \mathbb{S}^\xi_\zeta$ has the ccc. Note that also $|\prod_{\zeta<\omega_2} \mathbb{S}^\xi_\zeta| \le \omega_2$ and for all $\gamma < \omega_2$, $|\prod^\gamma_{\zeta<\omega_2} \mathbb{S}^\xi_\zeta| \le \omega_1$. Let τ^ξ be a $\mathbb{P}_\xi$-name for the partial order $\prod_{\zeta<\omega_2} \mathbb{S}^\xi_\zeta$ and τ^ξ_γ a $\mathbb{P}_\xi$-name for the partial order $\prod^\gamma_{\zeta<\omega_2} \mathbb{S}^\xi_\zeta$ arranged in such a way that $\tau^\xi_\gamma \subseteq \tau^\xi_\delta \subseteq \tau^\xi$ as names, for $\gamma < \delta < \omega_2$. Let $\mathbb{Q}_\xi$ be τ^ξ and $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \mathbb{Q}_\xi$. Then $\mathbb{P}_{\xi+1}$ has the ccc and with the help of Lemmas 1.12 and 2.6 extending by $\mathbb{Q}_\xi$ does not add any new paths through ω_1-trees and hence no new (ω_1, ω_1)-gaps in $\mathbb{L}_\xi$. At this stage no new elements are added to $\mathbb{L}$ so that $\mathbb{L}_{\xi+1} = \mathbb{L}_\xi$.

Finally, I show how to treat the remaining successor stages, namely, stages where ξ is an even ordinal and $\mathrm{cf}(\xi) \ne \omega_1$ (i.e. $\xi \in \Lambda$). At these stages I extend with a splitting order for some pregap in $\mathbb{L}$. However, I make sure that such a splitting order has the ccc so that by Lemmas 1.9 and 2.6 no new paths are added through ω_1-trees and hence by Lemma 2.7 no new (ω_1, ω_1)-gaps are added in $\mathbb{L}$. So fix $\xi \in \Lambda$ and let $\langle C^\xi_\gamma : \gamma < \omega_2\rangle$ be an enumeration, in N_ξ, of all pregaps in $\mathbb{L}_\xi$. Let $g(\xi) = \langle\eta, \gamma\rangle$. Since $\eta \le \xi$, the pregap C^η_γ in $\mathbb{L}_\eta$ has been defined. Let C be that C^ξ_δ whose restriction to $\mathbb{L}_\eta$ is equivalent to C^η_γ and let $\mathbb{S}_\xi$ be its splitting order in N_ξ. By the treatment of earlier successor stages and by Lemma 2.8, (ω_1, ω_1)-gaps in $\mathbb{L}_\xi$ can only occur at stages of cofinality ω_1. But at these stages all such gaps are filled by a single ccc partial order and no new (ω_1,ω_1)-gaps are added in $\mathbb{L}$ by this partial order. Therefore C cannot be an (ω_1,ω_1)-gap (i.e. an (ω_1,ω_1)-pregap which is not split) so that by Lemma 2.5, $\mathbb{S}_\xi$ has the ccc. Let $\mathbb{Q}_\xi$ be a nice $\mathbb{P}_\xi$-name for the partial order representing $\mathbb{S}_\xi$ and let $\mathbb{P}_{\xi+1}=\mathbb{P}_\xi*\mathbb{Q}_\xi$. At these stages no new paths are added through ω_1-trees and hence no new (ω_1,ω_1)-gaps are added to $\mathbb{L}_\xi$, as indicated earlier. Let f_ξ be the element of $\mathbb{Z}^\omega$ added by extending with $\mathbb{Q}_\xi$ and let $\mathbb{L}_{\xi+1}=\mathbb{L}_\xi\cup\{f_\xi\}$. This finishes the treatment of the successor stages and since the limit stages are handled in the usual way this also finishes the construction.

Let J be $\mathbb{P}_{\omega_2}$-generic over N. Now I show that

$$\mathrm{N[J]}\models \text{“}\mathrm{MA}+\neg\mathrm{wKH}+\mathfrak{c}=\omega_2+\varphi_{\mathfrak{c}}\text{”}.$$

It is straightforward to show $\mathfrak{c}=\omega_2$ in N[J]. For MA, let $\mathbb{R}$ be a ccc partial order of size ω_1 and $\langle D_\zeta:\zeta<\omega_1\rangle$ a sequence of dense subsets of $\mathbb{R}$ in N[J]. Then there is a $\xi<\omega_2$ such that $\mathbb{R}$ and $\langle D_\zeta:\zeta<\omega_1\rangle$ are all in N[J_ξ]. Now at some later odd stage η, $\mathbb{Q}_\eta$ was a $\mathbb{P}_\eta$-name for $\mathbb{R}$. However, since $\mathbb{R}$ has the ccc in N[J], $\mathbb{Q}_\eta$ must satisfy Case 2 in the treatment of stage η. Therefore, at that stage a filter is added in $\mathbb{R}$ that intersects all $\langle D_\zeta:\zeta<\omega_1\rangle$. This shows that MA holds in N[J].

For $\varphi_{\mathfrak{c}}$, let $\langle f,g\rangle$ be a (λ,μ)-pregap in $\mathbb{L}$, where λ and μ are cardinals with $\lambda,\mu<\omega_2$. Then there is a $\xi<\omega_2$ such that $\langle f,g\rangle\subseteq\mathbb{L}_\xi$ and $\langle f,g\rangle\in\mathrm{N}_\xi$. Note that because of Case 3 in the construction of odd stages of the iteration either $\langle f,g\rangle$ is a non-(ω_1,ω_1)-gap, in which case its splitting order has the ccc, or $\langle f,g\rangle$ is an (ω_1,ω_1)-gap, in which case some equivalent gap had to be constructed at some earlier stage θ with $\mathrm{cf}(\theta)=\omega_1$. But then at that stage its splitting order has the ccc. Therefore, at the next stage the gap was split so that its splitting order remains to have the ccc in all further extensions. Then at some later even stage η, an element is added to $\mathbb{L}$ which splits $\langle f,g\rangle$. Therefore N[J] $\models$ “$\varphi_{\mathfrak{c}}$”.

Finally, I show N[J] $\models$ “$\neg$wKH”. Let $\mathbb{T}$ be an ω_1-tree in N[J]. I may assume $\mathbb{T}=\langle\omega_1,\le_{\mathbb{T}}\rangle$ where $\le_{\mathbb{T}}$ is some subset of $\omega_1\times\omega_1$. Let $\sigma=\bigcup\{\{\check{s}\}\times A_s:s\in\omega_1\times\omega_1\}$ be a nice $\mathbb{P}_{\omega_2}$-name for a subset of $(\omega_1\times\omega_1)\check{}$ with $\sigma_J=\le_{\mathbb{T}}$. Then there is a $\mu<\omega_2$ such that $A=\bigcup\{A_s:s\in\omega_1\times\omega_1\}\subseteq\mathbb{P}_\mu$ so that σ is actually a nice $\mathbb{P}_\mu$-name. I may assume that $\mathrm{cf}(\mu)=\omega$. Recall that τ^ξ is a nice $\mathbb{P}_\xi$-name for the product of the splitting partial orders of all (ω_1,ω_1)-gaps in $\mathbb{L}_\xi$ constructed by stage ξ. Then since $|A|=\omega_1$, for each $\xi<\mu$ with $\mathrm{cf}(\xi)=\omega_1$, only a subset of $\mathrm{dom}(\tau^\xi)$ of size ω_1 is used in defining σ and all $\mathbb{Q}_\eta$ with $\mathrm{cf}(\eta)\neq\omega_1$ and $\xi<\eta<\mu$. With this in mind I construct, in N, a finite support ccc iteration

$$\langle\langle\mathbb{X}_\xi:\xi\le\mu\rangle,\langle\mathbb{Y}_\xi:\xi<\mu\rangle\rangle$$

such that $|\mathbb{X}_\mu| = \omega_1$, there is a complete embedding, i, from $\mathbb{X}_\mu$ into $\mathbb{P}_\mu$, and σ is in fact a nice $\mathbb{X}_\mu$-name for a subset of $(\omega_1 \times \omega_1)\check{}$. If $\mathrm{cf}(\xi) \neq \omega_1$, then let $\mathbb{Y}_\xi = \mathbb{Q}_\xi$. Otherwise let $\mathbb{Y}_\xi = \tau^\xi_{\gamma^\xi_\mu}$ where $\gamma^\xi_\mu < \omega_2$ is large enough to make $\mathbb{X}_\mu$ have the properties indicated in the previous sentence.

The construction is fairly straightforward. For each $\xi < \mu$ with $\mathrm{cf}(\xi) = \omega_1$ let

$$B_\xi = \{(p)_\xi : \exists s \in \omega_1 \times \omega_1\, (p \in A_s)\},$$

where $(p)_\xi$ denotes the ξth component of p. Choose $\gamma^\xi_\mu < \omega_2$ so large that $B_\xi \subseteq \mathrm{dom}(\tau^\xi_{\gamma^\xi_\mu})$ and if $\mathbb{Y}_\xi = \tau^\xi_{\gamma^\xi_\mu}$ for each $\xi < \mu$ with $\mathrm{cf}(\xi) = \omega_1$ and $\mathbb{Y}_\xi = \mathbb{Q}_\xi$ for all the other $\xi < \mu$ with $\mathrm{cf}(\xi) \neq \omega_1$ then the sequence

$$\langle\langle \mathbb{X}_\xi : \xi \le \mu\rangle, \langle \mathbb{Y}_\xi : \xi < \mu\rangle\rangle$$

obtained in such a way is in fact a finite support ccc iteration.

Clearly $|\mathbb{X}_\mu| = \omega_1$ and note that in view of Lemma 2.15 there is a complete embedding $i : \mathbb{X}_\mu \to \mathbb{P}_\mu$. Furthermore, σ is actually an $\mathbb{X}_\mu$-name. Then $i^{-1}(\mathrm{J}_\mu)$ is $\mathbb{X}_\mu$-generic over N and $\le_{\mathbb{T}} \in \mathrm{N}[i^{-1}(\mathrm{J}_\mu)]$ so that $\mathbb{T}$ is an ω_1-tree in $\mathrm{N}[i^{-1}(\mathrm{J}_\mu)]$.

Now since $|\mathbb{X}_\mu| = \omega_1$ there is a regular cardinal ν in M, with $\omega_1 < \nu < \kappa$, such that $\mathbb{X}_\mu \in \mathrm{M}[\mathrm{K}_\nu]$ and $i^{-1}(\mathrm{J}_\mu)$ is $\mathbb{X}_\mu$-generic over $\mathrm{M}[\mathrm{K}_\nu]$. Now, in $\mathrm{M}[\mathrm{K}_\nu][i^{-1}(\mathrm{J}_\mu)]$, κ is still an inaccessible cardinal and $\mathbb{T}$ is an ω_1-tree so that $\mathbb{T}$ has less than κ paths, say λ many. But in $\mathrm{M}[\mathrm{K}_\nu][i^{-1}(\mathrm{J}_\mu)][\mathrm{K}^\nu]$, λ is collapsed onto a cardinal less than $\omega_2 = \kappa$. However,

$$\mathrm{M}[\mathrm{K}_\nu][i^{-1}(\mathrm{J}_\mu)][\mathrm{K}^\nu] = \mathrm{M}[\mathrm{K}_\nu][\mathrm{K}^\nu][i^{-1}(\mathrm{J}_\mu)] = \mathrm{M}[\mathrm{K}][i^{-1}(\mathrm{J}_\mu)] = \mathrm{N}[i^{-1}(\mathrm{J}_\mu)].$$

Hence, by Lemma 3.1, in going from $\mathrm{M}[\mathrm{K}_\nu][i^{-1}(\mathrm{J}_\mu)]$ to $\mathrm{M}[\mathrm{K}_\nu][i^{-1}(\mathrm{J}_\mu)][\mathrm{K}^\nu]$, no new paths are added through $\mathbb{T}$. Hence $\mathbb{T}$ has at most ω_1 paths in $\mathrm{N}[i^{-1}(\mathrm{J}_\mu)]$, since λ is collapsed onto ω_1.

Now I show that $\mathbb{T}$ has at most ω_1 paths in N[J]. Let $\{\delta_\zeta : \zeta \le \varepsilon\}$ be an enumeration in the increasing order of all ordinals $\delta < \mu$ with $\mathrm{cf}(\delta) = \omega_1$. I construct, in N, a sequence of partial orders $\langle \mathbb{Z}_\xi : \xi \le \omega_2 \cdot \varepsilon\rangle$, where $\omega_2 \cdot \varepsilon$ denotes a product of ordinals, together with complete embeddings $i_{\xi\eta} : \mathbb{Z}_\xi \to \mathbb{Z}_\eta$, for $0 \le \xi \le \eta \le \omega_2 \cdot \varepsilon$, such that $\mathbb{Z}_0 = \mathbb{X}_\mu$ and $\mathbb{Z}_{\omega_2\cdot\varepsilon} = \mathbb{P}_\mu$. In addition, because of the complete embeddings, $i_{\xi\eta}$, the sequence $\langle \mathbb{Z}_\xi : \xi \le \omega_2 \cdot \varepsilon\rangle$ can be viewed as a finite support ccc iteration where $\mathbb{Z}_{\xi+1}$ is obtained from $\mathbb{Z}_\xi$ by extending $\mathbb{Z}_\xi$ with a ccc splitting order for some pregap. Therefore, by Lemma 1.9 as well as 2.6, no new paths are added through $\mathbb{T}$ in going from $\mathbb{Z}_\xi$ to $\mathbb{Z}_{\xi+1}$. And since by Lemma 1.12 no new paths can be added through $\mathbb{T}$ at limit stages, it follows that $\mathbb{T}$ has as many paths in $\mathrm{N}[\mathrm{J}_\mu]$ as it does in $\mathrm{N}[i^{-1}(\mathrm{J}_\mu)]$, namely at most ω_1.

The construction of the sequence $\langle \mathbb{Z}_\xi : \xi \le \omega_2 \cdot \varepsilon\rangle$ is fairly easy, so I only give an outline. Start with $\mathbb{Z}_0 = \mathbb{X}_\mu$. For any $\zeta \le \varepsilon$, $\alpha \le \omega_2$ let $\theta = \gamma^{\delta_\zeta}_\mu + \alpha$.

In order to get $\mathbb{Z}_{\omega_2\cdot\zeta+\alpha}$, in the definition of $\mathbb{X}_\mu$ replace all $\mathbb{Y}_{\delta_\eta}$, for $\eta < \zeta$, by τ^{δ_η}, $\mathbb{Y}_{\delta_\zeta}$ by $\tau_\theta^{\delta_\zeta}$ and keep all the other $\mathbb{Y}_\xi$ the same. Then $\mathbb{Z}_{\omega_2\cdot\zeta+\alpha}$ is the partial order obtained in such a way. Then clearly $\mathbb{Z}_0 = \mathbb{X}_\mu$ and $\mathbb{Z}_{\omega_2\cdot\varepsilon} = \mathbb{P}_\mu$. By Lemma 2.15 it is clear that for each $0 \le \xi < \eta \le \omega_2\cdot\varepsilon$ there is a complete embedding $i_{\xi\eta} : \mathbb{Z}_\xi \to \mathbb{Z}_\eta$.

Thus the sequence $\langle \mathbb{Z}_\xi : \xi \le \omega_2 \cdot \varepsilon\rangle$ can be viewed as a finite support ccc iteration with splitting orders for some pregaps. Then by Lemma 1.9 as well as 2.6 no new paths are added through ω_1-trees at successor stages. And since, by the remark following Lemma 1.12, no new paths are added through ω_1-trees at limit stages, it follows that $\mathbb{T}$ still has at most ω_1 paths in $\mathrm{N}[\mathrm{J}_\mu]$. But now, in the construction of the iteration

$$\langle\langle \mathbb{P}_\xi : \xi \le \omega_2\rangle, \langle \mathbb{Q}_\xi : \xi < \omega_2\rangle\rangle$$

the partial orders $\mathbb{Q}_\xi$ were chosen in such a way that no new paths were added through ω_1-trees in extensions by $\mathbb{Q}_\xi$. And since by Lemma 1.12 no new paths were added at limit stages, it follows that $\mathbb{T}$ has at most ω_1 paths in N[J]. Therefore $\mathbb{T}$ cannot be a weak Kurepa tree in N[J]. Hence, $\mathrm{N}[\mathrm{J}] \models$ "$\neg$wKH", which completes the proof that N[J] is a model for

$$\mathrm{ZFC} + \mathrm{MA} + \neg\mathrm{wKH} + \mathfrak{c} = \omega_2 + \varphi_{\mathfrak{c}},$$

which in turn proves the theorem. ■

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN-MADISON
MADISON, WISCONSIN 53706
U.S.A.

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