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Cantor manifolds in the theory of transfinite dimension

by

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Abstract. For every countable non-limit ordinal α we construct an α-dimensional Cantor ind-manifold, i.e., a compact metrizable space Z_α such that $\operatorname{ind} Z_\alpha = \alpha$, and no closed subset L of Z_α with $\operatorname{ind} L$ less than the predecessor of α is a partition in Z_α. An α-dimensional Cantor Ind-manifold can be constructed similarly.

1. Introduction. Unless otherwise stated all spaces considered are metrizable and separable. Our terminology and notation follow [2] and [4] with the exception of the boundary, closure, and interior of a subset A of a topological space X, which are denoted by $\operatorname{bd} A$, $\operatorname{cl} A$, and $\operatorname{int} A$, respectively.

We denote by I the unit closed interval, and by I^n the standard n-dimensional cube, i.e., the Cartesian product of n copies of I. By an *arc* we mean every space homeomorphic to I, and by an *n-dimensional cube* every space homeomorphic to I^n; we identify every such space with I^n by a "canonical" homeomorphism. This allows us to apply geometrical notions, e.g., broken line or parallelism, to spaces homeomorphic to I^n. In the sequel, we assume that "canonical" homeomorphisms are always defined in a natural way, and they are not described; we will simply apply geometrical notions to the cubes.

We denote by $a^\wedge b$ any arc with endpoints a and b, i.e., an arc J such that $h(0) = a$ and $h(1) = b$, where $h : I \to J$ is the "canonical" homeomorphism.

A *partition* in a space X between a pair of disjoint sets A and B is a closed set L such that $X - L = U \cup V$, where U and V are disjoint open sets with $A \subseteq U$ and $B \subseteq V$.

The small transfinite dimension ind and the large transfinite dimension Ind are the extension by transfinite induction of the classical Menger–Urysohn dimension and the classical Brouwer–Čech dimension:

- $\operatorname{ind} X = -1$ as well as $\operatorname{Ind} X = -1$ means $X = \emptyset$,

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• $\operatorname{ind} X \le \alpha$ (resp. $\operatorname{Ind} X \le \alpha$), where α is an ordinal, if and only if for every $x \in X$ and each closed set $B \subseteq X$ such that $x \notin B$ (resp. for every pair A, B of disjoint closed subsets of X), there exists a partition L between x and B (resp. a partition L between A and B) such that $\operatorname{ind} L < \alpha$ (resp. $\operatorname{Ind} L < \alpha$),

• $\operatorname{ind} X$ is the smallest ordinal α with $\operatorname{ind} X \le \alpha$ if such an ordinal exists, and $\operatorname{ind} X = \infty$ otherwise,

• $\operatorname{Ind} X$ is the smallest ordinal α with $\operatorname{Ind} X \le \alpha$ if such an ordinal exists, and $\operatorname{ind} X = \infty$ otherwise.

The transfinite dimension ind was first discussed by W. Hurewicz [6] and the transfinite dimension Ind by Yu. M. Smirnov [12]; a comprehensive survey of the topic is given by R. Engelking [3].

The small transfinite dimension of a space X at a point x does not exceed an ordinal α (written briefly $\operatorname{ind}_x X \le \alpha$) if for each closed set $B \subseteq X$ not containing x there exists a partition L between x and B such that $\operatorname{ind} L < \alpha$. The small transfinite dimension of a space X at a point x, denoted by $\operatorname{ind}_x X$, is the smallest ordinal α such that $\operatorname{ind}_x X \le \alpha$ if such an ordinal exists, and ∞ otherwise.

If $X \ne \emptyset$, then $\operatorname{ind} X$ and $\operatorname{Ind} X$ are either countable ordinals or equal to infinity (see [6] and [12], or [3], Theorems 3.5 and 3.8). Obviously, $\operatorname{ind} X \le \operatorname{Ind} X$, but the reverse inequality does not hold; there exists a compact space X such that $\operatorname{ind} X < \operatorname{Ind} X$ (see [9]).

For a long time all known compact spaces X with $\operatorname{ind} X = \alpha \ge \omega_0$ had the property that $\operatorname{ind}_x X = \alpha$ only for some distinguished points x; B. A. Pasynkov asked whether there exist compact spaces X with $\operatorname{ind}_x X = \alpha$ for every $x \in X$, or even with a stronger property that X is an α-dimensional Cantor manifold (see [1]).

1.1. Definition. Let $\alpha = \beta + 1$ be a non-limit ordinal. A compact metrizable space X such that $\operatorname{ind} X = \alpha$ (resp. $\operatorname{Ind} X = \alpha$) is an *$\alpha$-dimensional Cantor* ind-*manifold* (resp. Ind-*manifold*) if no closed set $L \subseteq X$ with $\operatorname{ind} L < \beta$ (resp. $\operatorname{Ind} L < \beta$) is a partition in X between any pair of points.

For $\alpha = n < \omega_0$, the above notions and the classical notion of a Cantor manifold (see [2], Definition 1.9.5) are equivalent; the n-dimensional cube I^n is an example of an α-dimensional Cantor manifold. Of course, every α-dimensional Cantor ind-manifold has the property that $\operatorname{ind}_x X = \alpha$ for each $x \in X$. In [1], V. A. Chatyrko gave examples of a non-metrizable α-dimensional ind-manifold and a non-metrizable α-dimensional Ind-manifold for every non-limit ordinal α such that $\omega_0 < \alpha < \omega_1$; he also constructed, for every infinite $\alpha < \omega_1$, a compact metrizable space X_α with $\operatorname{ind} X_\alpha < \infty$, and a compact metrizable space Y_α with $\operatorname{Ind} Y_\alpha < \infty$ such that $\operatorname{ind} L \ge \alpha$

for every partition L in X_α between any pair of points, and $\operatorname{Ind} L \geq \alpha$ for every partition L in Y_α between any pair of points.

In the present paper, we construct a metrizable α-dimensional Cantor ind-manifold Z_α for every non-limit infinite ordinal $\alpha < \omega_1$. Slightly modifying this construction, one can also define metrizable Cantor Ind-manifolds. However, we will restrict the discussion to the small transfinite dimension, and in the sequel "Cantor manifold" will mean "Cantor ind-manifold".

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2. Henderson's spaces. Yu. M. Smirnov defined a sequence $\{S_\alpha : \alpha < \omega_1\}$ of compact metrizable spaces with the property that $\operatorname{Ind} S_\alpha = \alpha$ for every $\alpha < \omega_1$ (see [12]). Slightly modifying his construction, D. W. Henderson defined a sequence $\{H_\alpha : \alpha < \omega_1\}$ of absolute retracts with the same property (see [5]).

Let us recall the definitions of S_α and H_α.

We apply induction on α; we simultaneously distinguish a point $p_\alpha \in H_\alpha$ and, for $\alpha > 0$, a covering $\mathcal{C}_\alpha$ of H_α by cubes of positive dimensions.

Let $S_0 = H_0 = \{p_0\}$ be the one-point space, $S_1 = H_1 = I$, $p_1 = 0$, and $\mathcal{C}_1 = \{I\}$. Assume that $S_\beta, H_\beta, p_\beta$, and $\mathcal{C}_\beta$ are defined for every $\beta < \alpha$.

If $\alpha = \beta + 1$ for some β, then set $S_\alpha = S_\beta \times I$, $H_\alpha = H_\beta \times I$, $p_\alpha = (p_\beta, 0)$, $\mathcal{C}_\alpha = \{C \times I : C \in \mathcal{C}_\beta\}$.

If α is a limit ordinal, then let S_α be the one-point compactification of the topological sum $\bigoplus\{S_\beta : \beta < \alpha\}$. In order to define H_α take a half-open arc A'_β with endpoint p_β such that $H_\beta \cap A'_\beta = \{p_\beta\}$, and set $K'_\beta = H_\beta \cup A'_\beta$ for every $\beta < \alpha$. Let H_α be the one-point compactification of the topological sum $\bigoplus\{K'_\beta : \beta < \alpha\}$; let p_α stand for the unique point of the remainder. Set $A_\beta = A'_\beta \cup \{p_\alpha\}$ and $K_\beta = K'_\beta \cup \{p_\alpha\}$ for $\beta < \alpha$. Let

$$\mathcal{C}_\alpha = \{A_\beta : \beta < \alpha\} \cup \bigcup\{\mathcal{C}_\beta : 0 < \beta < \alpha\}.$$

For simplicity of notation, we will identify the spaces $H_n = I^n$ and $H_0 \times I^n$, where $H_0 = \{p_0\}$.

The spaces H_{ω_0} and H_{ω_0+1} are exhibited in Fig. 2.1.

Observe that S_α is embeddable in H_α for every $\alpha < \omega_1$, and that if $\beta < \alpha$, then S_β is embeddable in S_α and H_β is embeddable in H_α.

Henderson's spaces play an important role in our considerations, whereas Smirnov's spaces will only be used in the proof of Theorem 2.1.

Both Henderson's and Smirnov's spaces are also a source of examples of compact metrizable spaces with given small transfinite dimension.

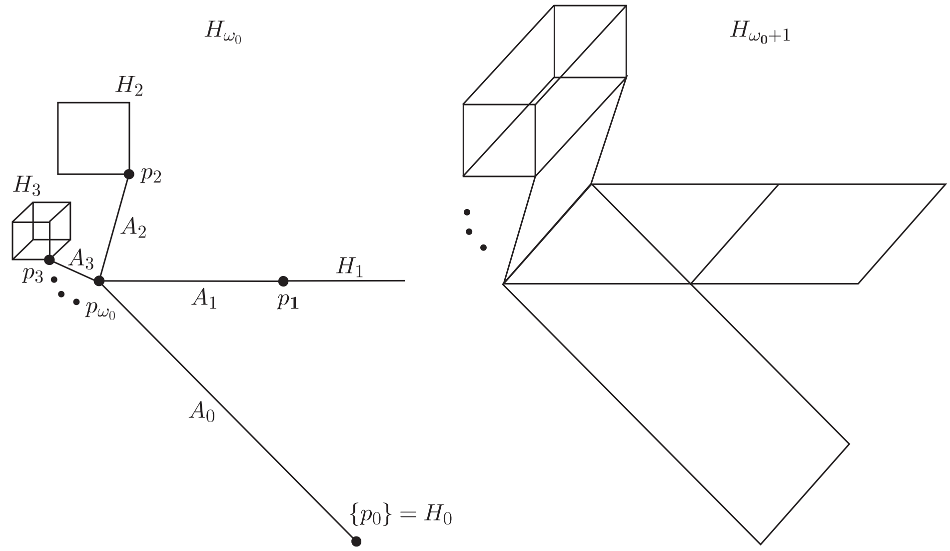

Fig. 2.1

Indeed, one proves that

$$\operatorname{Ind} X \le \omega_0 \cdot \operatorname{ind} X \quad \text{for every hereditarily normal compact space } X$$

(see [8]),

$$\operatorname{ind}(X \times I) \le \operatorname{ind} X + 1 \quad \text{for every metrizable space } X$$

(see [13]).

From the theorems mentioned above and the easy equality

$$\operatorname{ind} H_\lambda = \sup\{\operatorname{ind} H_\alpha : \alpha < \lambda\}, \tag{2.1}$$

where λ is any countable limit ordinal, it follows that for every ordinal $\alpha < \omega_1$, there exists an ordinal $\beta \ge \alpha$ such that $\operatorname{ind} H_\beta = \alpha$.

The least ordinal β with $\operatorname{ind} H_\beta = \alpha$ will be denoted by $\beta(\alpha)$. Note that $\beta(\alpha) > \alpha$ for some α (see [9]), and $\operatorname{ind} H_\alpha$ is unknown for some α (see [3], Problems 2.3 and 2.4). From (2.1) it follows immediately that

(2.2) if α is a non-limit ordinal, then so is $\beta(\alpha)$.

The remaining part of this section is devoted to some notions and results concerning the spaces H_α.

Every ordinal α can be uniquely represented as the sum $\lambda + n$ of a limit ordinal λ or $\lambda = 0$ and a natural number n. From the construction of H_α it follows that $H_\alpha = H_\lambda \times I^n$. Let B_α denote the set $\{p_\lambda\} \times I^n$; we call it the *base* of H_α. Sometimes, we will identify the base B_α and the cube I^n; in particular, we will write $H_\alpha = H_\lambda \times B_\alpha$. Thus for every ordinal $\alpha < \omega_1$, the base B_α of H_α is a finite-dimensional cube, and it has positive dimension whenever α is a non-limit ordinal.

In the sequel, we need the following two theorems. The first theorem for ind is a consequence of a theorem of G. H. Toulmin ([13], or [3], Theorem 5.16), and for Ind it is a consequence of a theorem obtained independently by M. Landau and A. R. Pears ([7] and [11], or [3], Theorem 5.17); the theorem for Ind also follows from a theorem of B. T. Levshenko ([8], or [3], Theorem 5.15). The second theorem was established by G. H. Toulmin ([13]).

2.A. THEOREM ([13], resp. [7] and [11]). *If a hereditarily normal space X can be represented as the union of closed subspaces A_1 and A_2 such that* $\operatorname{ind} A_i \le \alpha \ge \omega_0$ (*resp.* $\operatorname{Ind} A_i \le \alpha \ge \omega_0$) *for $i = 1, 2$, and $A_1 \cap A_2$ is finite-dimensional*, *then* $\operatorname{ind} X \le \alpha$ (*resp.* $\operatorname{Ind} X \le \alpha$).

2.B. THEOREM ([13]). *If a hereditarily normal space X can be represented as the union of closed subspaces A_1 and A_2 with the property that there is a homeomorphism $h : A_1 \to A_2$ such that $f(x) = x$ for every $x \in A_1 \cap A_2$, then*

$$\operatorname{ind} X = \operatorname{ind} A_1 = \operatorname{ind} A_2 .$$

2.1. THEOREM. *Let $\alpha = \lambda + n$, where λ is* 0 *or a countable limit ordinal and $n \ge 1$ is a natural number. For every partition K in H_α between any pair of distinct points $a, b \in B_\alpha$, we have*

$$\operatorname{ind} K \ge \operatorname{ind} H_\alpha - 1 \quad \textit{and} \quad \operatorname{Ind} K \ge \lambda + (n-1) .$$

P r o o f. As in the proof of Theorem 2.1 of [10] we can see that

(2.3) for every $x \in B_\alpha$ and each closed set $F \subseteq B_\alpha$ not containing x, there exists a partition L in H_α between x and F such that $\operatorname{ind} L \le \operatorname{ind} K$.

We first prove that

$$\operatorname{ind}_x H_\alpha \le \operatorname{ind} K + 1 \quad \text{for every } x \in B_\alpha . \tag{2.4}$$

Let $x \in B_\alpha$ and let $F \subseteq H_\alpha$ be a closed set not containing x. Since $H_\alpha = B_\alpha$ for $\alpha < \omega_0$, we can assume that $\alpha \ge \omega_0$. Let $E = F \cap B_\alpha$; by (2.3), there exists a partition M in H_α between x and E such that $\operatorname{ind} M \le \operatorname{ind} K$. Let $U, V \subseteq H_\alpha$ be disjoint open sets such that $x \in U$, $E \subseteq V$, and $M = H_\alpha - (U \cup V)$.

By construction, we have

$$H_\alpha = B_\alpha \cup \bigcup \{(K_\beta - \{p_\lambda\}) \times I^n : \beta < \lambda\} ;$$

from the definition of H_α it also follows that each closed subset of H_α disjoint from B_α meets only a finite number of the sets $(K_\beta - \{p_\lambda\}) \times I^n$. Thus

$$[F \cap (U \cup M)] \cap [(K_\beta - \{p_\lambda\}) \times I^n] \ne \emptyset$$

only for finitely many β, say for $\beta = \beta_1, \ldots, \beta_k$.

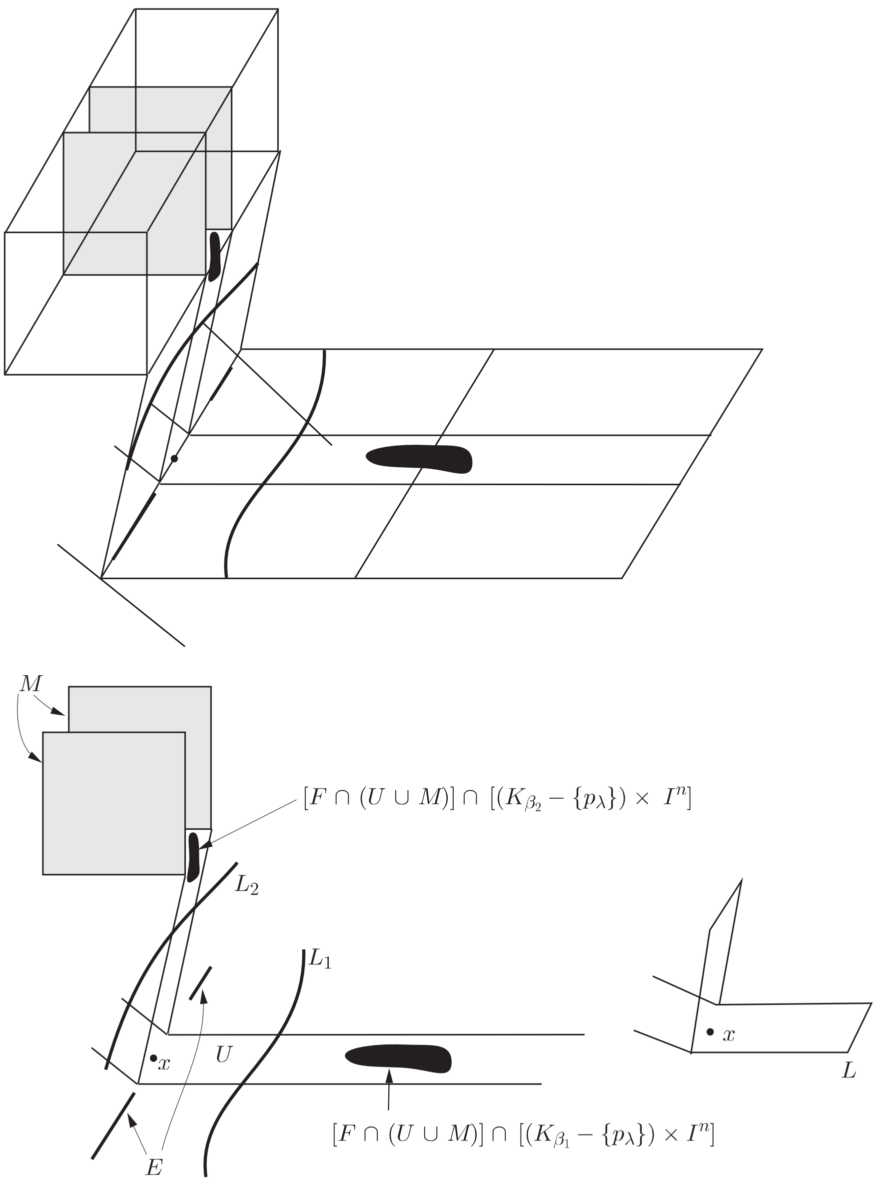

Fig. 2.2

Let $L_i \subseteq A_{\beta_i} \times I^n$, for $i = 1, \ldots, k$, be a partition in K_{β_i} between B_α and $[F \cap (U \cup M)] \cap [(K_{\beta_i} - \{p_\lambda\}) \times I^n]$ (see Fig. 2.2, where $k = 2$); since $\operatorname{ind}(A_{\beta_i} \times I^n) = n + 1$, we have $\operatorname{ind} L_i \leq n + 1$.

It is easily seen that $M \cup \bigcup_{i=1}^{\infty} L_i$ contains a partition L in H_α between x and E (see Fig. 2.2); by Theorem 2.A and monotonicity of ind, we have $\operatorname{ind} L \leq \operatorname{ind} K$.

Thus the proof of (2.4) is concluded. Applying (2.4) we prove the first inequality of our theorem by induction on α. For every $\alpha < \omega_0$, the hypothesis of the theorem is equivalent to (2.4). Assume, therefore, that $\alpha \geq \omega_0$; that is, $\alpha = \lambda + n$, where λ is a limit ordinal and n is a natural number. Obviously, $\operatorname{ind} K \geq \omega_0$. Assume the inequality holds for each $\beta < \alpha$. By (2.4) it suffices to show that $\operatorname{ind}_x H_\alpha \leq \operatorname{ind} K + 1$ for each $x \in H_\alpha - B_\alpha$.

Observe that K contains a partition in $H_\beta \times I^n = H_{\beta+n}$ between a pair of distinct points of the base $B_{\beta+n}$ for all but a finite number of $\beta < \lambda$. Thus, by the inductive assumption, $\operatorname{ind} H_{\beta+n} \leq \operatorname{ind} K + 1$ for those β; since $\operatorname{ind} H_\nu \leq \operatorname{ind} H_\mu$ whenever $\nu \leq \mu$, we have $\operatorname{ind} H_{\beta+n} \leq \operatorname{ind} K + 1$ for all $\beta < \lambda$. By Theorem 2.A, every $x \in H_\alpha - B_\alpha$ has a neighbourhood U in H_α with $\operatorname{ind} U \leq \operatorname{ind} K + 1$, which completes the proof of the inequality $\operatorname{ind} H_\alpha \leq \operatorname{ind} K + 1$.

Just as the base B_α of H_α, one can define the base B'_α of S_α (see [10]). The inequality $\operatorname{Ind} K \geq \lambda + (n-1)$ follows from Theorem 2.1 of [10], because there exists an embedding of S_α in H_α mapping B'_α onto B_α.

2.2. Lemma. *Let $\beta > 0$ be a countable ordinal. For every $x \in H_\beta$ and each closed set $E \subseteq H_\beta$ not containing x, there exists a partition Y in H_β between x and E such that*

(2.5) *for every cube $C \in \mathcal{C}_\beta$, the set $Y \cap C$ is the union of a finite number of cubes of dimension less than that of C, each parallel to a proper face of C; furthermore, if $\beta = \beta(\alpha)$ for some α, then $\operatorname{ind} Y < \alpha$.*

Proof. For $\beta < \omega_0$ the lemma is obvious. Thus assume that $\beta \geq \omega_0$. Represent β as the sum $\lambda + n$ of a limit ordinal λ and a natural number n.

Let $H_{\beta,k}$, $k = 0, 1, \ldots, n$, be the space obtained by sticking the $(k+1)$-dimensional cube $C_{\beta,k} = I^{k+1}$ to a k-dimensional face D of the base $B_\beta \subseteq H_\beta$ along its k-dimensional face (see Fig. 2.3, where $\beta = \omega_0 + 1$ and $k = 1$). Precisely, define $H_{\beta,k}$ to be the subspace of $H_\beta \times I$ consisting of all (y, z) such that either $z = 0$ or $y \in D$; let $\mathcal{C}_{\beta,k} = \mathcal{C}_\beta \cup \{C_{\beta,k}\}$. Observe that from Theorem 2.A it follows that $\operatorname{ind} H_{\beta,k} = \operatorname{ind} H_\beta$.

We apply induction on β. Since $H_\beta \subseteq H_{\beta,k}$ and $\mathcal{C}_\beta \subseteq \mathcal{C}_{\beta,k}$, it is sufficient to prove the counterpart of the lemma for each $H_{\beta,k}$, $k = 0, 1, \ldots, n$, and its covering $\mathcal{C}_{\beta,k}$.

Assume that $\lambda = \omega_0$ or $\lambda > \omega_0$ and the modified lemma holds for every ordinal $\beta' = \lambda' + n'$ such that $\lambda' < \lambda$. Fix $k \in \{1, \ldots, n\}$. Then

$$\begin{aligned} H_{\beta,k} &= H_\beta \cup C_{\beta,k} = (H_\lambda \times I^n) \cup C_{\beta,k} \\ &= \Big(\bigcup\{(A_\gamma \cup H_\gamma) \times I^n : \gamma < \lambda\}\Big) \cup C_{\beta,k}\,, \end{aligned}$$

$$C_{\beta,k} \cap \bigcup\{(A_\gamma \cup H_\gamma) \times I^n : \gamma < \lambda\} = D\,,$$

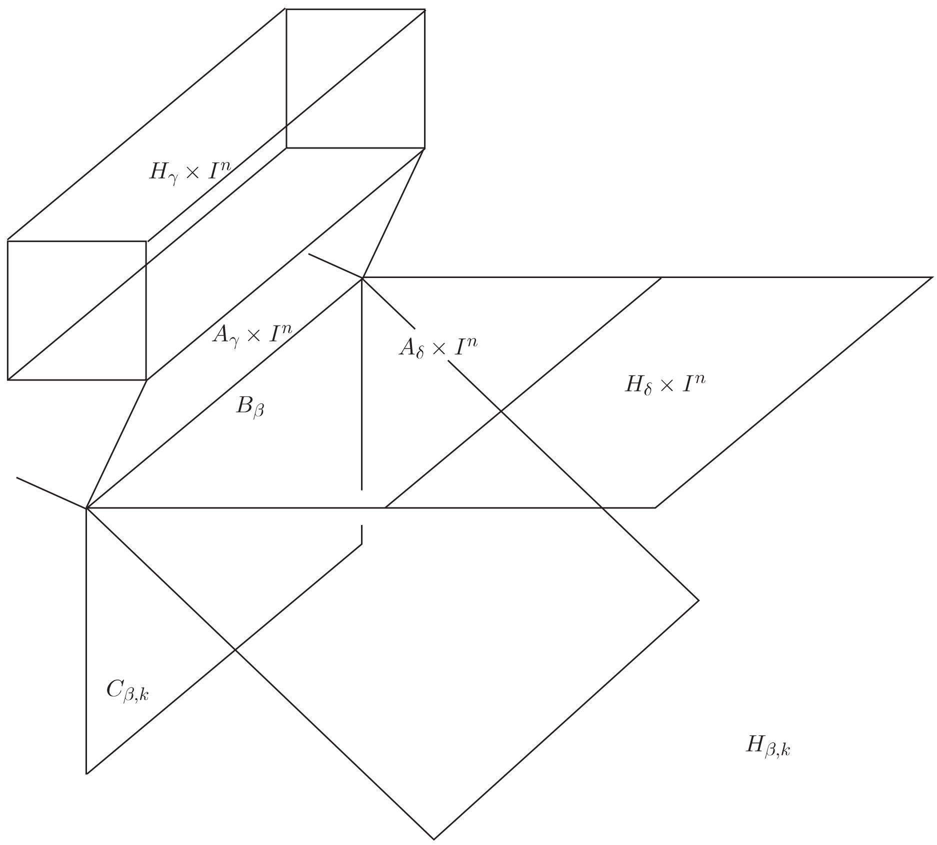

Fig. 2.3

and

$$[(A_\gamma \cup H_\gamma) \times I^n] \cap [(A_\delta \cup H_\delta) \times I^n] = B_\beta$$

for distinct $\gamma, \delta < \lambda$ (see Fig. 2.3).

Let $x \in H_{\beta,k}$ and let $E \subseteq H_{\beta,k}$ be a closed set not containing x. If $x \notin B_\beta$, then $x \in C_{\beta,k} - B_\beta$ or $x \in (A_\gamma \cup H_\gamma) \times I^n - B_\beta$ for some $\gamma < \lambda$. Since $C_{\beta,k} - B_\beta$ and $(A_\gamma \cup H_\gamma) \times I^n - B_\beta$ are open subsets of $H_{\beta,k}$, the existence of a partition Y with the suitably modified property (2.5) is obvious whenever $x \in C_{\beta,k} - B_\beta$ or $x \in (A_\gamma \cup H_\gamma) \times I^n - B_\beta$ and $\gamma < \omega_0$, and it follows from the inductive assumption if $x \in (A_\gamma \cup H_\gamma) \times I^n - B_\beta$ and $\gamma \geq \omega_0$. Obviously, we can assume that Y is contained either in $C_{\beta,k} - B_\beta$ or in $(A_\gamma \cup H_\gamma) \times I^n - B_\beta$; thus $\operatorname{ind} Y < \alpha$ for $\beta = \beta(\alpha)$.

Suppose now that $x \in B_\beta$. Assume that β is a non-limit ordinal; for limit β the proof is straightforward. Let $Q \subseteq B_\beta$ be an n-dimensional cube with faces parallel to the faces of $B_\beta = I^n$ such that $x \in \operatorname{int} Q$, where $\operatorname{int} Q$ stands for the interior of Q in B_β, and $E \cap Q = \emptyset$ (see Fig. 2.4, where $\beta = \omega_0 + 1$

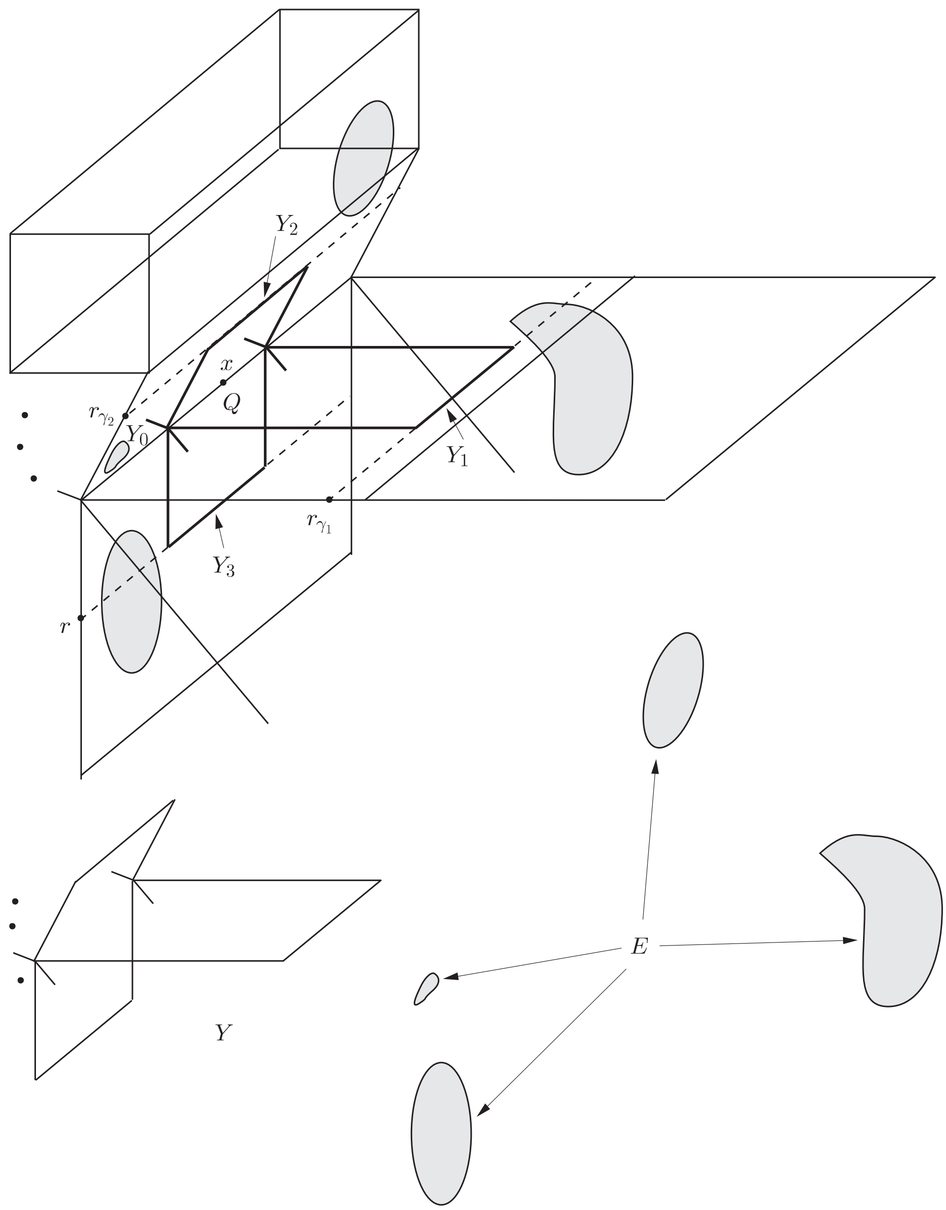

Fig. 2.4

and $k = 1$). It follows that $E \cap [(A_\gamma \cup H_\gamma) \times Q] \neq \emptyset$ only for finitely many $\gamma < \lambda$, say for $\gamma = \gamma_1, \ldots, \gamma_m$ (in Fig. 2.4, $m = 2$). For $i = 1, \ldots, m$, take $r_{\gamma_i} \in A_{\gamma_i}$ such that $(p_\lambda{}^\wedge r_{\gamma_i} \times Q) \cap E = \emptyset$; recall that p_λ is an endpoint of A_{γ_i}, and $p_\lambda{}^\wedge r_{\gamma_i}$ is the arc with endpoints p_λ and r_{γ_i} contained in A_{γ_i}.

Let

$$Y_i = \{r_{\gamma_i}\} \times Q \cup p_\lambda{}^\wedge r_{\gamma_i} \times \operatorname{bd} Q$$

where $\operatorname{bd} Q$ is the boundary of Q in B_β (see Fig. 2.4). Next, take $r \in I$ such that $\{(y,z) \in D \times I : y \in Q \text{ and } z \le r\} \subseteq C_{\beta,k}$ does not meet E, and set

$$Y_{m+1} = (Q \cap D) \times \{r\} \cup (\operatorname{bd} Q \cap D) \times [0,r]$$

(see Fig. 2.4). Let

$$Y_0 = \Big(\bigcup\{A_\gamma \cup H_\gamma : \gamma < \lambda \text{ and } \gamma \ne \gamma_1, \ldots, \gamma_m\}\Big) \times \operatorname{bd} Q\,,$$

and $Y = \bigcup_{i=0}^{m+1} Y_i$ (see Fig. 2.4).

It is easily seen that Y is a partition in $H_{\beta,k}$ between x and E with the modified property (2.5).

It remains to show that if $\beta = \beta(\alpha)$ for some α, then $\operatorname{ind} Y < \alpha$. Let ν stand for the predecessor of $\beta = \beta(\alpha)$. Since $\operatorname{ind}(\bigcup_{i=1}^{m+1} Y_i) < \omega_0$, it remains to verify that $\operatorname{ind} Y_0 < \alpha$ (see Theorem 2.A).

The set $\operatorname{bd} Q$ is homeomorphic either to the $(n-1)$-dimensional sphere or to the $(n-1)$-dimensional cube, and so it can be represented as the union of subspaces B_1 and B_2 homeomorphic to the $(n-1)$-dimensional cube such that there exists a homeomorphism f of B_1 onto B_2 with $f(x) = x$ for every $x \in B_1 \cap B_2$. For $i = 1, 2$, let

$$A_i = \Big(\bigcup\{A_\gamma \cup H_\gamma : \gamma < \lambda \text{ and } \gamma \ne \gamma_1, \ldots, \gamma_m\}\Big) \times B_i\,.$$

Then $Y_0 = A_1 \cup A_2$ and there exists a homeomorphism $h : A_1 \to A_2$ such that $h(x) = x$ for every $x \in A_1 \cap A_2$; since A_1 and A_2 are homeomorphic to a subspace of H_ν, by Theorem 2.A, we have

$$\operatorname{ind} Y_0 = \operatorname{ind} A_1 = \operatorname{ind} A_2 = \operatorname{ind} H_\nu < \beta$$

(see the definition of $\beta(\alpha)$).

2.3. Lemma. *Let J be a segment contained in an edge of the base B_β, and $E \subseteq H_\beta$ a closed set such that $E \cap J = \emptyset$; let b_1 and b_2 be the endpoints of J. Then there exist a closed set $Y \subseteq H_\beta$ with the property* (2.5) *and open sets $U, V \subseteq H_\beta$ such that $Y = H_\beta - (U \cup V)$, $J - \{b_1, b_2\} \subseteq U$, $E \subseteq V$, and for $i = 1, 2$, we have*

$$\begin{aligned} &b_i \in U \quad \text{if } b_i \text{ is a vertex of the cube } B_\beta,\\ &b_i \in Y \quad \text{otherwise;}\end{aligned}$$

furthermore, if $\beta = \beta(\alpha)$ for some α, then $\operatorname{ind} Y < \alpha$.

Proof. Let $Q \subseteq B_\beta$ be an n-dimensional cube with faces parallel to the faces of B_β and with the property that J is an edge of Q and $E \cap Q = \emptyset$; let $\operatorname{bd} Q$ stand for the boundary of Q in B_β. A reasoning similar to that in the proof of Lemma 2.2 shows that $E \cap [(A_\gamma \cup H_\gamma) \times Q] \ne \emptyset$ only for a finite number of $\gamma < \lambda$, say for $\gamma_1, \ldots, \gamma_m$. For $i = 1, \ldots, m$, take $r_{\gamma_i} \in A_{\gamma_i}$ such that $(p_\lambda{}^\wedge r_{\gamma_i} \times Q) \cap E = \emptyset$. Let

$$Y_i = \{r_{\gamma_i}\} \times Q \cup p_\lambda{}^\wedge r_{\gamma_i} \times \operatorname{bd} Q \quad \text{for } i = 1, \ldots, m,$$
$$Y_0 = \Big(\bigcup \{A_\gamma \cup H_\gamma : \gamma < \lambda \text{ and } \gamma \neq \gamma_1, \ldots, \gamma_m\}\Big) \times \operatorname{bd} Q,$$

and

$$Y = \bigcup_{i=0}^{m} Y_i$$

(see Fig. 2.5). Just as in the proof of Lemma 2.2 one can show that Y has the required properties.

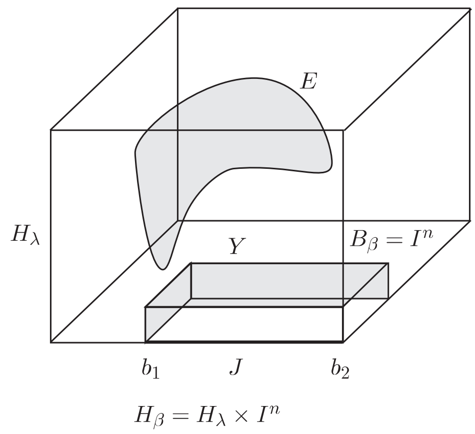

Fig. 2.5

3. Examples of Cantor ind-manifolds. For each non-limit ordinal $\alpha = \beta + 1$ such that $\omega_0 \leq \alpha < \omega_1$, we describe an α-dimensional Cantor ind-manifold Z_α. For the convenience of the reader some technical reasonings showing that the construction is feasible are deferred to the Appendix.

First, we define an inverse sequence $\{Z_n, r_n^{n+1}\}$ consisting of compact metrizable spaces Z_n and retractions r_n^{n+1}; simultaneously, we define countable coverings $\mathcal{D}_n$ of Z_n by cubes with dimension greater than 1.

Let $Z_1 = H_{\beta(\alpha)}$ and $\mathcal{D}_1 = \mathcal{C}_{\beta(\alpha)}$ (see Section 2); recall that the covering $\mathcal{C}_\beta$ consists of cubes of positive dimension for every ordinal β, and so it consists of cubes of dimension greater than 1 whenever β is a non-limit ordinal. Suppose that we have already defined the space Z_n and its covering $\mathcal{D}_n$. Let ϱ_n be any metric on Z_n compatible with its topology. Assume additionally that for $k = 1, 2, \ldots$, there exists an arc $L_{n,k} \subseteq Z_n$ with the following properties:

(3.1) $\quad \varrho_n(x, L_{n,k}) \leq 1/k$ for every $x \in Z_n$,

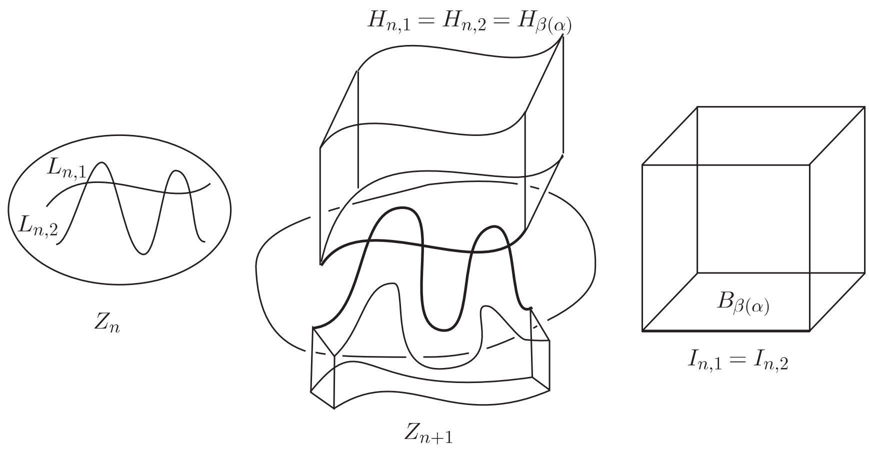

Fig. 3.1

(3.2) $L_{n,k}$ is contained in the union of a finite number of cubes belonging to $\mathcal{D}_n$,

(3.3) if J is a cube contained in a cube $D \in \mathcal{D}_n$ and parallel to a proper face of D, then $J \cap L_{n,k}$ is finite.

For $k = 1, 2, \ldots$, denote by $H_{n,k}$ a copy of Henderson's space $H_{\beta(\alpha)}$ and by $I_{n,k}$ an arbitrary edge of $B_{\beta(\alpha)}$ (see Section 2), and set $\mathcal{D}_{n,k} = \mathcal{C}_{\beta(\alpha)}$. Loosely speaking, in order to obtain Z_{n+1} we stick a copy $H_{n,k}$ of Henderson's space to each arc $L_{n,k}$ along the edge $I_{n,k}$ in such a way that the sets $H_{n,k} - I_{n,k}$ are pairwise disjoint, and the space so obtained is compact, i.e., $H_{n,k}$ is contained in an arbitrarily small neighbourhood of $L_{n,k}$ for sufficiently large k's (see Fig. 3.1). Strictly speaking, the space Z_{n+1} can be defined as follows.

Let γ stand for the predecessor of $\beta(\alpha)$ (see (2.2)); then $H_{\beta(\alpha)} = H_\gamma \times I$. Set $Z'_n = Z_n \times \{(p_\gamma, p_\gamma, \ldots)\} \subset Z_n \times (H_\gamma)^{\aleph_0}$, where p_γ denotes the distinguished point of H_γ (see Section 2). Next, let $H'_{n,k}$ consist of all $(x, (y_m)_{m=1}^\infty) \in Z_n \times (H_\gamma)^{\aleph_0}$ such that $x \in L_{n,k}$ and $y_m = p_\gamma$ for $m \neq k$. Put

$$Z_{n+1} = Z'_n \cup \bigcup_{k=1}^{\infty} H'_{n,k}\,.$$

Since $L_{n,k}$ is an arc, $H_{n,k}$ and $H'_{n,k}$ are homeomorphic; obviously, so are Z_n and Z'_n. In the sequel, we identify $H_{n,k}$ and $H'_{n,k}$ as well as Z_n and Z'_n.

Let $\mathcal{D}_{n+1} = \mathcal{D}_n \cup \bigcup_{k=1}^\infty \mathcal{D}_{n,k}$ and r_n^{n+1} be the retraction of Z_{n+1} onto Z_n determined by the "orthogonal projections" of the spaces $H_{n,k}$ onto the edges $I_{n,k}$ of their bases, i.e.,

$$r_n^{n+1}((x, (y_m)_{m=1}^\infty)) = x \quad \text{for } (x, (y_m)_{m=1}^\infty) \in Z_{n+1} \subseteq Z_n \times (H_\gamma)^{\aleph_0}\,.$$

It is easy to see that Z_{n+1} is a closed subspace of $Z_n \times (H_\gamma)^{\aleph_0}$, and so it is a compact metrizable space, and $\mathcal{D}_{n+1}$ is a countable covering of Z_{n+1} consisting of cubes with dimension greater than 1.

To complete our construction, we should check that for $n = 1, 2, \ldots$ and any metric ϱ_n on Z_n, there exist arcs $L_{n,k} \subseteq Z_n$ with properties (3.1)–(3.3); in the Appendix we show that there exist arcs $L_{1,k} \subseteq Z_{1,k}$ which have, apart from (3.1)–(3.3), some additional properties, and if we assume that there exist arcs $L_{n,k}$ with these additional properties, then there exist arcs $L_{n+1,k} \subseteq Z_{n+1}$ with these properties.

Now, assume that the inverse sequence $\{Z_n, r_n^{n+1}\}$ is defined.

Let $Z_\alpha = \varprojlim\{Z_n, r_n^{n+1}\}$; denote by r_n the projection of Z_α onto Z_n. Obviously, Z_α is a compact metrizable space. Since each bonding mapping r_n^{n+1} is a retraction, we can assume that $Z_n \subseteq Z_\alpha$ and r_n is a retraction for every $n = 1, 2, \ldots$

We now show that

(3.4) if K is a partition in Z_α between any pair of distinct points, then $\operatorname{ind} K$ is not less than the predecessor of α.

Let $U, V \subseteq Z_\alpha$ be disjoint open sets with $K = Z_\alpha - (U \cup V)$. Take an n such that

$$Z_n \cap U \neq \emptyset \neq Z_n \cap V\,;$$

then, by (3.1), $L_{n,k} \cap U \neq \emptyset \neq L_{n,k} \cap V$ for a $k \in \mathbb{N}$, and thus $K \cap H_{n,k}$ is a partition in $H_{n,k}$ between a pair of distinct points from $I_{n,k}$. By Theorem 2.1, $\operatorname{ind}(K \cap H_{n,k})$ is not less than the predecessor of α and so is $\operatorname{ind} K$.

It remains to prove that

$$\operatorname{ind} Z_\alpha \leq \alpha\,. \tag{3.5}$$

To this end, we need the following technical lemma; the situation concerned by the lemma is illustrated in Fig. 3.2.

3.1. Lemma. *Let $\{Z_n, r_n^{n+1}\}$ be a sequence of compact spaces such that $Z_n \subseteq Z_{n+1}$ and r_n^{n+1} is a retraction for every $n \in \mathbb{N}$. Suppose $Y_n \subseteq Z_n, n = 1, 2, \ldots,$ are closed subspaces with*

$$Y_{n+1} = Y_n \cup \bigcup\{A_s : s \in S_n\}\,, \tag{3.6}$$

where

(3.7) *A_s is closed and $A_s - Y_n$ is open in Y_{n+1},*

(3.8) *$A_s \cap A_t \subseteq Y_n$ for distinct $s, t \in S_n$,*

and there is a natural number m such that:

(3.9) *$|A_s \cap Y_n| < \aleph_0$ for every $s \in S_n$, and $|A_s \cap Y_n| > 1$ only for a finite number of $s \in S_n$ provided $n < m$,*

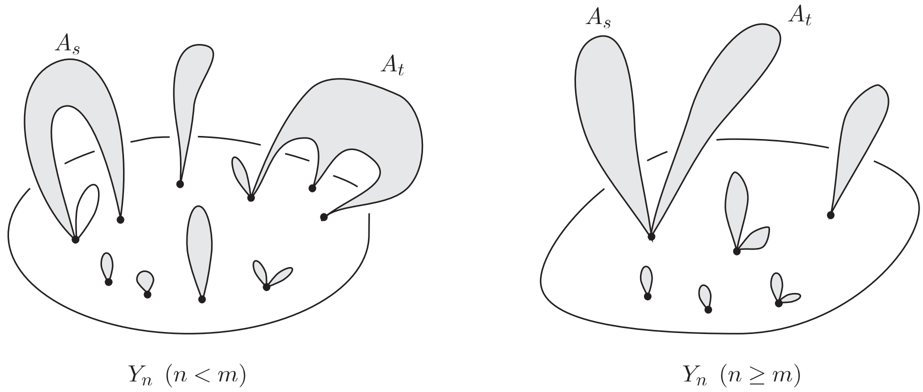

Fig. 3.2

(3.10) $|A_s \cap Y_n| = 1$ *for every* $s \in S_n$ *provided* $n \geq m$,

(3.11) $r_n^{n+1}(A_s) = A_s \cap Y_n$ *for any* $n \in \mathbb{N}$ *and* $s \in S_n$ *such that* $|A_s \cap Y_n| = 1$.

Let $Y = \varprojlim\{Y_n, r_n^{n+1}|Y_{n+1},\ n \geq m\}$. *If* $\operatorname{ind} Y_1 \leq \gamma$ *and* $\operatorname{ind} A_s \leq \gamma$ *for every* $s \in \bigcup_{n=1}^{\infty} S_n$, *then* $\operatorname{ind} Y \leq \gamma$.

Proof. We first show by induction that

$$\operatorname{ind} Y_n \leq \gamma \quad \text{for every } n \in \mathbb{N}\,. \tag{3.12}$$

For $n = 1$, this is one of our assumptions. Assume (3.12) holds for an n; we will prove it for $n + 1$. Let $y \in Y_{n+1}$, and let $F \subseteq Y_{n+1}$ be any closed set not containing y.

If $y \in A_s - Y_n$ for some $s \in S_n$, then the existence of a partition between y and F with small transfinite dimension less than γ follows from (3.7) and the inequality $\operatorname{ind} A_s \leq \gamma$. Assume therefore that $y \in Y_n$ (see (3.6)).

Set $Z = \bigcup\{A_s \cap Y_n : s \in S_n \text{ and } |A_s \cap Y_n| > 1\} - \{y\}$ (see Fig. 3.3); then Z is finite by (3.9) and (3.10) ($Z = \emptyset$ whenever $n \geq m$). By the inductive assumption, there exists a partition K_0 in Y_n between y and $(F \cap Y_n) \cup Z$ such that $\operatorname{ind} K_0 < \gamma$ (see Fig. 3.3); let $U, V \subseteq Y_n$ be disjoint open sets with $y \in U$, $(F \cap Y_n) \cup Z \subseteq V$, and $K_0 = Y_n - (U \cup V)$.

Let $s_1, \ldots, s_j$ be all $s \in S_n$ such that $y \in A_s$ and $|A_s \cap Y_n| > 1$ (see (3.9)). For $i = 1, \ldots, j$, $\operatorname{ind} A_{s_i} \leq \gamma$, and so there exists a partition K_i in A_{s_i} between y and $(F \cap A_{s_i}) \cup (A_{s_i} \cap Y_n - \{y\})$ such that $\operatorname{ind} K_i < \gamma$ (see Fig. 3.3, where $j = 2$).

We now show that

$$T = \{s \in S_n : |A_s \cap Y_n| = 1,\ A_s \cap Y_n \subseteq U,\ \text{and } F \cap A_s \neq \emptyset\}$$

is finite.

Indeed, suppose that T is infinite. For every $s \in T$, choose $x_s \in F \cap A_s$; since $F \cap U = \emptyset$, we have $x_s \in A_s - Y_n$. Let x be an accumulation point of

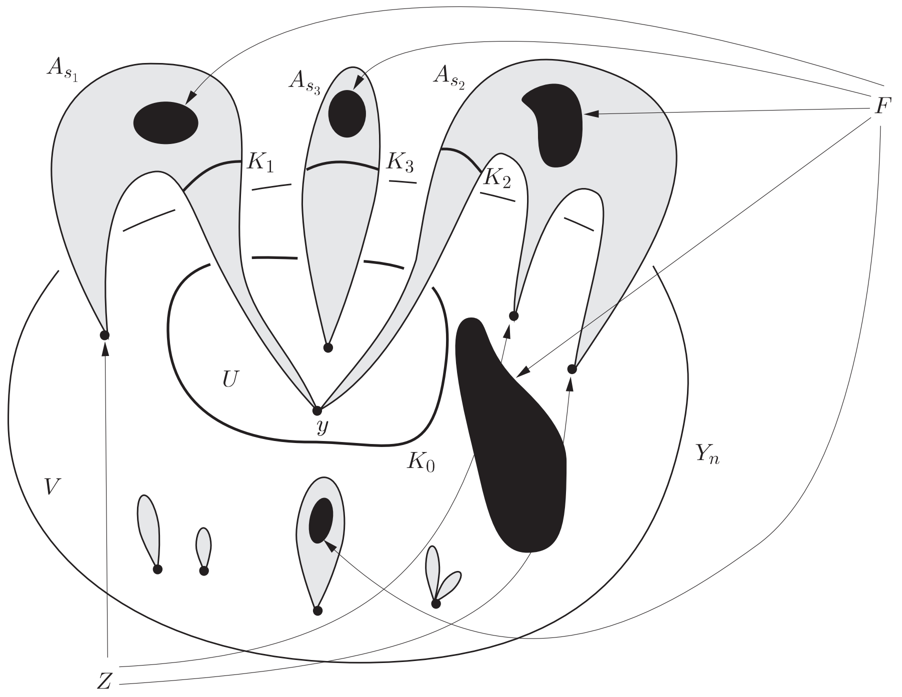

Fig. 3.3

$\{x_s : s \in T\}$. From (3.6)–(3.8) it follows that $x \in Y_n \cap F \subseteq V$; on the other hand, since $r_n^{n+1}(x_s) \in U$ for every $s \in T$ (see (3.11)) and r_n^{n+1} is a retraction onto Y_n, we have $r_n^{n+1}(x) = x \in U \cup K_0$, contrary to $V \cap (U \cup K_0) = \emptyset$.

Let $s_{j+1}, \ldots, s_p$ be all elements of T. For every $i = j+1, \ldots, p$, we have $\operatorname{ind} A_{s_i} \le \gamma$, and so there exists a partition K_i in A_{s_i} between $A_{s_i} \cap Y_n$ and $F \cap A_{s_i}$ such that $\operatorname{ind} K_i < \gamma$ (see Fig. 3.3, where $p = 3$).

It is easy to check that $K = \bigcup_{i=0}^{p} K_i$ is a partition in Y_{n+1} between y and F. The sets K_i, $i = 0, 1, \ldots, p$, are compact; since $K_0 \subseteq Y_n$ and $K_i \subseteq A_{s_i} - Y_n$ for $i = 1, \ldots, p$, they are pairwise disjoint (see (3.8)). Thus

$$\operatorname{ind} K \le \max\{\operatorname{ind} K_i : i = 0, 1, \ldots, p\} < \gamma .$$

Therefore the proof of (3.12) is concluded. We are now in a position to show that $\operatorname{ind} Y \le \gamma$. Denote by r_n the projection of Y onto Y_n. Let $y \in Y$, and $F \subseteq Y$ a closed set not containing y. Take $n \ge m$ such that $r_n(y) \notin r_n(F)$.

Since $\operatorname{ind} Y_n \le \gamma$ (see (3.12)), there exists a partition K_n in Y_n between $r_n(y)$ and $r_n(F)$ with $\operatorname{ind} K_n < \gamma$; consider disjoint open sets $U_n, V_n \subseteq Y_n$ such that $r_n(y) \in U_n$, $r_n(F) \subseteq V_n$, and $K_n = Y_n - (U_n \cup V_n)$. Set

$$K = K_n, \quad U = r_n^{-1}(U_n), \quad V = r_n^{-1}(V_n \cup K_n) - K_n .$$

We prove that K is a partition in Y between y and F. Obviously, $y \in U$, $F \subseteq V$, U is open and K is closed in Y, and $U \cap K = \emptyset = V \cap K$, $U \cap V = \emptyset$. We only need to show that V is open in Y.

Take $z \in V$. If $r_n(z) \in V_n$, then $r_n^{-1}(V_n)$ is a neighbourhood of z containing in V. Thus assume that $r_n(z) \in K_n$. Since $z \notin K_n$, we have $z \notin Y_n$. If $r_{k+1}(z)$ belonged to Y_k for every $k \geq n$, then z would belong to Y_n.

Indeed, suppose that $r_{k+1}(z) \in Y_k$ for $k \geq n$. Then, in particular, $r_{n+1}(z) \in Y_n$. Assuming that $r_{k+1}(z) \in Y_n$ for some $k \geq n$, we obtain (recall that r_{k+1}^{k+2} is a retraction) $r_{k+2}(z) = r_{k+1}^{k+2}(r_{k+2}(z)) = r_{k+1}(z) \in Y_n$. Hence, by induction, $r_k(z)$ is in Y_n for every $k \geq n$, and so is z.

Thus $r_{k+1}(z) \notin Y_k$ for some $k \geq n$. Then by (3.6), $r_{k+1}(z) \in A_s - Y_k$ for some $s \in S_k$, and by (3.7), $r_{k+1}^{-1}(A_s - Y_k)$ is open. We now show that $r_{k+1}^{-1}(A_s - Y_k) \subseteq V$.

Indeed, since $k \geq n \geq m, r_k^{k+1}(A_s)$ is a one-point set (see (3.10) and (3.11)); thus

$$r_k^{k+1}(A_s) = \{r_k^{k+1}(r_{k+1}(z))\} = \{r_k(z)\} \subseteq K_n\,.$$

Obviously, $r_{k+1}^{-1}(A_s - Y_k) \cap K_n = \emptyset$, and so $r_{k+1}^{-1}(A_s - Y_k) \subseteq V$.

Having proved the lemma, we can turn to the proof of inequality (3.5); recall that β stands for the predecessor of α. Let $z \in Z_\alpha$, and $F \subseteq Z_\alpha$ a closed set not containing z. We prove that there exists a partition in Z_α between z and F of dimension not greater than β.

Take m such that $r_m(z) \notin r_m(F)$, and $p \leq m$ such that $r_m(z) \in Z_p - Z_{p-1}$; we assume that $Z_0 = \emptyset$, that is, if $r_m(z) \in Z_1$, then $p = 1$. We shall define by induction for $n = p, p+1, \ldots, m$ a partition Y_n in Z_n between $r_m(z)$ and $r_m(F) \cap Z_n$ with the following property:

(3.13) for every cube $D \in \mathcal{D}_n$ the set $D \cap Y_n$ is the union of a finite number of cubes of dimension less than that of D, each parallel to a proper face of D;

moreover, we will require $\operatorname{ind} Y_p \leq \beta$. Simultaneously, we shall define sets S_n and A_s for $s \in S_n$ and $n = p, p+1, \ldots, m-1$ satisfying (3.6)–(3.9), (3.11) and

$$\operatorname{ind} A_s \leq \beta \quad \text{for every } s \in S_n\,. \tag{3.14}$$

Since $Z_p - Z_{p-1}$ is a neighbourhood of $r_m(z)$ homeomorphic to an open subset of $H_{\beta(\alpha)}$, the existence of a partition Y_p with the required properties follows from Lemma 2.2. Assume that we have defined a partition Y_n with the required properties for an $n < m$.

Let $U_n, V_n \subseteq Z_n$ be open sets such that $r_m(z) \in U_n$, $r_m(F) \cap Z_n \subseteq V_n$ and $Y_n = Z_n - (U_n \cup V_n)$. Set

$$Y'_{n+1} = (r_n^{n+1})^{-1}(Y_n), \quad U'_{n+1} = (r_n^{n+1})^{-1}(U_n), \quad V'_{n+1} = (r_n^{n+1})^{-1}(V_n)\,.$$

Then Y'_{n+1} is a partition in Z_{n+1} between $r_m(z)$ and $r_m(F) \cap Z_n$.

Since $L_{n,k}$ has properties (3.2)–(3.3) and Y_n satisfies (3.13),

$$Y_n \cap L_{n,k} \text{ is finite for every } k = 1, 2, \ldots \tag{3.15}$$

Hence $Y'_{n+1} \cap H_{n,k}$ is the union of a finite number of pairwise disjoint sets homeomorphic to H_ν, where $\beta(\alpha) = \nu + 1$, for every $k = 1, 2, \ldots$ Denote these sets by $A_s, s \in T_k$ (see Fig. 3.4). Observe that $A_s \cap Y_n$ is a one-point set for every $s \in T_k$ and $k = 1, 2, \ldots$

Since $r_m(F) \cap (U_n \cup Y_n) = \emptyset$, it follows that $r_m(F) \cap (U'_{n+1} \cup Y'_{n+1}) \subseteq \bigcup\{H_{n,k} - L_{n,k} : k \in \mathbb{N}\}$; furthermore, since the sets $H_{n,k} - L_{n,k}$ are pairwise disjoint and $r_m(F) \cap (U'_{n+1} \cup Y'_{n+1})$ is compact, there exists $l \in \mathbb{N}$ such that $r_m(F) \cap (U'_{n+1} \cup Y'_{n+1}) \subseteq \bigcup\{H_{n,k} - L_{n,k} : k = 1, \ldots, l\}$.

Fix $k \le l$, and an orientation of $L_{n,k}$. Then $L_{n,k} = \bigcup_{i=1}^{j} a_{i-1}{}^\wedge a_i$, where $a_0, a_1, \ldots, a_j$ are ordered consistently with the orientation, and either

$$a_{i-1}{}^\wedge a_i \subseteq U_n \cup Y_n, \quad \text{whereas} \quad a_i{}^\wedge a_{i+1} \subseteq V_n \cup Y_n\,,$$

or

$$a_{i-1}{}^\wedge a_i \subseteq V_n \cup Y_n, \quad \text{whereas} \quad a_i{}^\wedge a_{i+1} \subseteq U_n \cup Y_n$$

for $i = 1, \ldots, j-1$, that is, $\{a_1, \ldots, a_{j-1}\}$ is the set of all points at which $L_{n,k}$ goes across Y_n; of course, a_0 and a_j are the endpoints of $L_{n,k}$ (see Fig. 3.4).

Let $T'_k = \{(i,k) : i = 1, \ldots, j \text{ and } a_{i-1}{}^\wedge a_i \subseteq U_n \cup Y_n\}$. For every $s = (i,k) \in T'_k$, the arc $a_{i-1}{}^\wedge a_i$ is identified with a segment contained in the edge $I_{n,k}$ of the base of $H_{n,k} = H_{\beta(\alpha)}$. Let A_s, U_s, V_s stand for sets Y, U, V with the properties described in Lemma 2.3 for $J = a_{i-1}{}^\wedge a_i$ and $E = r_m(F) \cap H_{n,k}$ (see Fig. 3.4).

The set

$$\begin{aligned} Y_{n+1} &= \Big(Y'_{n+1} - \bigcup\{H_{n,k} - L_{n,k} : k \le l\}\Big) \cup \bigcup\{A_s : k \le l \text{ and } s \in T'_k\} \\ &= Y_n \cup \bigcup\{A_s : k > l \text{ and } s \in T_k\} \cup \bigcup\{A_s : k \le l \text{ and } s \in T'_k\} \end{aligned}$$

is a partition in Z_{n+1} between $r_m(z)$ and $r_m(F) \cap Z_{n+1}$; indeed,

$$U_{n+1} = \Big(U'_{n+1} - \bigcup\{H_{n,k} - L_{n,k} : k \le l\}\Big) \cup \bigcup\{U_s : k \le l \text{ and } s \in T'_k\}$$

and

$$V_{n+1} = \Big(V'_{n+1} - \bigcup\{H_{n,k} - L_{n,k} : k \le l\}\Big) \cup \bigcup\{V_s : k \le l \text{ and } s \in T'_k\}$$

are open sets in Z_{n+1} such that $r_m(z) \in U_{n+1}$, $r_m(F) \cap Z_{n+1} \subseteq V_{n+1}$, $U_{n+1} \cap V_{n+1} = \emptyset$ and $Y_{n+1} = Z_{n+1} - (U_{n+1} \cup V_{n+1})$.

Let $S_{n+1} = \bigcup\{T'_k : k \le l\} \cup \bigcup\{T_k : k > l\}$. It is easy to check that our sets have the required properties. Thus we have constructed inductively the sets $Y_p, Y_{p+1}, \ldots, Y_m$ and the sets S_n and A_s, $s \in S_n$, for $n = p, p+1, \ldots, m-1$.

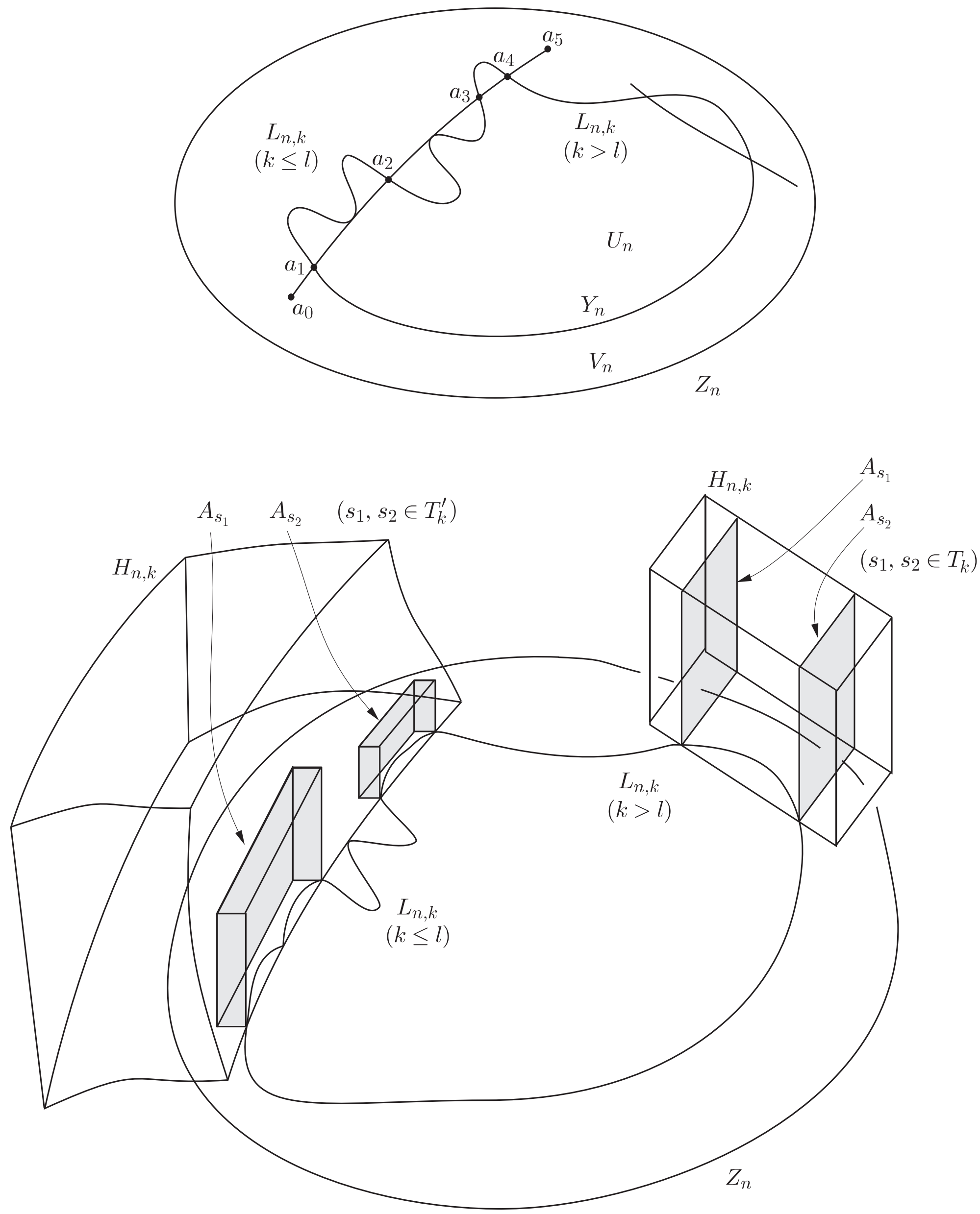

Fig. 3.4

We define Y_n for $n = m+1, m+2, \ldots$ by induction setting $Y_{n+1} = (r_n^{n+1})^{-1}(Y_n)$ (see Fig. 3.5).

Let $S_{n,k} = (L_{n,k} \cap Y_n) \times \{k\}$ for $n = m, m+1, \ldots$ and $k = 1, 2, \ldots$, and

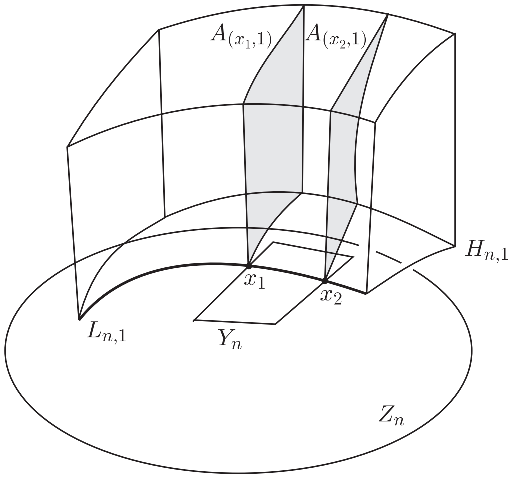

Fig. 3.5

next $S_n = \bigcup_{k=1}^{\infty} S_{n,k}$; let

$$A_s = (r_n^{n+1})^{-1}(x) \cap H_{n,k} \quad \text{for } s = (x, s) \in S_{n,k} \text{ (see Fig. 3.5).}$$

One can check by induction that Y_n satisfies (3.13) for $n = m$, $m + 1, \ldots$, and hence $Y_n \cap L_{n,k}$ is finite for $k = 1, 2, \ldots$ (see (3.2) and (3.3)). By construction and the above observation, it follows that (3.6)–(3.8), (3.10)–(3.11) are also satisfied for $n \geq m$; since each A_s, $s \in S_n$, is homeomorphic to Henderson's space H_ν, where ν is the predecessor of $\beta(\alpha)$, condition (3.14) is also satisfied (recall that $\operatorname{ind} H_\mu < \alpha$ for every $\mu < \beta(\alpha)$, see Section 2).

Since Y_m is a partition in Z_m between $r_m(z)$ and $r_m(F)$, it follows that $r_m^{-1}(Y_m)$ is a partition in Z_α between z and F. It is easily seen that $r_m^{-1}(Y_m)$ is homeomorphic to $\varprojlim\{Y_n, r_n^{n+1}|Y_{n+1}, n \geq m\}$; thus $\operatorname{ind} r_m^{-1}(Y_m) \leq \beta$ by Lemma 3.1.

4. Appendix. We complete the description of the construction of $\{Z_n, r_n^{n+1}\}$. To wit, we show that there exist arcs $L_{1,k}$ in Z_1 satisfying (3.1)–(3.3), and having some additional properties: each $L_{1,k}$ is a $\mathcal{D}_1$-broken line (see Definition 4.1). We also show that if each $L_{n,k} \subseteq Z_n$ is a $\mathcal{D}_n$-broken line, then there exist $\mathcal{D}_{n+1}$-broken lines $L_{n+1,k} \subseteq Z_{n+1}$ with properties (3.1)–(3.3).

First, we have to prepare an auxiliary apparatus.

4.1. Definition. Let $\mathcal{D}$ be a countable covering of a topological space X by cubes. An arc L is said to be a $\mathcal{D}$-*broken line* in X if it is contained in the union of a finite number of cubes belonging to $\mathcal{D}$, and for every $D \in \mathcal{D}$, $L \cap D$ is the union of a finite number of segments and one-point sets.

4.2. Definition. Let $\mathcal{D}$ be a countable covering of a topological space X by cubes of dimension greater than 1. We say that $\mathcal{D}$ has *property* $(*)$ if the following conditions are satisfied:

(4.1) for every pair of distinct cubes $C, D \in \mathcal{D}$, $C \cap D$ is either a proper face of C and a proper face of D, or is the union of a finite number of segments and one-point sets contained either in a proper face of C or in a proper face of D,

(4.2) for every pair of cubes $C, D \in \mathcal{D}$, there exists a sequence of cubes $D_1, \ldots, D_n \in \mathcal{D}$ such that $C = D_1$, $D = D_n$, and $|D_i \cap D_{i+1}| \geq \aleph_0$ for $i = 1, \ldots, n-1$.

Note that (4.1) does not exclude that $C \cap D = \emptyset$ for some $C, D \in \mathcal{D}$, and it implies that if $|C \cap D| \geq \aleph_0$, then $C \cap D$ contains a segment.

4.3. Lemma. *For every countable non-limit ordinal $\alpha > 1$, the covering $\mathcal{C}_\alpha$ of H_α has property $(*)$.*

The proof is by induction on α.

4.4. Lemma. *Let Y be a topological space. For $k = 0, 1, \ldots$, let X_k be a subspace of Y, $\mathcal{E}_k$ a covering of X_k by cubes with property $(*)$, and $(L_k)_{k=1}^{\infty}$ a sequence of $\mathcal{E}_0$-broken lines in X_0. Furthermore, suppose that*

(4.3) *$X_0 \cap X_k = L_k$, and $X_k \cap X_m = L_k \cap L_m$ for distinct $k, m = 1, 2, \ldots$,*

(4.4) *for every cube $D \in \mathcal{E}_k$, $L_k \cap D$ is the union of a finite number of segments and one-point sets contained in a proper face of D.*

Then $\mathcal{E} = \bigcup_{k=0}^{\infty} \mathcal{E}_k$ is a covering of $X = \bigcup_{k=0}^{\infty} X_k$ by cubes with property $(*)$.

Proof. Obviously, $\mathcal{E}$ is a countable covering of X by cubes of dimension greater than 1. It is a simple matter to check that (4.1) is satisfied. We now show that (4.2) is also satisfied.

Let $C, D \in \mathcal{E}$. If $C, D \in \mathcal{E}_k$ for some $k = 0, 1, \ldots$, then the existence of a sequence $D_1, \ldots, D_n$ with the required properties follows from the assumption that $\mathcal{E}_k$ has property $(*)$; thus suppose that $C \in \mathcal{E}_k$ and $D \in \mathcal{E}_m$, where $k \neq m$. We only consider the case when $k, m > 0$; if $k = 0$ or $m = 0$, the reasoning is similar.

Since $X_0 \cap X_k = L_k$, and $\mathcal{E}_0$ and $\mathcal{E}_k$ are countable, $|C' \cap C''| \geq \aleph_0$ for some $C' \in \mathcal{E}_0$ and $C'' \in \mathcal{E}_k$; by a similar argument, there exist $D' \in \mathcal{E}_0$ and $D'' \in \mathcal{E}_m$ such that $|D' \cap D''| \geq \aleph_0$. Let

- $D_1, \ldots, D_j \in \mathcal{E}_k$ be such that $D_1 = C$, $D_j = C''$, and $|D_i \cap D_{i+1}| \geq \aleph_0$ for $i = 1, \ldots, j-1$,

- $D_{j+1}, \ldots, D_l \in \mathcal{E}_0$ be such that $D_{j+1} = C'$, $D_l = D'$, and $|D_i \cap D_{i+1}| \geq \aleph_0$ for $i = j+1, \ldots, l-1$, and
- $D_{l+1}, \ldots, D_n \in \mathcal{E}_m$ be such that $D_{l+1} = D''$, $D_n = D$, and $|D_i \cap D_{i+1}| \geq \aleph_0$ for $i = l+1, \ldots, n$.

Then the sequence $D_1, \ldots, D_n$ has the required properties.

4.5. Lemma. *Let (X, ϱ) be a totally bounded metric space, and $\mathcal{D}$ its covering by cubes with property $(*)$. Then for every $\varepsilon > 0$, there exists a $\mathcal{D}$-broken line L in X with the following properties*:

(4.5) *for every $x \in X$, the distance between x and L is not greater than ε,*

(4.6) *if J is a cube contained in a cube $D \in \mathcal{D}$, the dimension of J is less than that of D, and J is parallel to a proper face of D, then $L \cap D$ is finite.*

The proof of Lemma 4.5 will be preceded by two preliminary lemmas, both concerning the situation described in Lemma 4.5.

4.6. Lemma. *Let $C \in \mathcal{D}$; suppose $T \subseteq C$ is finite and $x, y \in C - T$. Then for every $\delta > 0$, there exists a broken line $K \subseteq C - T$ satisfying* (4.6) *with endpoints x and y and such that*

(4.7) $$\varrho(z, K) \leq \delta \quad \textit{for every } z \in C\,.$$

Proof. Since the dimension of C is not less than 2, it is a simple matter to find a broken line $K \subseteq C - T$ satisfying (4.7) with endpoints x and y (see Fig. 4.1).

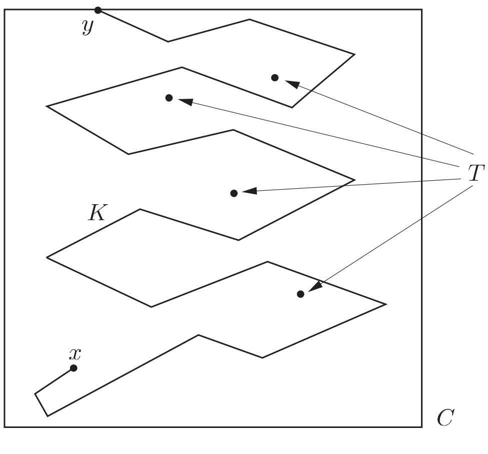

Fig. 4.1

Since $\mathcal{D}$ is countable and satisfies (4.1), $C \cap [\bigcup(\mathcal{D} - \{C\})]$ is the union of a number of faces of C, a countable number of segments (say $F_1, F_2, \ldots$), and a countable number of one-point sets. In order to show that (4.6) is also satisfied, it suffices to ensure that K is the union of segments $K_1, \ldots, K_m$ none of which is parallel either to one of $F_1, F_2, \ldots$ or to a proper face of C.

Indeed, for every cube $J \subseteq C$ with dimension less than that of C and parallel to a proper face of C, and each $i = 1, \ldots, m$, the set $K_i \cap J$ consists of at most one point; for every cube $D \neq C$, the set $C \cap D$ is, by (4.1), either a proper face of C or the union of a finite number of the segments $F_1, F_2, \ldots$ and a finite number of one-point sets, and thus $D \cap K_i$ is finite for each $i = 1, \ldots, m$.

4.7. Lemma. *Let $C \in \mathcal{D}$; suppose U is a connected open subset of C and $K_1, K_2 \subseteq C$ are disjoint broken lines such that $K_i \cap U \neq \emptyset$ for $i = 1, 2$. Then there exist disjoint broken lines $M_1, M_2 \subseteq U$ both with property* (4.6) *and such that*

(4.8) *$M_j \cap K_1 = \{c_j\}$ and $M_j \cap K_2 = \{d_j\}$, where c_j and d_j are the endpoints of M_j, for $j = 1, 2$.*

Proof. Since U is a connected subset of a cube, there exists an arc $J \subseteq U$ such that $J \cap K_i \neq \emptyset$ for $i = 1, 2$; without loss of generality we can assume that $J \cap K_1 = \{x\}$ and $J \cap K_2 = \{y\}$, where x and y stand

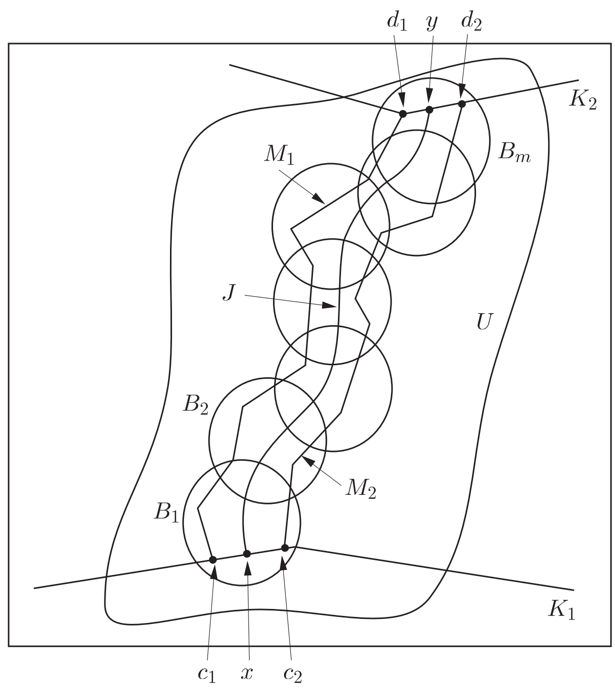

Fig. 4.2

for the endpoints of J. Let $J \subseteq V \subseteq U$ with $V = \bigcup_{i=1}^{m} B_i$, where each B_i is an open ball and $K_1 \cap V \subseteq B_1 - \bigcup_{k=2}^{m} B_k$, $K_2 \cap V \subseteq B_m - \bigcup_{k=1}^{m-1} B_k$, and $B_k \cap B_l = \emptyset$ whenever $|k - l| \geq 2$ (see Fig. 4.2). Since C is a cube of dimension not less than 2, it is a simple matter to find disjoint broken lines $M_1, M_2 \subseteq V$ satisfying (4.8) (see Fig. 4.2).

Just as in the proof of Lemma 4.6 we can ensure that M_1 and M_2 satisfy (4.6).

Proof of Lemma 4.5. Let $S \subseteq X$ be a finite $\varepsilon/2$-dense set. Since $\mathcal{D}$ is a countable covering of X and satisfies (4.2), there exist $D_1, \ldots, D_m \in \mathcal{D}$, not necessarily distinct, such that $S \subseteq \bigcup_{i=1}^{m} D_i$ and $|D_i \cap D_{i+1}| \geq \aleph_0$ for $i = 1, \ldots, m-1$; let $Y = \bigcup_{i=1}^{m} D_i$. Then

$$\varrho(x, Y) \leq \varepsilon/2 \quad \text{for every } x \in X\,. \tag{4.9}$$

We now show that for $n = 1, \ldots, m$, there exists a $\mathcal{D}$-broken line $L_n \subseteq \bigcup_{i=1}^{n} D_i$ satisfying (4.6) and such that

(4.10) $\varrho(z, L_n) \leq \varepsilon/2^{m-n+1}$ for every $z \in \bigcup_{i=1}^{n} D_i$,

(4.11) L_n intersects the geometrical interior of D_i for $i = 1, \ldots, n$.

We apply induction on n. The existence of an $\varepsilon/2^m$-dense $\mathcal{D}$-broken line $L_1 \subseteq D_1$ satisfying (4.6) follows from Lemma 4.6, because $\mathcal{D}$ satisfies (4.1), and thus each broken line contained in a cube of $\mathcal{D}$ is $\mathcal{D}$-broken; since L_1 cannot be contained in the geometrical boundary of D_1 by (4.6), it follows that (4.11) is satisfied. Assume that there exists a $\mathcal{D}$-broken line L_n with the required properties.

If $D_{n+1} = D_i$ for some $i = 1, \ldots, n$, then $L_{n+1} = L_n$ has the required properties. Thus suppose that $D_{n+1} \neq D_i$ for $i = 1, \ldots, n$.

Since $|D_n \cap D_{n+1}| \geq \aleph_0$, there exists a closed segment $K_1 \subseteq D_n \cap D_{n+1}$ (see the remark following (4.2)). Without loss of generality we can assume that $K_1 \cap L_n = \emptyset$ (see Fig. 4.3). Indeed, $D_n \cap D_{n+1}$ is contained either in a proper face of D_n or in a proper face of D_{n+1} (see (4.1)); thus $K_1 \cap L_n$ is finite (see (4.6)), and we can consider a segment contained in K_1 with the required property instead of K_1.

Take y in the intersection of L_n and the geometrical interior of D_n (see (4.11)), and $x \in K_1$. Consider an arc J joining x and y; without loss of generality we can assume that $J \cap L_n = \{y\}$. Let K_2 be a broken line containing y, contained in the intersection of L_n and the geometrical interior of D_n, such that $L_n - K_2$ intersects the geometrical interior of D_n and

$$\operatorname{diam} K_2 \leq \varepsilon/2^{m-n+1} \tag{4.12}$$

(see Fig. 4.3). Consider a connected open set $U \subseteq D_n$ containing J and such that $U \cap L_n \subseteq K_2$.

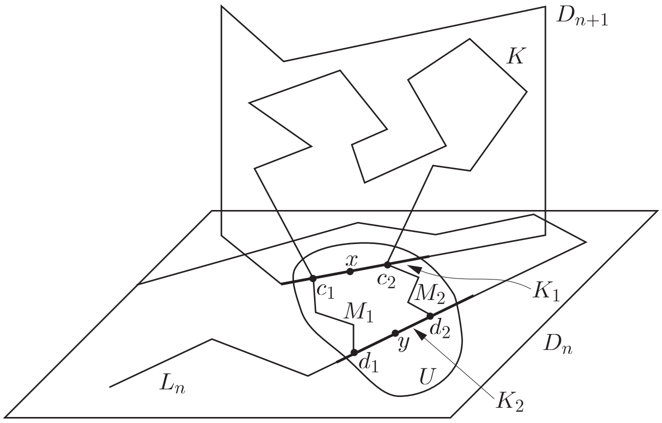

Fig. 4.3

By Lemma 4.7, there exist disjoint broken lines $M_1, M_2 \subseteq U$ with properties (4.6) and (4.8) (see Fig. 4.3); since $\mathcal{D}$ satisfies (4.1), M_1 and M_2 are $\mathcal{D}$-broken lines.

Let $T' = (L_n \cup M_1 \cup M_2) \cap D_{n+1}$; since $D_{n+1} \cap D_i$ is contained either in a proper face of D_{n+1} or in a proper face of D_i for $i = 1, \ldots, n$ (see (4.1)) and each of the $\mathcal{D}$-broken lines $L_n, M_1, M_2 \subseteq \bigcup_{i=1}^n D_i$ has property (4.6), the set T' is finite. Let $T = T' - \{c_1, c_2\}$, where c_1 and c_2 are the endpoints of M_1 and M_2, respectively, belonging to K_1.

By Lemma 4.6, there exists a broken line $K \subseteq D_{n+1} - T$ with endpoints c_1 and c_2 such that

$$\varrho(z, K) \le \varepsilon/2^{m-n} \quad \text{for every } z \in D_{n+1} \tag{4.13}$$

(see Fig. 4.3); since $\mathcal{D}$ satisfies (4.1), K is a $\mathcal{D}$-broken line.

Let $L_{n+1} = [L_n - (d_1{}^\wedge d_2 - \{d_1, d_2\})] \cup M_1 \cup M_2 \cup K$, where d_1 and d_2 are the endpoints of M_1 and M_2, respectively, belonging to $K_2 \subseteq L_n$.

Obviously, $L_{n+1} \subseteq \bigcup_{i=1}^{n+1} D_i$. Since

$$\begin{aligned}
\{d_1, d_2\} &\subseteq [L_n - (d_1{}^\wedge d_2 - \{d_1, d_2\})] \cap (M_1 \cup M_2) \\
&\subseteq L_n \cap (M_1 \cup M_2) \cap U \subseteq K_2 \cap (M_1 \cup M_2) = \{d_1, d_2\}, \\
\{c_1, c_2\} &\subseteq K \cap ([L_n - (d_1{}^\wedge d_2 - \{d_1, d_2\})] \cup M_1 \cup M_2) \\
&\subseteq K \cap D_{n+1} \cap (L_n \cup M_1 \cup M_2) \subseteq K \cap T' = \{c_1, c_2\}
\end{aligned}$$

(see (4.8)), and M_1, M_2 are disjoint, it follows that L_{n+1} is a $\mathcal{D}$-broken line.

By the inductive assumption, L_n has property (4.10); hence by (4.12) and (4.13), so does L_{n+1}. Moreover, L_{n+1} satisfies (4.6) since M_1, M_2, L_n, and K do. It remains to show that L_{n+1} has property (4.11).

The set $L_n - K_2$ meets the geometrical interior of D_n, and so does $L_n - (d_1{}^\wedge d_2 - \{d_1, d_2\}) \subseteq L_{n+1}$; since K has property (4.6), it meets the geometrical interior of D_{n+1}, and so does L_{n+1}. Consider the cube D_i, where $i \in \{1, \ldots, n-1\}$, and assume that $D_i \neq D_n$. By the inductive assumption, L_n meets the geometrical interior of D_i. From (4.1) it follows that the geometrical interiors of distinct cubes of a covering with property $(*)$ are disjoint. Since K_2 is contained in the geometrical interior of D_n and so is $d_1{}^\wedge d_2 \subseteq K_2, d_1{}^\wedge d_2$ does not intersect the geometrical interior of D_i. Thus $L_n - (d_1{}^\wedge d_2 - \{d_1, d_2\}) \subseteq L_{n+1}$ meets the geometrical interior of D_i.

This completes the inductive proof of the existence of the $\mathcal{D}$-broken lines $L_1, \ldots, L_m$. Obviously, $L = L_m$ satisfies (4.5) and (4.6).

Now, we can complete the description of the construction of the sequence $\{Z_n, r_n^{n+1}\}$. By Lemmas 4.3 and 4.5, there exist $\mathcal{D}_1$-broken lines $L_{1,k}$ in Z_1 with properties (3.1)–(3.3). Assume that the arcs $L_{n,k} \subseteq Z_n$ which appear in the construction are $\mathcal{D}$-broken lines. Then by Lemmas 4.4 and 4.5 applied to $Y = Z_n \times (H_\gamma)^{\aleph_0}$, $X_0 = Z_n$, $\mathcal{E}_0 = \mathcal{D}_n$, and $X_k = H_{n,k}$, $\mathcal{E}_k = \mathcal{D}_{n,k}$, $L_k = L_{n,k}$ for $k = 1, 2, \ldots$, there exist $\mathcal{E}$-broken lines with properties (4.5) and (4.6) in the space X described in Lemma 4.4. Since $X = Z_{n+1}$ and $\mathcal{E} = \mathcal{D}_{n+1}$, there exist $\mathcal{D}_{n+1}$-broken lines $L_{n+1,k}$ in Z_{n+1} with properties (3.1)–(3.3).

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