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Normal numbers and subsets of $\mathbb{N}$ with given densities

by

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Abstract. For $X \subseteq [0,1]$, let D_X denote the collection of subsets of $\mathbb{N}$ whose densities lie in X. Given the exact location of X in the Borel or difference hierarchy, we exhibit the exact location of D_X. For $\alpha \geq 3$, X is properly $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_\alpha)$ iff D_X is properly $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_{1+\alpha})$. We also show that for every nonempty set $X \subseteq [0,1]$, D_X is $\boldsymbol{\Pi}^0_3$-hard. For each nonempty $\boldsymbol{\Pi}^0_2$ set $X \subseteq [0,1]$, in particular for $X = \{x\}$, D_X is $\boldsymbol{\Pi}^0_3$-complete. For each $n \geq 2$, the collection of real numbers that are normal or simply normal to base n is $\boldsymbol{\Pi}^0_3$-complete. Moreover, $D_{\mathbb{Q}}$, the subsets of $\mathbb{N}$ with rational densities, is $\mathcal{D}_2(\boldsymbol{\Pi}^0_3)$-complete.

Introduction. The collection of "naturally arising" or "non-ad hoc" sets that are properly located in the Borel hierarchy (meaning for example $\boldsymbol{\Pi}^0_2$ non-$\boldsymbol{\Sigma}^0_2$), is relatively small. In fact, only a small number of specific examples of any sort are known to be properly located above the third level of the Borel hierarchy. Recently, A. Kechris asked whether the set of real numbers that are normal to base two is $\boldsymbol{\Pi}^0_3$-complete. A. Ditzen then conjectured that if this was true for each base $n \geq 2$, then the set of real numbers that are normal to at least one base $n \geq 2$ should be $\boldsymbol{\Sigma}^0_4$-complete. Certainly this example is non-ad hoc. We found this set extremely difficult to manage, and hence we are inclined to agree with Ditzen's conjecture. There is some evidence supporting this conjecture. Namely results as in [6] which suggest that normality to base two and normality to base three have a weak form of independence. Unfortunately, such proofs are nonconstructive and the conjecture appears to be more number theoretic than set theoretic. It seemed reasonable to replace the set in the conjecture with the easier to manage collection of subsets of $\mathbb{N}$ with density $1/n$, for some (varying) $n \in \mathbb{N}$. However, in this case, the limit one computes is the same for all n, and the set is too simple. We then looked at the subsets of $\mathbb{N}$ with rational densities, $D_{\mathbb{Q}}$, and were able to show it was properly the difference of two $\boldsymbol{\Pi}^0_3$ sets, i.e. $\mathcal{D}_2(\boldsymbol{\Pi}^0_3)$-complete. As $D_{\mathbb{Q}}$ is at least somewhat natural, this is rather surpris-

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ing, since it lies above the third level of the Borel hierarchy. In continuing the study of the relationship between the Borel class of $X \subseteq [0,1]$, and that of $D_X \subseteq 2^{\mathbb{N}}$, the collection of subsets of $\mathbb{N}$ whose densities lie in X, we were able to show that if X is properly $\boldsymbol{\Pi}^0_n$ ($\boldsymbol{\Sigma}^0_n$), then D_X is properly $\boldsymbol{\Pi}^0_{n+1}$ ($\boldsymbol{\Sigma}^0_{n+1}$) for $n \geq 3$. Furthermore, the relationship extended to the difference hierarchy of $\boldsymbol{\Delta}^0_{n+1}$ sets. If X is properly $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_n)$, then D_X is properly $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_{n+1})$, so long as $n \geq 2$. However, on the dual side, at the finite levels of the difference hierarchy for $n = 2$, an interesting phenomenon arises. For $m < \omega$, if X is properly $\widetilde{\mathcal{D}}_m(\boldsymbol{\Pi}^0_2)$, then D_X is properly $\mathcal{D}_{m+1}(\boldsymbol{\Pi}^0_3)$. So the analogy of $\mathbb{Q}$ to $D_{\mathbb{Q}}$ extends to all finite levels of the difference hierarchy, and no D_X can be properly $\widetilde{\mathcal{D}}_m(\boldsymbol{\Pi}^0_3)$. If $\xi \geq \omega$ and X is properly $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_2)$, then D_X is properly $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_3)$. For $\alpha \geq 3$, if $\Gamma = \boldsymbol{\Pi}^0_\alpha$, $\boldsymbol{\Sigma}^0_\alpha$, $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_\alpha)$, or $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_\alpha)$, and Γ^* is the class where the α in Γ is replaced by $1+\alpha$, then X is properly Γ iff D_X is properly Γ^*. In particular, we are able to show that for every nonempty set $X \subseteq [0,1]$, D_X is $\boldsymbol{\Pi}^0_3$-hard; for each nonempty $\boldsymbol{\Pi}^0_2$ set $X \subseteq [0,1]$, D_X is $\boldsymbol{\Pi}^0_3$-complete; for each $n \geq 2$, the collection of real numbers that are normal or simply normal to base n is $\boldsymbol{\Pi}^0_3$-complete; and as mentioned above, $D_{\mathbb{Q}}$, or D_X for any $\boldsymbol{\Sigma}^0_2$-complete set X, is $\mathcal{D}_2(\boldsymbol{\Pi}^0_3)$-complete.

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Notation and background information. For sets A and B, $|A|$ is the cardinality of A, $\overline{A}$ denotes the topological closure of A, and we denote the set of all functions from B into A by A^B. If $X \subseteq A$, we denote the preimage under f of X by $f^{\leftarrow}(X)$. We sometimes identify $n \in \mathbb{N} = \{0,1,2,\ldots\}$ with the set $\{0,1,\ldots,n-1\}$. Thus $2^{\mathbb{N}}$ is the collection of functions $f\colon \mathbb{N} \to \{0,1\}$. We let $A^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} A^n$ denote all finite sequences on A, and $A^{\leq\mathbb{N}} = A^{<\mathbb{N}} \cup A^{\mathbb{N}}$. $\vec{0}$ and $\vec{1}$ denote the constant zero and constant one functions in $2^{\mathbb{N}}$. If $f \in A^{\leq\mathbb{N}}$ and $n \in \mathbb{N}$, $f{\restriction}_n = \langle f(0),\ldots,f(n-1)\rangle$, and for $s,t \in A^{<\mathbb{N}}$, $|s|$ denotes the length of s (the unique n for which $s \in A^n$), $s \subseteq t$ (t extends s) means $t{\restriction}_{|s|} = s$, and $s^\frown t$ is the sequence s followed by the sequence t. We use $\mathbb{R}$ and $\mathbb{Q}$ to denote the reals and rationals, and $\mathbb{P}$ denotes the irrationals between zero and one.

We describe the Borel hierarchy using the standard modern terminology of Addison, and define the difference hierarchy, on the ambiguous classes of $\boldsymbol{\Delta}^0_{\alpha+1}$ sets, based on decreasing sequences of $\boldsymbol{\Pi}^0_\alpha$ sets. For Polish topological spaces X, let $\boldsymbol{\Sigma}^0_1(X)$ denote the collection of open subsets of X, and $\boldsymbol{\Pi}^0_1(X)$ denote the closed subsets of X. Inductively define for countable ordinals $\alpha \geq 2$,

$$\boldsymbol{\Sigma}^0_\alpha(X) = \Big\{ A \subseteq X \;\Big|\; A = \bigcup_{n\in\mathbb{N}} A_n, \text{ where each } A_n \in \boldsymbol{\Pi}^0_{\beta_n}(X) \text{ and } \beta_n < \alpha \Big\},$$

$$\boldsymbol{\Pi}^0_\alpha(X) = \Big\{ A \subseteq X \,\Big|\, A = \bigcap_{n\in\mathbb{N}} A_n, \text{ where each } A_n \in \boldsymbol{\Sigma}^0_{\beta_n}(X) \text{ and } \beta_n < \alpha \Big\},$$

$$\boldsymbol{\Delta}^0_\alpha(X) = \{ A \subseteq X \mid A \in \boldsymbol{\Pi}^0_\alpha(X) \cap \boldsymbol{\Sigma}^0_\alpha(X) \}.$$

If X is known by context or irrelevant we frequently drop it for notational convenience. Thus, $\boldsymbol{\Sigma}^0_1 = \text{Open}$, $\boldsymbol{\Pi}^0_1 = \text{Closed}$, $\boldsymbol{\Sigma}^0_2 = F_\sigma$, $\boldsymbol{\Pi}^0_2 = G_\delta$, and so on. The difference hierarchy, which is a finer two-sided hierarchy on the $\boldsymbol{\Delta}^0_\alpha$ sets, extends the Borel hierarchy by including it as the first level ($\xi = 1$) for each countable ordinal α. For ξ a countable ordinal and any sequence of subsets of X, $\langle A_\beta\rangle_{\beta<\xi}$, where $A_\beta \supseteq A_{\beta'}$ if $\beta < \beta'$ (so $\langle A_\beta\rangle$ is decreasing) and for limit $\lambda < \xi$, $A_\lambda = \bigcap_{\beta<\lambda} A_\beta$ (so the sequence is continuous), define a set $A = \mathcal{D}_\xi(\langle A_\beta\rangle_{\beta<\xi})$, the *difference* of the $\langle A_\beta\rangle_{\beta<\xi}$, by

$$x \in A \Leftrightarrow \exists\beta < \xi\ (x \in A_\beta), \text{ and the largest such } \beta \text{ is even.}$$

A countable ordinal β is *even* if when we write $\beta = \lambda + n$, with $\lambda = 0$ or a limit ordinal, n is even. Let $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_\alpha)$ be the collection of sets of the form $\mathcal{D}_\xi(\langle A_\beta\rangle_{\beta<\xi})$, where $\langle A_\beta\rangle_{\beta<\xi}$ is a decreasing, continuous sequence of $\boldsymbol{\Pi}^0_\alpha$ sets (for $\xi < \omega$, the decreasing requirement is redundant). So $\mathcal{D}_1(\boldsymbol{\Pi}^0_\alpha) = \boldsymbol{\Pi}^0_\alpha$, $\mathcal{D}_2(\boldsymbol{\Pi}^0_\alpha) = \{A - B \mid A, B \in \boldsymbol{\Pi}^0_\alpha, \text{ and } A \supseteq B\}$ (so in $\mathbb{R}$, $[0,2)$ is a typical $\mathcal{D}_2(\boldsymbol{\Pi}^0_1)$ set), and $\mathcal{D}_3(\boldsymbol{\Pi}^0_\alpha)$ is the collection of sets of the form

$$(A - B) \cup C \quad \text{where} \quad A, B, C \in \boldsymbol{\Pi}^0_\alpha \text{ and } A \supseteq B \supseteq C.$$

For any class of sets Γ, let the dual class, $\widetilde{\Gamma}$, be the collection of complements of sets in Γ (so $\widetilde{\mathcal{D}}_1(\boldsymbol{\Pi}^0_\alpha) = \boldsymbol{\Sigma}^0_\alpha$), and say A is *properly* Γ if $A \in \Gamma - \widetilde{\Gamma}$. We need the following elementary facts about the difference hierarchy classes.

The $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_\alpha)$ sets are closed under:

(i) intersections with $\boldsymbol{\Pi}^0_\alpha$ sets;
(ii) intersections with $\boldsymbol{\Sigma}^0_\alpha$ sets if ξ is even;
(iii) unions with $\boldsymbol{\Pi}^0_\alpha$ sets if ξ is odd;
(iv) unions with $\boldsymbol{\Sigma}^0_\alpha$ sets if $\xi \geq \omega$.

Each of these implies a dual property for the $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_\alpha)$ sets. For example, (i) says that the $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_\alpha)$ sets are closed under unions with $\boldsymbol{\Sigma}^0_\alpha$ sets. By combining the above properties with the fact that if A is $\boldsymbol{\Pi}^0_\alpha$, and B is $\boldsymbol{\Pi}^0_\beta$ ($\beta < \alpha$), then both $A - B$ and $A \cup B$ are $\boldsymbol{\Pi}^0_\alpha$, we also have (for $\alpha > \beta$, or $\alpha = \beta$ and $\xi \geq \omega$):

(v) the $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_\alpha)$ sets are closed under intersections with $\boldsymbol{\Pi}^0_\beta$ sets.

We will need this for $\beta = 3$ later.

In order to determine the exact location of a set in the above hierarchy, one must produce an upper bound, or prove membership in the class Γ, and then a lower bound, showing the set is not in $\widetilde{\Gamma}$. In general the lower bounds are more difficult, but since these classes are closed under continuous preimages, the notion of a continuous or Wadge reduction yields a powerful technique for producing lower bounds. The idea is to take a set C that is known to be a non-$\widetilde{\Gamma}$ set, and find a continuous function f such that $f^{\leftarrow}(A) = C$. Then A cannot be in $\widetilde{\Gamma}$ either. The Cantor space $2^{\mathbb{N}}$ (with the usual product topology and $2 = \{0,1\}$ discrete) is known to contain sets that are proper, in all the classes above. A subset, A, of a Polish topological space, X, is called *Γ-hard* (for $\Gamma = \mathcal{D}_\xi(\boldsymbol{\Pi}^0_\alpha)$ or $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_\alpha)$) if for every $C \in \Gamma(2^{\mathbb{N}})$ there is a continuous function, $f : 2^{\mathbb{N}} \to X$, such that $x \in C \Leftrightarrow f(x) \in A$, that is, $f^{\leftarrow}(A) = C$. Thus if A is Γ-hard, then $A \notin \widetilde{\Gamma}$. If in addition to being Γ-hard, A is also in Γ, we say A is *Γ-complete*. Wadge [7] (using Borel determinacy [3]) showed that in zero-dimensional Polish spaces, there is no difference between a set being Γ-complete or properly Γ. Let X and Y be Polish spaces, $C \subseteq X$, A and B disjoint subsets of Y, let $C \leq_{\mathbf{W}} (A; B)$ assert that there is a continuous function $f : X \to Y$ where

$$x \in C \Rightarrow f(x) \in A, \quad \text{and} \quad x \notin C \Rightarrow f(x) \in B\,.$$

If $B = \neg A = Y - A$, we write $C \leq_{\mathbf{W}} A$ for $C \leq_{\mathbf{W}} (A; \neg A)$, and say C is *Wadge reducible* to A. Wadge's result mentioned above was that for all Borel subsets A and B of zero-dimensional Polish spaces, either $A \leq_{\mathbf{W}} B$ or $\neg B \leq_{\mathbf{W}} A$. Louveau and Saint-Raymond [2] later showed a similar result (which Wadge obtained using analytic determinacy) for $C \leq_{\mathbf{W}} (A; B)$, using closed games. It implies that for each class $\Gamma = \mathcal{D}_\xi(\boldsymbol{\Pi}^0_\alpha)$ or $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_\alpha)$ ($\alpha \geq 2$), there is a Γ-complete set $H_\Gamma \subseteq 2^{\mathbb{N}}$ such that for all disjoint analytic A and B (in any Polish space), either $H_\Gamma \leq_{\mathbf{W}} (A; B)$ (by a one-to-one continuous function), or there is a $\widetilde{\Gamma}$ set S such that $A \subseteq S$ and $B \cap S = \emptyset$. For our classes (since we only work in Polish spaces), being Γ-hard and being a non-$\widetilde{\Gamma}$ set are the same thing. Hence a set will be properly Γ iff it is Γ-complete. Notice also that if C is Γ-hard, and $C \leq_{\mathbf{W}} B$, then B is Γ-hard. And if $C \leq_{\mathbf{W}} (A; B)$, and D is any set containing A and disjoint from B, then $C \leq_{\mathbf{W}} D$.

Subsets of $\mathbb{N}$ with given densities. We describe here the basic facts and properties about the densities of subsets of the natural numbers, that we need. This topic is covered in detail in [1], for example.

Definition 1. For $A \subseteq \mathbb{N}$, let $\delta(A) = \lim_{n\to\infty} |A \cap [0,n)|/n$ if the limit exists, and say $\delta(A)$ does not exist otherwise. We call $\delta(A)$ the *density* of A.

Thus, whenever it exists, $\delta(A) \in \mathbb{R} \cap [0,1]$, and is roughly the frequency of occurrences of A in $\mathbb{N}$. For nonempty $X \subseteq [0,1] \cap \mathbb{R}$, let

$$D_X = \{A \subseteq \mathbb{N} \mid \delta(A) \in X\}$$

(if $X = \{r\}$ we write D_r for $D_{\{r\}}$). Let $DE = D_{[0,1]}$ denote the collection of subsets of $\mathbb{N}$ whose densities exist. If we identify $A \subseteq \mathbb{N}$ with its characteristic function $\chi_A : \mathbb{N} \to 2$, given by

$$\chi_A(n) = \begin{cases} 1 & \text{if } n \in A; \\ 0 & \text{if } n \notin A, \end{cases}$$

then D_X becomes a subset of the Cantor space $2^{\mathbb{N}}$ (with the usual product topology). For $s \in 2^{<\mathbb{N}}$, let $\|s\| = |\{i \in \mathrm{Dom}(s) : s(i) = 1\}|$ and let $|s|$ denote the length of s. We can then define the density of s as

$$\delta(s) = \frac{\|s\|}{|s|} \in \mathbb{Q} \cap [0,1]\,.$$

For $\alpha \in 2^{\mathbb{N}}$, the density of α exists iff the sequence $\{\delta(\alpha\restriction_n)\}_{n\in\mathbb{N}}$ converges, in which case the limit of the sequence is the density of α. This shows that DE and D_0 are $\boldsymbol{\Pi}^0_3$, since

$$\begin{aligned} \alpha \in DE &\Leftrightarrow \forall n\ \exists N\ \forall k\ (|\delta(\alpha\restriction_N) - \delta(\alpha\restriction_{N+k})| < 1/n) \\ &\qquad \text{(that is, the sequence of partial densities of } \alpha \text{ is Cauchy)} \\ &\Leftrightarrow \alpha \in \bigcap_{n\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} C(n,N,k)\,, \end{aligned}$$

where $C(n,N,k)$ is the collection of $\alpha \in 2^{\mathbb{N}}$ such that $|\delta(\alpha\restriction_N) - \delta(\alpha\restriction_{N+k})| < 1/n$, which is clopen (both closed and open). In the future we will not bother rewriting number quantifiers as countable intersections or countable unions, nor will we verify that the sets similar to $C(n,N,k)$ above are clopen, if it is clear that they are. D_0 is also $\boldsymbol{\Pi}^0_3$ since

$$\alpha \in D_0 \Leftrightarrow \forall n\ \exists N\ \forall k \geq N\ (\delta(\alpha\restriction_k) < 1/n)\,.$$

The sequence $\{\delta(\alpha\restriction_n)\}_{n\in\mathbb{N}}$ is very close to being a Cauchy sequence, meaning that if n is large, then $\delta(\alpha\restriction_n)$ and $\delta(\alpha\restriction_{n+1})$ are very close. In fact,

$$|\delta(\alpha\restriction_n) - \delta(\alpha\restriction_{n+1})| < \frac{1}{n+1}\,. \tag{2}$$

This shows that if $I = \liminf_{n\to\infty}\{\delta(\alpha\restriction_n)\}$, and $S = \limsup_{n\to\infty}\{\delta(\alpha\restriction_n)\}$, then for every real number $r \in [I,S]$, r is a limit point or cluster value of the sequence $\{\delta(\alpha\restriction_n)\}_{n\in\mathbb{N}}$.

Two methods for producing subsets with nice densities. We now give two methods for producing $\alpha \in 2^{\mathbb{N}}$ so that the density of α is easy to compute. The first involves copying the values of a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in (0,1)$. The idea is to define α as a union, $\alpha = \bigcup_{n\in\mathbb{N}} \alpha_n$, where for all $n \in \mathbb{N}$, α_{n+1} is a finite proper extension of $\alpha_n \in 2^{<\mathbb{N}}$, $\delta(\alpha_n) \approx x_n$, and $\delta(\alpha\restriction_{k+1})$ is between $\delta(\alpha_n)$ and $\delta(\alpha_{n+1})$ whenever k is between $|\alpha_n|$ and $|\alpha_{n+1}|$. Thus the density of α will exist iff the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges, and the limit of this sequence will be the density of α. Given any sequence $\{x_n\}_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ we define α, the *result of running the canonical construction with input* $\{x_n\}_{n\in\mathbb{N}}$, inductively as follows:

Let $\alpha_0 = \langle 0,1\rangle$. Given α_n, if $\delta(\alpha_n) < x_{n+1}$, fix the least $k\in\mathbb{N}$ such that

$$\delta(\alpha_n{}^\frown 1^k) = \frac{\|\alpha_n\|+k}{|\alpha_n|+k} \geq x_{n+1}$$

(k exists since $\{\delta(\alpha_n{}^\frown 1^k)\}_{k\in\mathbb{N}}$ starts at $\delta(\alpha_n)$ and increases to 1). Set $\alpha_{n+1} = \alpha_n{}^\frown 1^k$. If $\delta(\alpha_n) \geq x_{n+1}$, fix the least $k\in\mathbb{N}-\{0\}$ such that

$$\delta(\alpha_n{}^\frown 0^k) = \frac{\|\alpha_n\|}{|\alpha_n|+k} \leq x_{n+1}\,,$$

and set $\alpha_{n+1} = \alpha_n{}^\frown 0^k$. Let $\alpha = \bigcup_{n\in\mathbb{N}} \alpha_n \in 2^{\mathbb{N}}$. Clearly, $|\alpha_{n+1}| \geq |\alpha_n|+1 > n+1$, for all $n\in\mathbb{N}$. Using the minimality of k and (2), we see that

$$|x_{n+1} - \delta(\alpha_{n+1})| < \frac{1}{|\alpha_n|} < 1/n\,.$$

Since α_{n+1} is α_n followed by k zeros or k ones, it is clear that $\delta(\alpha\restriction_{m+1})$ is between $\delta(\alpha_n)$ and $\delta(\alpha_{n+1})$ whenever m is between $|\alpha_n|$ and $|\alpha_{n+1}|$. So the density of α exists iff the sequence of partial densities of α is Cauchy iff the sequence of the densities of the α_n's is Cauchy iff $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. More precisely, for any convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$,

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} \delta(\alpha_{n_k})\,.$$

This gives then a canonical way to produce α with $\delta(\alpha) = r$, for any $r\in[0,1]$. Notice also that if x_n happened to be zero or one, we could replace x_n with $1/n$ or $1-1/n$ respectively, and hence we can run this construction for sequences in $[0,1]^{\mathbb{N}}$.

The second construction involves partitioning $\mathbb{N}$ into a finite or countably infinite collection of sets with positive densities, and placing a copy of some $\alpha_n \in 2^{\mathbb{N}}$ on the nth set in the partition. Then even when the partition is infinite, one can basically add the densities. In general this is not true, since the union of the singletons has density one, whereas each singleton has density zero. But when the pieces being combined are contained in disjoint sets with positive densities, everything works out fine. Let $I\subseteq\mathbb{N}$, and $\{A_n\}_{n\in I}$ be a family of pairwise disjoint subsets of $\mathbb{N}$ such that $\bigcup_{n\in I} A_n = \mathbb{N}$ (i.e. a partition of $\mathbb{N}$); for each $n\in I$, the density of A_n exists and is positive; and $\lim_{N\to\infty}\sum_{n=0}^{N}\delta(A_n) = 1$. For each $n\in I$, let $\alpha_n\in 2^{\mathbb{N}}$ be such that $\delta(\alpha_n)$ exists. Define $C\subseteq\mathbb{N}$, the *set obtained by playing a copy of* α_n *on* A_n, as follows:

First, since $\delta(A_n) > 0$, A_n is infinite. Let $\{a_k^n\}_{k\in\mathbb{N}}$ be a one-to-one increasing enumeration of A_n. Then for each $m\in\mathbb{N}$ there are unique n and k such that $m = a_k^n$. We put $m\in C$ iff $m = a_k^n$ and $\alpha_n(k) = 1$. It is straightforward to check that

$$\delta(C) = \sum_{n\in I} \delta(A_n)\delta(\alpha_n)\,.$$

Thus, whenever we say "let α be the result of playing α_n on A_n", we mean that α is the characteristic function of the set C defined above. Of course we can still make this definition even if $\delta(\alpha_n)$ does not exist. If at least two of the α_n's have divergent densities, the density of C may or may not exist. However, if exactly one of α_n's has a divergent density, then the density of C does not exist.

Some $\boldsymbol{\Pi}^0_3$-complete sets. In this section we establish a strong reduction of a $\boldsymbol{\Pi}^0_3$-complete set to the set D_0. Thus we have an affirmative answer to a question of Kechris, who asked if D_0 was $\boldsymbol{\Pi}^0_3$-complete. Once this is done, we are able to show hardness for numerous other sets, including the collections of normal and simply normal numbers. It is known that the set

$$\mathcal{C}_3 = \{\beta \in \mathbb{N}^{\mathbb{N}} \mid \forall n,\ \beta^{\leftarrow}(n) \text{ is finite}\} = \{\beta \in \mathbb{N}^{\mathbb{N}} \mid \liminf_{n\to\infty} \beta(n) = \infty\}$$

is $\boldsymbol{\Pi}^0_3$-complete (see for example (24) in [4] for a proof).

THEOREM 3. $\mathcal{C}_3 \le_{\mathbf{W}} (D_0; \neg DE)$. *In particular*, *both D_0 and DE are $\boldsymbol{\Pi}^0_3$-complete.*

Proof. The second part of the theorem follows from the first, because $D_0 \subseteq DE$ and $DE \cap \neg DE = \emptyset$. The idea is to associate with $\beta \in \mathbb{N}^{\mathbb{N}}$ a sequence $\{x_n\}_{n\in\mathbb{N}}$ so that x_n depends only on a finite initial segment of β. We then produce the canonical α with input $\{x_n\}_{n\in\mathbb{N}}$. The function $\beta \mapsto \alpha$ (from $\mathbb{N}^{\mathbb{N}}$ into $2^{\mathbb{N}}$) will then be continuous since the first N values of α depend only on the first N values of $\{x_n\}_{n\in\mathbb{N}}$, which depend only on a finite initial segment of β. The sequence $x_n = 1/\beta(n)$ almost works, but we must first fix β so that $\beta(n) \ge 2$, and β is not eventually constant (so that $\lim_{n\to\infty} 1/\beta(n)$ exists iff it is zero). For $\beta \in \mathbb{N}^{\mathbb{N}}$, define $\beta' \in \mathbb{N}^{\mathbb{N}}$ by

$$\beta'(n) = \begin{cases} \beta(n/2) + 2 & \text{if } n \text{ is even;} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Then $\beta \mapsto \beta'$ is continuous and $\beta \in \mathcal{C}_3 \Leftrightarrow \beta' \in \mathcal{C}_3$. Given $\beta \in \mathbb{N}^{\mathbb{N}}$, let $\alpha \in 2^{\mathbb{N}}$ be the result of running the canonical construction on input $\{1/\beta'(n)\}_{n\in\mathbb{N}}$. Then the sequence $\{\delta(\alpha{\restriction}_n)\}_{n\in\mathbb{N}}$ always contains a subsequence which converges to zero, since $\beta'(2n+1) = 2n+2$. Hence the density of α exists iff it is zero. Thus,

$$\beta \in \mathcal{C}_3 \Leftrightarrow \beta' \in \mathcal{C}_3 \Leftrightarrow \lim_{n\to\infty} \frac{1}{\beta'(n)} = 0 \Leftrightarrow \delta(\alpha) = 0 \Leftrightarrow \alpha \in D_0 \Leftrightarrow \alpha \in DE\,.$$

This shows $\mathcal{C}_3 \le_{\mathbf{W}} (D_0; \neg DE)$ and completes the proof. ∎

Corollary 4. *For any nonempty $X \subseteq [0,1]$, D_X is $\boldsymbol{\Pi}^0_3$-hard. In particular, for each $r \in [0,1]$, D_r is $\boldsymbol{\Pi}^0_3$-complete.*

Proof. It is clear that D_r is $\boldsymbol{\Pi}^0_3$ for each $r \in [0,1]$, so it suffices to prove the first statement. Let f denote the continuous function from Theorem 3. If $0 \in X$, then f shows $\mathcal{C}_3 \leq_{\mathbf{W}} D_X$. If $1 \in X$ let $g(\beta) = \phi(f(\beta))$, where ϕ is the bit switching homeomorphism of $2^{\mathbb{N}}$,

$$\phi(\alpha)(n) = \begin{cases} 0 & \text{if } \alpha(n) = 1\,; \\ 1 & \text{if } \alpha(n) = 0\,. \end{cases}$$

Then g shows $\mathcal{C}_3 \leq_{\mathbf{W}} D_X$. Finally, if $X \subseteq (0,1)$, let $x \in X$ be arbitrary. Let $A_0 \subseteq \mathbb{N}$ have density x. Fix $n \in \mathbb{N}$ such that $x + 1/n < 1$. Let A_1 be disjoint from A_0 and have density $1/n$. Given $\beta \in \mathbb{N}^{\mathbb{N}}$, let α be the characteristic function of $C = A_0 \cup C_1$, where C_1 is the result of playing $f(\beta)$ on A_1. Then $\beta \mapsto \alpha$ is continuous, $\delta(C)$ exists iff $\delta(C_1)$ exists, and

$$\beta \in \mathcal{C}_3 \Leftrightarrow \delta(C_1) \text{ exists} \Leftrightarrow \delta(C_1) = 0 \Leftrightarrow \delta(\alpha) = x + \frac{1}{n} \cdot 0 \Leftrightarrow \alpha \in D_x\,.$$

Hence $\mathcal{C}_3 \leq_{\mathbf{W}} (D_x; \neg DE)$, so $\mathcal{C}_3 \leq_{\mathbf{W}} D_X$, because $x \in X$ and $D_X \cap \neg DE = \emptyset$. ∎

Normal numbers. For $x \in [0,1]$ and $n \geq 2$, the base n expansion of x is the sequence $\{d_i\}_{i\in\mathbb{N}} \in n^{\mathbb{N}}$ such that $x = \sum_{i=1}^{\infty} d_i/n^i$, and $d_i \neq n-1$ for infinitely many i. For $x \in [0,1]$ and $n \geq 2$, say x is *simply normal to base* n, and write $x \in SN_n$, if for each $k = 0, 1, \ldots, n-1$,

$$\delta(\{i \in \mathbb{N} \mid d_i = k\}) = 1/n\,.$$

Say $x \in [0,1]$ is *normal to base* n, and write $x \in N_n$, if for each $m \in \mathbb{N}$ and each $s \in n^{m+1}$,

$$\delta(\{i \in \mathbb{N} \mid d_i = s(0), d_{i+1} = s(1), \ldots, d_{i+m} = s(m)\}) = 1/n^{m+1}\,.$$

Thus, x is normal to base n if in the base n expansion of x, all the digits $k < n$ appear with equal frequency, all the pairs $\langle k, j\rangle$ appear with equal frequency, etc. It is known that the set of numbers in $[0,1]$ that are normal to all bases $n \geq 2$ simultaneously has Lebesgue measure one (see for example 8.11 in [5]). It is straightforward to see that SN_n and N_n are $\boldsymbol{\Pi}^0_3$, since each $D_{1/n}$ is $\boldsymbol{\Pi}^0_3$. One of the main questions that motivated this study was to try to show that $N_{\mathbb{N}} = \bigcup_{n\in\mathbb{N}} N_n$ was $\boldsymbol{\Sigma}^0_4$-complete. We were unable to answer this, but did manage to show that N_n and SN_n are each $\boldsymbol{\Pi}^0_3$-complete. As $N_n \subseteq SN_n$ the following result shows both of these simultaneously.

Theorem 5. *For each $n \in \mathbb{N} - \{0,1\}$, $D_0 \leq_{\mathbf{W}} (N_n; \neg SN_n)$. In particular, both SN_n and N_n are $\boldsymbol{\Pi}^0_3$-complete.*

Proof. Let $x = \sum_{i=1}^{\infty} d_i/n^i$ be any fixed number that is normal to base n. Let $\{i_k\}_{k\in\mathbb{N}}$ be an increasing enumeration of the set $I_0 = \{i \in \mathbb{N} \mid$

$d_i = 0\}$. Then I_0 has density $1/n$ since $x \in N_n$. Given $\alpha \in 2^{\mathbb{N}}$ let $x' \in [0,1]$ be given by the base n expansion,

$$d'_i = \begin{cases} 1 & \text{if } i = i_k \in I_0 \text{ and } \alpha(k) = 1\,; \\ d_i & \text{otherwise}\,. \end{cases}$$

That is,

$$x' = x + \sum_{k \in \alpha^{\leftarrow}(1)} \frac{1}{n^{i_k}}\,.$$

The function $\alpha \mapsto x'$ is continuous. If $\alpha \in D_0$, then x' is the result of changing a subset of density zero of the 0's in the base n expansion of x to ones, leaving the rest of the base n expansion of x unchanged. Hence, x' is still normal to base n. And if $\alpha \notin D_0$, then $x' \notin SN_n$, since 0 and 1 no longer occur with density $1/n$ in the base n expansion of x'. ∎

The Borel classes of D_X. We now turn to the problem of classifying the Borel class of D_X in terms of the class of X. The fact that such an exact relationship exists is surprising. Basically D_X has one more quantifier, and lies on the same side of the hierarchy as X. We start with the upper bounds.

PROPOSITION 6. *For nonempty $X \subseteq [0,1]$, if X is $\boldsymbol{\Pi}^0_2$, then $D_X \subseteq 2^{\mathbb{N}}$ is $\boldsymbol{\Pi}^0_3$-complete.*

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable basis of open sets for $\mathbb{R} \cap [0,1]$ (with the usual topology). Let $X = \bigcap_{k\in\mathbb{N}} G_k$ be any nonempty $\boldsymbol{\Pi}^0_2$ subset of $[0,1]$, where each G_k is open. A moment's reflection shows that

$$\alpha \in D_X \Leftrightarrow \alpha \in DE \text{ and } \forall k\ \exists n\ \exists m\ \forall p \ge m\ (\delta(\alpha{\restriction}_p) \in U_n \subseteq \overline{U}_n \subseteq G_k)\,.$$

Since membership in $N(k,n,p) = \{\alpha \in 2^{\mathbb{N}} \mid \delta(\alpha{\restriction}_p) \in U_n \subseteq \overline{U}_n \subseteq G_k\}$ is completely determined by $\alpha{\restriction}_p$ (and whether or not $\overline{U}_n \subseteq G_k$, which is independent of $\alpha{\restriction}_p$), $N(k,n,p)$ is clopen for each $(k,n,p) \in \mathbb{N}^3$. Thus D_X is $\boldsymbol{\Pi}^0_3$, and by Corollary 4, D_X is $\boldsymbol{\Pi}^0_3$-complete. Furthermore, if we denote by $P(X)$ the set of $\alpha \in 2^{\mathbb{N}}$ such that

$$\forall k\ \exists n\ \exists m\ \forall p \ge m\ (\delta(\alpha{\restriction}_p) \in U_n \subseteq \overline{U}_n \subseteq G_k)\,,$$

then $P(X)$ is $\boldsymbol{\Pi}^0_3$ and $\alpha \in D_X \Leftrightarrow \alpha \in DE \cap P(X)$. ∎

COROLLARY 7. *Let $X \subseteq [0,1]$ be nonempty.*

(i) *If X is $\boldsymbol{\Sigma}^0_2$, then D_X is $\mathcal{D}_2(\boldsymbol{\Pi}^0_3)$.*
(ii) *If X is $\boldsymbol{\Pi}^0_\alpha$ ($\boldsymbol{\Sigma}^0_\alpha$) for $\alpha \ge 3$, then D_X is $\boldsymbol{\Pi}^0_{1+\alpha}$ ($\boldsymbol{\Sigma}^0_{1+\alpha}$).*
(iii) *If X is $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_\alpha)$ for α and $\xi \ge 2$, then D_X is $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_{1+\alpha})$.*
(iv) *If X is $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_\alpha)$ for $\alpha \ge 3$, or $\alpha = 2$ and $\xi \ge \omega$, then D_X is $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_{1+\alpha})$.*
(v) *If X is $\widetilde{\mathcal{D}}_m(\boldsymbol{\Pi}^0_2)$ for $m < \omega$, then D_X is $\mathcal{D}_{m+1}(\boldsymbol{\Pi}^0_3)$.*

Proof. If X is $\boldsymbol{\Sigma}^0_2$, then $\neg X$ is $\boldsymbol{\Pi}^0_2$ and $D_X = DE - D_{\neg X} \in \mathcal{D}_2(\boldsymbol{\Pi}^0_3)$ by Proposition 6, so (i) holds. More precisely, for each $X \in \boldsymbol{\Sigma}^0_2([0,1])$, there is a $\boldsymbol{\Sigma}^0_3$ set $P'(X)$ (namely $\neg P(\neg X)$ from Proposition 6) such that $\alpha \in D_X \Leftrightarrow \alpha \in DE \cap P'(X)$. Clearly, for $W = X - X'$, $Y = \bigcup_{n\in\mathbb{N}} X_n$ and $Z = \bigcap_{n\in\mathbb{N}} X_n$,

$$D_W = D_X - D_{X'}, \quad D_Y = \bigcup_{n\in\mathbb{N}} D_{X_n}, \quad \text{and} \quad D_Z = \bigcap_{n\in\mathbb{N}} D_{X_n}\,. \tag{8}$$

An easy induction then shows, for $n \geq 2$, that for each $\boldsymbol{\Pi}^0_n$ ($\boldsymbol{\Sigma}^0_n$) set $X \subseteq [0,1]$, there is a $\boldsymbol{\Pi}^0_{n+1}$ ($\boldsymbol{\Sigma}^0_{n+1}$) set $P(X) \subseteq 2^{\mathbb{N}}$ such that

$$\alpha \in D_X \Leftrightarrow \alpha \in DE \cap P(X)\,. \tag{9}$$

This then gives (ii) for $\alpha < \omega$, since the classes $\boldsymbol{\Pi}^0_k$ and $\boldsymbol{\Sigma}^0_k$, for $k \geq 4$, are closed under intersections with $\boldsymbol{\Pi}^0_3$ sets. The function $f : 2^{\mathbb{N}} \to \mathbb{R}$ given by

$$f(\alpha) = \begin{cases} \delta(\alpha) & \text{if } \delta(\alpha) \text{ exists}\,; \\ 2 & \text{otherwise}\,, \end{cases}$$

is a Baire class 4 function by Proposition 6. As $f^{\leftarrow}(X) = D_X$, for all $X \subseteq [0,1]$, if X is $\boldsymbol{\Pi}^0_\alpha$ or $\boldsymbol{\Sigma}^0_\alpha$, then D_X is $\boldsymbol{\Pi}^0_{4+\alpha}$ or $\boldsymbol{\Sigma}^0_{4+\alpha}$. Thus if $\alpha \geq \omega$, then $4+\alpha = 1+\alpha = \alpha$, so (ii) holds (also the levels of the projective hierarchy do not increase from X to D_X). Using (8) and (9), one shows that for each $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_\alpha)$ set $X \subseteq [0,1]$ (with α and $\xi \geq 2$), there is a $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_{1+\alpha})$ set $P(X) \subseteq 2^{\mathbb{N}}$ such that

$$D_X = DE \cap P(X)\,.$$

Thus (iii) follows, since $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_{1+\alpha})$ sets are closed under intersections with $\boldsymbol{\Pi}^0_3$ sets (as long as $\alpha \geq 2$). If X is $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_\alpha)$ for α and $\xi \geq 2$, then $D_X = DE \cap \neg D_{\neg X}$, which is the intersection of a $\boldsymbol{\Pi}^0_3$ set with a $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_{1+\alpha})$ set. For $\alpha \geq 3$, or $\alpha = 2$ and $\xi \geq \omega$, the class $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_{1+\alpha})$ is closed under intersections with $\boldsymbol{\Pi}^0_3$ sets, and hence (iv) follows. Finally, if X is $\widetilde{\mathcal{D}}_m(\boldsymbol{\Pi}^0_2)$, then $D_X = DE \cap \neg D_{\neg X}$, which by (iii) and the definition is a $\mathcal{D}_{m+1}(\boldsymbol{\Pi}^0_3)$ set, so (v) holds. ∎

The upper bound of $\mathcal{D}_2(\boldsymbol{\Pi}^0_3)$ for $X = \mathbb{Q}$, the rationals, turns out to be a lower bound also. This is rather surprising, since very few sets are known to be properly located above the third level of the Borel hierarchy. We now show that $D_{\mathbb{Q}}$ is $\boldsymbol{\Sigma}^0_3$-hard. It turns out that no D_X is $\boldsymbol{\Sigma}^0_3$-complete (except of course for $X = \emptyset$ in which case one might say $D_X = \neg DE$, which is $\boldsymbol{\Sigma}^0_3$-complete by Theorem 3), and our proof of this fact will be the second half of the proof that $D_{\mathbb{Q}}$ is $\mathcal{D}_2(\boldsymbol{\Pi}^0_3)$-complete. Our reduction here uses a nonstandard $\boldsymbol{\Sigma}^0_3$-complete set, namely,

$$S_3 = \{\alpha \in 2^{\mathbb{N}\times\mathbb{N}} \mid \exists R\ \forall r \geq R\ \exists c\ (\alpha(r,c) = 1)\}\,.$$

If one views α as an $\mathbb{N}\times\mathbb{N}$ matrix of zeros and ones whose entry in row r and column c is $\alpha(r,c)$, then S_3 is the set of matrices where all but finitely many rows contain a one, or equivalently with finitely many "all zero" rows. To prove that $\neg\mathcal{C}_3 \le_{\mathbf{W}} S_3$, $\beta\in\mathbb{N}^{\mathbb{N}}\mapsto\alpha\in 2^{\mathbb{N}\times\mathbb{N}}$, one attempts to define $\alpha\restriction_{n\times n}$ so that it contains $\beta(n)$ partial "all zero" rows. With a little organization, this makes the number of rows in α without any ones equal to the lim inf of β, so $\beta\notin\mathcal{C}_3 \Leftrightarrow \liminf_{n\to\infty}\beta(n)$ is finite $\Leftrightarrow \alpha\in S_3$. We define inductively $\alpha\restriction_{n\times n}$ from $\beta\restriction_n$ (so that $\beta\mapsto\alpha$ is continuous, and at stage n we must define α's entries in column n for the rows $0,1,2,\dots,n-1$, as well as row n, columns $0,1,\dots,n$), as follows:

Stage 0: If $\beta(0)=0$, set $\alpha(0,0)=1$ and if $\beta(0)>0$, set $\alpha(0,0)=0$. Thus Z_1, the number of "all zero" rows in $\alpha\restriction_{1\times 1}$, is either $0=\beta(0)$ or $1\le\beta(0)$, and hence $Z_1\le\beta(0)$.

Stage n: We are given $\alpha\restriction_{n\times n}$ and $\beta(n)$, and must define the first $n+1$ entries, in both column n and row n, of α. Let Z_n be the number of partial rows in $\alpha\restriction_{n\times n}$ that are all zeros.

If $\beta(n)\le Z_n$, extend the "first" $\beta(n)$ "all zero" rows of $\alpha\restriction_{n\times n}$ by adding a zero in column n; all the remaining rows (with index less than n) receive a one in column n; and define the first $n+1$ entries in row n to be ones. Here, "first" is defined from the indices of the rows, so the first 5 rows refers to the 5 rows with lowest indices.

If $\beta(n)>Z_n$, extend every "all zero" row by adding a zero in column n; every row that already has a one gets a one in column n; and make row n begin with $n+1$ zeros. Hence Z_{n+1}, the number of "all zero" rows in $\alpha\restriction_{(n+1)\times(n+1)}$, is either $\beta(n)$ or $1+Z_n\le\beta(n)$, and again $Z_{n+1}\le\beta(n)$. More precisely, for $r<n$, let $Z_n(r)$ denote the number of rows in $\alpha\restriction_{n\times n}$, with index $r'<r$, that are all zeros. Then (for $r<n$) we set

$$\alpha(r,n)=\begin{cases}0 & \text{if } \forall c<n\ [\alpha(r,c)=0] \text{ and } Z_n(r)<\beta(n)\,;\\ 1 & \text{otherwise}\,.\end{cases}$$

And for $c\le n$ set

$$\alpha(n,c)=\begin{cases}0 & \text{if } \beta(n)>Z_n\,;\\ 1 & \text{otherwise}\,.\end{cases}$$

One sees that if for all $n\ge N$, $\beta(n)\ge k$, then $Z_m\ge k$ for all $m>N+k$, and the first k "all zero" rows in $\alpha\restriction_{(N+k+1)\times(N+k+1)}$ always receive a zero. Thus,

$$\liminf_{n\to\infty}\beta(n)\le \text{ the number of "all zero" rows in } \alpha\,.$$

The reverse inequality is trivial if $\liminf\beta{=}\infty$, so assume $\liminf\beta{=}k<\infty$. Then β takes the value k infinitely often. Let $n_1<\ldots<n_{k+1}$ be any collection of $k+1$ natural numbers. We show that for some $i=1$ to $k+1$, row n_i of α contains a one. Since $\beta(n)=k$ infinitely often, fix $n>n_{k+1}$

such that $\beta(n) = k$. Then $Z_{n+1} \le k$, so at least one of the rows with indices n_i gets a one at stage n. Thus $\liminf_{n\to\infty} \beta(n) =$ the number of "all zero" rows in α, which directly translates to

$$\beta \notin \mathcal{C}_3 \Leftrightarrow \alpha \in S_3\,,$$

and $\neg\mathcal{C}_3 \le_{\mathbf{W}} S_3$. So S_3 is $\boldsymbol{\Sigma}^0_3$-hard and it is straightforward to see that $S_3 \in \boldsymbol{\Sigma}^0_3$.

PROPOSITION 10. *$S_3 \le_{\mathbf{W}} (D_{\mathbb{Q}}; D_{\mathbb{P}})$, and hence $D_{\mathbb{Q}}$ is $\boldsymbol{\Sigma}^0_3$-hard.*

Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ with $\delta(A_n) = 1/2^{n+1}$ (for example one can take $\{2^n(2p+1)-1\}_{p\in\mathbb{N}}$ for A_n). Let $\{a^n_k\}_{k\in\mathbb{N}}$ be an increasing enumeration of A_n. Let $B_n \subseteq A_n$ be the set $\{a^n_{k(n!)} \mid k \in \mathbb{N}\}$, so that $\delta(B_n) = 1/(n!\,2^{n+1})$. Let $B^* = \mathbb{N} - \bigcup_{n\in\mathbb{N}} B_n$, so that $B^* \cup \{B_n\}_{n\in\mathbb{N}}$ is a partition of $\mathbb{N}$ suitable for our second canonical construction. Given $\alpha \in 2^{\mathbb{N}\times\mathbb{N}}$, let

$$\alpha^*_r(c) = \begin{cases} 1 & \text{if } \alpha(r,c') = 0 \text{ for all } c' \le c\,;\\ 0 & \text{otherwise}\,. \end{cases}$$

Then $\alpha \mapsto \{\alpha^*_r\}_{r\in\mathbb{N}}$ is continuous (from $2^{\mathbb{N}\times\mathbb{N}}$ to $(2^{\mathbb{N}})^{\mathbb{N}}$) and α^*_r is eventually zero (hence $\delta(\alpha^*_r) = 0$) iff row r of α has a one, and α^*_r is identically one (hence $\delta(\alpha^*_r) = 1$) iff row r of α is identically zero. Let $f(\alpha) \in 2^{\mathbb{N}}$ be the result of playing α^*_r on B_r, for $r \in \mathbb{N}$, and $\vec{0}$ on B^*. Then

$$\delta(f(\alpha)) = \sum_{r=0}^{\infty} \frac{\delta(\alpha^*_r)}{r!\,2^{r+1}}$$

(which always exists), and it is rational iff $\delta(\alpha^*_r)$ is nonzero finitely often (see for example 1.7 in [5] or a proof that e is irrational). Thus, $\alpha \in S_3 \Leftrightarrow$ all but finitely many rows of α contain a 1 $\Leftrightarrow$ for all but finitely many r, $\delta(\alpha^*_r) = 0$ $\Leftrightarrow$ $\delta(f(\alpha)) \in \mathbb{Q}$. Thus $S_3 \le_{\mathbf{W}} (D_{\mathbb{Q}}; D_{\mathbb{P}})$, and $D_{\mathbb{Q}}$ is $\boldsymbol{\Sigma}^0_3$-hard, provided f is continuous. Since $f(\alpha)\restriction_n$ is completely determined by $\alpha\restriction_{M\times n}$, where $M = \max\{m \in \mathbb{N} \mid A_m \cap [0,n) \ne \emptyset\}$, f is continuous and we are done. ∎

LEMMA 11. *For any set C, if $C \le_{\mathbf{W}} (D_X; D_{\neg X})$, then $\mathcal{C}_3 \times C \le_{\mathbf{W}} D_X$.*

Proof. Let f be continuous and witness $C \le_{\mathbf{W}} (D_X; D_{\neg X})$. Assume $C \subseteq Y$ (some topological space). Then $\delta(f(y))$ exists for all $y \in Y$. As in Theorem 3, we replace $\beta \in \mathbb{N}^{\mathbb{N}}$ with β', where $\beta'(2n) = \beta(n) + 2$, and $\beta'(2n+1) = 2n+2$. So that $\beta \mapsto \beta'$ is continuous and does not alter membership in $\mathcal{C}_3$. We show $\mathcal{C}_3 \times C \le_{\mathbf{W}} D_X$ by defining $\phi(\beta, y)$ to be the result of playing $f(y)$ (whose density always exists) on A_0, the evens, and α' on A_1, the odds, where α' comes from the canonical construction with input $\{x_n\}_{n\in\mathbb{N}}$, where

$$x_n = (1 - 1/\beta'(n))(\delta(f(y)\restriction_n) + 1/\beta'(n))\,.$$

This defines a continuous function, since f is continuous and $\alpha'\restriction_n$ depends only on $\beta\restriction_n$ and the neighborhood of y that determines $f(y)\restriction_n$ (which exists since f is continuous). If $\beta \in \mathcal{C}_3$, then

$$\lim_{n\to\infty} x_n = \delta(f(y)).$$

Hence, when $\beta \in \mathcal{C}_3$,

$$\delta(\phi(\beta, y)) = \tfrac{1}{2}\delta(f(y)) + \tfrac{1}{2}\delta(f(y)) = \delta(f(y)) \in X \Leftrightarrow y \in C.$$

When $\beta \notin \mathcal{C}_3$ the sequence $\{x_n\}_{n\in\mathbb{N}}$ diverges. So $\delta(\alpha')$ does not exist and the density of $\phi(\beta, y)$ does not exist. Thus $\phi(\beta, y) \in D_X \Leftrightarrow (\beta, y) \in \mathcal{C}_3 \times C$. ∎

We now construct a sequence of complete sets for the differences of $\boldsymbol{\Pi}^0_3$ sets. Let $m \geq 1$ be a finite integer. In the space $(\mathbb{N}^{\mathbb{N}})^m$, consider the sets $A_0, A_1, \ldots, A_{m-1}$, where

$$\begin{aligned}
&A_0 = \mathcal{C}_3 \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \ldots \times \mathbb{N}^{\mathbb{N}},\\
&A_1 = \mathcal{C}_3 \times \mathcal{C}_3 \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \ldots \times \mathbb{N}^{\mathbb{N}},\\
&\ \vdots\\
&A_i = (\mathcal{C}_3)^{i+1} \times (\mathbb{N}^{\mathbb{N}})^{m-i-1},\\
&\ \vdots\\
&A_{m-1} = \mathcal{C}_3 \times \mathcal{C}_3 \times \ldots \times \mathcal{C}_3.
\end{aligned}$$

Then each $A_i \in \boldsymbol{\Pi}^0_3$ and $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_{m-1}$. Let $D^3_m = \mathcal{D}_m(\langle A_i\rangle_{i<m}) \in \mathcal{D}_m(\boldsymbol{\Pi}^0_3)$ and $\widetilde{D}^3_m = \neg D^3_m \in \widetilde{\mathcal{D}}_m(\boldsymbol{\Pi}^0_3)$. Then

$$\begin{aligned}
D^3_m = \{\langle\beta_i\rangle_{i<m} \in (\mathbb{N}^{\mathbb{N}})^m \mid \beta_0 \in \mathcal{C}_3 \text{ and}&\\
\max\{i < m : \beta_0, \beta_1, \ldots, \beta_i \in \mathcal{C}_3\} &\text{ is even}\},\\
\widetilde{D}^3_m = \{\langle\beta_i\rangle_{i<m} \in (\mathbb{N}^{\mathbb{N}})^m \mid \beta_0 \notin \mathcal{C}_3 \text{ or}&\\
\max\{i < m : \beta_0, \beta_1, \ldots, \beta_i \in \mathcal{C}_3\} &\text{ is odd}\}.
\end{aligned}$$

We now show that for any $\mathcal{D}_m(\boldsymbol{\Pi}^0_3)$ set $B \subseteq 2^{\mathbb{N}}$, $B \leq_{\mathbf{W}} D^3_m$, so D^3_m is $\mathcal{D}_m(\boldsymbol{\Pi}^0_3)$-complete and $\widetilde{D}^3_m$ is $\widetilde{\mathcal{D}}_m(\boldsymbol{\Pi}^0_3)$-complete. Given such a B, fix $B_0 \supseteq B_1 \supseteq \ldots \supseteq B_{m-1}$, $\boldsymbol{\Pi}^0_3$ subsets of $2^{\mathbb{N}}$ with $B = \mathcal{D}_m(\langle B_i\rangle_{i<m})$. Since $B_i \in \boldsymbol{\Pi}^0_3$ and $\mathcal{C}_3$ is $\boldsymbol{\Pi}^0_3$-complete, there is a continuous function $f_i : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\alpha \in B_i \Leftrightarrow f_i(\alpha) \in \mathcal{C}_3$. Define $f : 2^{\mathbb{N}} \to (\mathbb{N}^{\mathbb{N}})^m$ by

$$f(\alpha) = (f_0(\alpha), f_1(\alpha), \ldots, f_{m-1}(\alpha)).$$

Since the B_i's are decreasing, it is straightforward to check that $\alpha \in B_i \Leftrightarrow f_i(\alpha) \in \mathcal{C}_3 \Leftrightarrow f(\alpha) \in A_i$, which shows $B \leq_{\mathbf{W}} D^3_m$. Notice that $\mathcal{C}_3 \times \widetilde{D}^3_m = D^3_{m+1}$.

THEOREM 12. *For $1 \le m < \omega$, if D_X is $\widetilde{\mathcal{D}}_m(\boldsymbol{\Pi}^0_3)$-hard, then D_X is $\mathcal{D}_{m+1}(\boldsymbol{\Pi}^0_3)$-hard. Thus no D_X is $\widetilde{\mathcal{D}}_m(\boldsymbol{\Pi}^0_3)$-complete, and $D_{\mathbb{Q}}$ is $\mathcal{D}_2(\boldsymbol{\Pi}^0_3)$-complete.*

Proof. By Lemma 11 and the note above, it suffices to show $\widetilde{D}^3_m \le_{\mathbf{W}} (D_X; D_{\neg X})$. If this were not the case, then by the result of Louveau and Saint-Raymond [2] mentioned earlier, there would be a $\mathcal{D}_m(\boldsymbol{\Pi}^0_3)$ set S such that $D_X \subseteq S$ and $S \cap D_{\neg X} = \emptyset$. But then $D_X = S \cap DE$, a $\mathcal{D}_m(\boldsymbol{\Pi}^0_3)$ set, which is contrary to D_X being $\widetilde{\mathcal{D}}_m(\boldsymbol{\Pi}^0_3)$-hard. ∎

We shall now basically show that $X \le_{\mathbf{W}} D_X$. Literally this cannot be true because X lives in a connected space and D_X lives in a zero-dimensional space. However, for large enough ξ and α, intersecting a $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_\alpha)$ set X with $\mathbb{P}$, the irrationals in $[0,1]$, does not change the Borel class. Since $\mathbb{P}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, if $X \subseteq \mathbb{P}$ one can show $X \le_{\mathbf{W}} D_X$. The following material is well known (see [5], pp. 51–67 for example). Given $\beta \in \mathbb{N}^{\mathbb{N}}$, let $\beta^*(n) = \beta(n) + 1$. For each $n \in \mathbb{N}$, let

$$r_n(\beta) = r_n = \cfrac{1}{\beta^*(0) + \cfrac{1}{\beta^*(1) + \cfrac{1}{\ddots + \cfrac{1}{\beta^*(n)}}}} \tag{13}$$

so $r_n \in \mathbb{Q} \cap [0,1]$. Then $\phi(\beta) = \lim_{n\to\infty} r_n$ exists, is irrational, and ϕ is a homeomorphism onto $\mathbb{P}$. In a way, the next two results, as well as Lemma 11, show that our canonical construction can absorb continuous functions. We shall see later that it can actually absorb some Baire class one functions too.

LEMMA 14. *For nonempty $X \subseteq \mathbb{N}^{\mathbb{N}}$, $X \le_{\mathbf{W}} (D_{\phi(X)}; D_{\phi(\neg X)})$.*

Proof. Given $\beta \in \mathbb{N}^{\mathbb{N}}$, let $f(\beta) \in 2^{\mathbb{N}}$ be the result of running the canonical construction on input $\{x_n\}_{n\in\mathbb{N}}$, where x_n is the $r_n(\beta)$ in (13) (which only depends on $\beta\restriction_{n+1}$). Then $\delta(\alpha) = \lim_{n\in\mathbb{N}} r_n = \phi(\beta)$. Hence, since f is continuous, for any $X \subseteq \mathbb{N}^{\mathbb{N}}$, f shows

$$X \le_{\mathbf{W}} (D_{\phi(X)}; D_{\phi(\neg X)}),$$

and we are done. ∎

THEOREM 15. *Let Γ be one of the classes $\boldsymbol{\Pi}^0_\alpha$ or $\boldsymbol{\Sigma}^0_\alpha$ for $\alpha \ge 3$; $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_\alpha)$ for $\alpha \ge 2$; $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_\alpha)$ for $\alpha \ge 3$; or $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_2)$ for $\xi \ge \omega$. If $X \subseteq [0,1]$ is Γ-hard, then so is D_X. In particular, if $\alpha \ge \omega$ and X is Γ-complete, then so is D_X.*

Proof. The last part follows from the first and Corollary 7, since in this case $1+\alpha = \alpha$. For each such Γ, let $\widetilde{\Gamma}$ denote the dual class $\{\neg X \mid X \in \Gamma\}$. Then Γ is closed under intersections with $\boldsymbol{\Pi}^0_2$ sets and $\widetilde{\Gamma}$ is closed under unions with $\boldsymbol{\Sigma}^0_2$ sets. Since $X \subseteq [0,1]$ is Γ-hard, X is not in $\widetilde{\Gamma}$. If $X \cap \mathbb{P} \in \widetilde{\Gamma}$, then

$$X = (X \cap \mathbb{P}) \cup (X \cap \mathbb{Q})$$

is also in $\widetilde{\Gamma}$, since $X \cap \mathbb{Q}$ is countable and thus $\boldsymbol{\Sigma}^0_2$. So $X \cap \mathbb{P} \notin \widetilde{\Gamma}$, and hence $X \cap \mathbb{P}$ is Γ-hard. As ϕ^{-1} is a homeomorphism, $\phi^{-1}(X \cap \mathbb{P})$ is Γ-hard. By Lemma 14,

$$\phi^{-1}(X \cap \mathbb{P}) \leq_{\mathbf{W}} (D_{X\cap\mathbb{P}}; D_{\mathbb{P}\cap\neg X}).$$

Thus $\phi^{-1}(X \cap \mathbb{P}) \leq_{\mathbf{W}} D_X$, and D_X is Γ-hard. ∎

Notice that if Γ is one of the projective hierarchy classes, then X is Γ-complete iff D_X is. We have already seen D_X is $\boldsymbol{\Pi}^0_3$-complete, for any nonempty $\boldsymbol{\Pi}^0_2$ subset X of $[0,1]$. We shall now show that for $\alpha \geq 3$, if $X \subseteq [0,1]$ is $\boldsymbol{\Pi}^0_\alpha$-complete ($\boldsymbol{\Sigma}^0_\alpha$-complete), then D_X is $\boldsymbol{\Pi}^0_{1+\alpha}$-complete ($\boldsymbol{\Sigma}^0_{1+\alpha}$-complete). We need the complete sets, $\{H_n \subseteq 2^{\mathbb{N}} \mid n \in \mathbb{N}\}$, from [2], and some basic properties of their function ϱ. For n and $m \in \mathbb{N}$, let

$$\langle n, m\rangle = \tfrac{1}{2}(n+m)(n+m+1) + m.$$

Thus $\langle\ ,\ \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is one-to-one and onto. Define $\varrho : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$\varrho(\alpha)(n) = 1 \Leftrightarrow \forall m\ (\alpha(\langle n, m\rangle) = 0).$$

Again, if we think of α as an $\mathbb{N} \times \mathbb{N}$ matrix of zeros and ones, with the entry in row n and column m being $\alpha(\langle n, m\rangle)$, then

$$\varrho(\alpha)(n) = \begin{cases} 0 & \text{if row } n \text{ of } \alpha \text{ contains a } 1\,; \\ 1 & \text{if row } n \text{ of } \alpha \text{ is identically } 0\,. \end{cases}$$

Thus, if α_n is the binary sequence where $\alpha_n(m) = \alpha(\langle n, m\rangle)$, then $\varrho(\alpha)(n) = \chi_{\{\vec{0}\}}(\alpha_n)$. One can extend ϱ to $2^{\leq\mathbb{N}}$, by defining for $s \in 2^k$, $\varrho(s) = s^* \in 2^{<\mathbb{N}}$, where

$$\mathrm{Dom}(s^*) = \{n \in \mathbb{N} \mid \langle n, 0\rangle < k\}$$

(so that $\mathrm{Dom}(s^*)$ is an initial segment of $\mathbb{N}$), and for $n \in \mathrm{Dom}(s^*)$,

$$s^*(n) = 1 \Leftrightarrow \text{ for all } \langle n, m\rangle \in \mathrm{Dom}(s),\ s(\langle n, m\rangle) = 0\,.$$

The properties of ϱ that we need are the following, all appear in [2].

(i) $\forall \alpha \in 2^{\mathbb{N}}\ \forall n \in \mathbb{N}\ \exists k\ \forall m \geq k\ (\varrho(\alpha\restriction_m)\restriction_n = \varrho(\alpha)\restriction_n)$.

(ii) Let $H_1 = \{\vec{0}\} \subseteq 2^{\mathbb{N}}$ and $H_{n+1} = \varrho^{\leftarrow}(H_n)$. Then H_n is $\boldsymbol{\Pi}^0_n$-complete.

Thus (i) says that for each $i \in \mathbb{N}$, the approximations $\alpha^*_n(i) = \varrho(\alpha\restriction_n)(i)$ are eventually equal to $\varrho(\alpha)(i)$.

Lemma 16. *For $H \subseteq 2^{\mathbb{N}}$ and $X \subseteq [0,1]$, if $H \leq_{\mathbf{W}} X$, then $\varrho^{\leftarrow}(H) \leq_{\mathbf{W}} (D_X; D_{\neg X})$. In particular, for $n \geq 2$, and $X \subseteq [0,1]$, if $H_n \leq_{\mathbf{W}} X$, then $H_{n+1} \leq_{\mathbf{W}} D_X$, and if $\neg H_n \leq_{\mathbf{W}} X$, then $\neg H_{n+1} \leq_{\mathbf{W}} D_X$.*

Proof. Let $g : 2^{\mathbb{N}} \to [0,1]$ be a continuous function witnessing $H \leq_{\mathbf{W}} X$. Given $\alpha \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, apply ϱ to $\alpha\restriction_n$, yielding say $\alpha_n^* \in 2^{<\mathbb{N}}$ (this is a finitary process even though ϱ is Baire class one). Let $\alpha_n = \alpha_n^* {}^\frown \vec{0} \in 2^{\mathbb{N}}$, and set $x_n = g(\alpha_n) \in [0,1]$. We let $f(\alpha)$ be the canonical construction on input $\{x_n\}_{n\in\mathbb{N}}$. As usual, x_n depends only on $\alpha\restriction_n$, so f is continuous. Since g is continuous, $x_n = g(\alpha_n)$, and $\{\alpha_n\}_{n\in\mathbb{N}}$ converges pointwise to $\varrho(\alpha)$, we see that $\lim_{n\to\infty} x_n = g(\varrho(\alpha))$ (and $\delta(f(\alpha))$ always exists). Hence,

$$\begin{aligned}\alpha \in \varrho^{\leftarrow}(H) &\Leftrightarrow \varrho(\alpha) \in H \\ &\Leftrightarrow g(\varrho(\alpha)) = \lim_{n\to\infty} x_n = \delta(f(\alpha)) \in X \Leftrightarrow f(\alpha) \in D_X .\end{aligned}$$

So $\varrho^{\leftarrow}(H) \leq_{\mathbf{W}} (D_X; D_{\neg X})$.

THEOREM 17. (i) *If $X \subseteq [0,1]$ is $\boldsymbol{\Pi}^0_\alpha$-complete ($\boldsymbol{\Sigma}^0_\alpha$-complete) for $\alpha \geq 3$, then D_X is $\boldsymbol{\Pi}^0_{1+\alpha}$-complete ($\boldsymbol{\Sigma}^0_{1+\alpha}$-complete).*

(ii) *If $X \subseteq [0,1]$ is $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_\alpha)$-complete for $\alpha \geq 2$, then D_X is $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_{1+\alpha})$-complete.*

(iii) *If $X \subseteq [0,1]$ is $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_\alpha)$-complete for $\alpha \geq 3$, or for $\alpha = 2$ and $\xi \geq \omega$, then D_X is $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_{1+\alpha})$-complete.*

(iv) *If $X \subseteq [0,1]$ is $\widetilde{\mathcal{D}}_m(\boldsymbol{\Pi}^0_2)$-complete for $m<\omega$, then D_X is $\mathcal{D}_{m+1}(\boldsymbol{\Pi}^0_3)$-complete.*

Likewise all these hold with "hard" replacing "complete" and all implications reverse for $\alpha \geq 3$.

Proof. The upper bounds for D_X are from Proposition 6 and Corollary 7. They show the reverse implications hold for $\alpha \geq 3$. If $\alpha \geq \omega$, Theorem 15 gives the above statements. Hence we need only work with $\alpha < \omega$, which we will denote by n. Now, (i) is just the second part of Lemma 16. For the remaining cases consider the sets

$$\begin{aligned} D^n_\xi &= \{\langle \alpha_\beta\rangle_{\beta<\xi} \in (2^{\mathbb{N}})^\xi \mid \\ &\qquad \alpha_0 \in H_n \text{ and the least } \beta \text{ such that } \alpha_\beta \notin H_n \text{ is odd}\}, \\ \widetilde{D}^n_\xi &= \neg D^n_\xi = \{\langle \alpha_\beta\rangle_{\beta<\xi} \in (2^{\mathbb{N}})^\xi \mid \\ &\qquad \text{the least } \beta \text{ such that } \alpha_\beta \notin H_n \text{ is even}\},\end{aligned}$$

where $(H_n)^\xi$ is included in D^n_ξ if ξ is odd, and included in $\widetilde{D}^n_\xi$ when ξ is even. Then D^n_ξ is $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_n)$-complete, and $\widetilde{D}^n_\xi$ is $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_n)$-complete. Furthermore, by applying ϱ coordinatewise to D^{n+1}_ξ, we obtain D^n_ξ. That is, $\vec{\alpha} = \langle \alpha_\beta\rangle_{\beta<\xi} \in D^{n+1}_\xi \Leftrightarrow \langle \varrho(\alpha_\beta)\rangle_{\beta<\xi} \in D^n_\xi$. Thus, we simply mimic the proof of Lemma 16. Let Γ be any of the classes $\mathcal{D}_\xi(\boldsymbol{\Pi}^0_n)$ or $\widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_n)$, mentioned in the hypothesis, where X is Γ-complete or Γ-hard. Let Γ^* be the class where the n in Γ is replaced by $n+1$. Let

$$D_\Gamma = \begin{cases} D^n_\xi & \text{if } \Gamma = \mathcal{D}_\xi(\boldsymbol{\Pi}^0_n)\,; \\ \widetilde{D}^n_\xi & \text{if } \Gamma = \widetilde{\mathcal{D}}_\xi(\boldsymbol{\Pi}^0_n)\,. \end{cases}$$

Then the assumptions give a continuous function g witnessing $D_\Gamma \le_{\mathbf{W}} X$. Since ξ is countable, $(2^{\mathbb{N}})^\xi$ is homeomorphic to $2^{\mathbb{N}}$ by some function $\phi : (2^{\mathbb{N}})^\xi \to 2^{\mathbb{N}}$. In fact, if we take $\langle\ ,\ \rangle : \xi \times \mathbb{N} \to \mathbb{N}$ to be any bijection such that for each $\beta < \xi$, the sequence $\langle \beta, n\rangle_{n\in\mathbb{N}}$ is increasing, then we can take $\phi(\vec{\alpha})(\langle\beta, n\rangle) = \alpha_\beta(n)$. If we then let $\vec{\alpha}\restriction_n = \{\alpha_\beta(k) \mid \langle \beta, k\rangle < n\}$, this will be a finite set containing an initial segment of each α_β. We can then apply ϱ to each initial segment, obtaining say $\langle \alpha^*_{\beta,n}\rangle_{\beta<\xi}$. Let $\vec{\alpha}^*_n$ be the extension of $\langle \alpha^*_{\beta,n}\rangle_{\beta<\xi}$ by setting all undefined values to zero. Then $\{\vec{\alpha}^*_n\}_{n\in\mathbb{N}}$ converges pointwise to $\langle \varrho(\alpha_\beta)\rangle_{\beta<\xi}$. Given $\vec{\alpha} \in (2^{\mathbb{N}})^\xi$, let $x_n = g(\vec{\alpha}^*_n) \in [0,1]$ (where $\vec{\alpha}^*_n$ is as above). Then x_n depends only on a finite piece of $\vec{\alpha}$. If we set $f(\vec{\alpha})$ to be the result of running the canonical construction on $\{x_n\}_{n\in\mathbb{N}}$, then f is continuous and as in Lemma 16, f witnesses $D_{\Gamma^*} \le_{\mathbf{W}} (D_X; D_{\neg X})$. Hence D_X is Γ^*-hard. Thus we are done, except for the last case where $\Gamma^* = \widetilde{\mathcal{D}}_m(\boldsymbol{\Pi}^0_3)$, which follows immediately by Theorem 12. ∎

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