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# Hyperspaces of CW-complexes

by

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**Abstract.** It is shown that the hyperspace of a connected CW-complex is an absolute retract for stratifiable spaces, where the hyperspace is the space of non-empty compact (connected) sets with the Vietoris topology.

**0. Introduction.** The class $\mathcal{S}$ of stratifiable spaces ($M_3$-spaces) contains both metrizable spaces and CW-complexes and has many desirable properties (cf. [Ce] and [$\mathrm{Bo}_1$]). Moreover, any CW-complex is an ANR($\mathcal{S}$) (i.e., an absolute neighborhood retract for the class $\mathcal{S}$) [$\mathrm{Ca}_4$]. In [$\mathrm{Ca}_5$], it was shown that the space of continuous maps from a compactum to a CW-complex with the compact-open topology is stratifiable, whence it is an ANR($\mathcal{S}$) (cf. [$\mathrm{Bo}_4$] or [$\mathrm{Ca}_2$]). It is interesting to find hyperspaces which are ANR($\mathcal{S}$)'s (cf. [Wo], [Ke] and [Ta]). By $\mathfrak{K}(X)$, we denote the space of non-empty compact sets in a space $X$ with the Vietoris topology, i.e., the topology generated by the sets

$$\langle U_1, \ldots, U_n\rangle = \{A \in \mathfrak{K}(X) \mid A \subset U_1 \cup \ldots \cup U_n,\ \forall i,\ A \cap U_i \neq \emptyset\},$$

where $n \in \mathbb{N}$ and $U_1, \ldots, U_n$ are open in $X$. Let $\mathfrak{C}(X)$ denote the subspace of $\mathfrak{K}(X)$ consisting of compact connected sets. In this paper, we show the following:

Main Theorem. *For any connected CW-complex $X$, the hyperspaces $\mathfrak{K}(X)$ and $\mathfrak{C}(X)$ are AR($\mathcal{S}$)'s. Hence for any CW-complex $X$, $\mathfrak{K}(X)$ and $\mathfrak{C}(X)$ are ANR($\mathcal{S}$)'s.*

One should note that $\mathfrak{K}(X)$ is not stratifiable even if $X$ is stratifiable (cf. [MK] and [Mi]). Although Mizokami [Mi] gave a sufficient condition on $X$ for $\mathfrak{K}(X)$ to be stratifiable, this condition is not satisfied for any non-locally compact CW-complex (see §3). For a simplicial complex $K$, let $|K|$ denote

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the polyhedron of $K$, i.e., $|K| = \bigcup K$ with the *weak* (*Whitehead*) *topology.* Since any connected CW-complex $X$ can be embedded in $|K|$ as a retract for some connected simplicial complex $K$ (cf. [$\mathrm{Ca}_1$, Corollaire 2]), $\mathfrak{K}(X)$ and $\mathfrak{C}(X)$ can be considered as retracts of $\mathfrak{K}(|K|)$ and $\mathfrak{C}(|K|)$, respectively. Thus the main theorem reduces to the case $X = |K|$ for a connected simplicial complex $K$. By the same reason, the main theorem is valid for a (connected) ANR($\mathcal{S}$) $X$ which can be embedded in a simplicial complex as a closed set. Throughout the paper, we simply write $\mathfrak{K}(|K|) = \mathfrak{K}(K)$ and $\mathfrak{C}(|K|) = \mathfrak{C}(K)$ for any simplicial complex $K$.

In the case where $X$ is a *separable* CW-complex, it is easy to see that $\mathfrak{K}(X)$ is an ANR($\mathcal{S}$). In fact, let $\Delta^\infty$ be the countable full simplicial complex. Since $|\Delta^\infty|$ is the direct limit of $n$-simplexes $\Delta^n$, $\mathfrak{K}(\Delta^\infty)$ is homeomorphic to the direct limit of Hilbert cubes by [CP, Corollary 3.1], whence it is an AR($\mathcal{S}$) by [$\mathrm{Ca}_3$, Corollaire 4.2]. Since $X$ can be embedded in $|\Delta^\infty|$ as a closed set, it can be considered a neighborhood retract of $|\Delta^\infty|$, whence $\mathfrak{K}(X)$ is a neighborhood retract of $\mathfrak{K}(\Delta^\infty)$. Therefore $\mathfrak{K}(X)$ is an ANR($\mathcal{S}$).

**1. A particular base of neighborhoods of $A \in \mathfrak{K}(K)$.** Let $K$ be a simplicial complex. In this section, we construct a particular base of neighborhoods of $A \in \mathfrak{K}(K)$ in imitation of [$\mathrm{Ca}_5$]. For each $\sigma \in K$, the barycenter, the boundary and the interior of $\sigma$ are denoted by $\widehat{\sigma}$, $\partial\sigma$ and $\mathring{\sigma}$, respectively. Moreover, $\tau \le \sigma$ ($\tau < \sigma$) means that $\tau$ is a (proper) face of $\sigma$. The simplex with vertices $v_0, \ldots, v_n$ is denoted by $\langle v_0, \ldots, v_n\rangle$. We abuse the notation $\langle \ldots \rangle$, but it can be recognized from the context to stand for a simplex or a basic open set of the Vietoris topology.

For each $x \in |K|$, let $(x(\widehat{\sigma}))_{\sigma\in K}$ denote the barycentric coordinates of $x$ with respect to the barycentric subdivision $\operatorname{Sd} K$. Let $d$ be the barycentric metric on $|\operatorname{Sd} K|$ $(= |K|)$ defined by

$$d(x, y) = \sum_{\sigma\in K} |x(\widehat{\sigma}) - y(\widehat{\sigma})|\,,$$

and let $N_d(x, \varepsilon)$ denote the $\varepsilon$-neighborhood of $x \in |K|$ with respect to $d$. Let $d_{\mathrm{H}}$ be the Hausdorff metric on $\mathfrak{K}(K)$ induced by $d$, that is, for each $A, B \in \mathfrak{K}(K)$,

$$d_{\mathrm{H}}(A, B) = \inf\{\varepsilon > 0 \mid A \subset N_d(B, \varepsilon) \text{ and } B \subset N_d(A, \varepsilon)\},$$

where

$$N_d(C, \varepsilon) = \bigcup_{x\in C} N_d(x, \varepsilon) = \{y \in |K| \mid \operatorname{dist}_d(y, C) < \varepsilon\}\,.$$

One should not confuse $N_{d_{\mathrm{H}}}(C, \varepsilon)$ with $N_d(C, \varepsilon)$, where $N_{d_{\mathrm{H}}}(C, \varepsilon)$ denotes the $\varepsilon$-neighborhood of $C \in \mathfrak{K}(K)$ with respect to $d_{\mathrm{H}}$. Note that these metrics are continuous but they do not generate the topology of $|K|$ nor the

Vietoris topology of $\mathfrak{K}(K)$ if $K$ is infinite. For each finite subcomplex $L$ of $K$, they do.

For each $\sigma \in K$ and $0 < t \leq 1$, let

$$\sigma(t) = \{x \in \sigma \mid 0 \leq x(\widehat{\sigma}) < t\} \quad \text{and} \quad \sigma[t] = \{x \in \sigma \mid 0 \leq x(\widehat{\sigma}) \leq t\}\,.$$

Then each $\sigma(t)$ is an open neighborhood of $\partial\sigma$ in $\sigma$ and $\sigma[t] = \mathrm{cl}_\sigma\, \sigma(t)$. Each $x \in \sigma(1) = \sigma \setminus \{\widehat{\sigma}\}$ can be uniquely written as

$$x = (1 - x(\widehat{\sigma}))\pi_\sigma(x) + x(\widehat{\sigma})\widehat{\sigma}, \quad \pi_\sigma(x) \in \partial\sigma\,.$$

Then for each $\sigma \in K$, we have a map $\pi_\sigma : \sigma(1) \to \partial\sigma$, called the *radial projection*.

1.1. Lemma. *Let $\sigma_0 < \ldots < \sigma_n = \sigma \in K$. For each $x \in \langle\widehat{\sigma}_0, \ldots, \widehat{\sigma}_n\rangle \cap \sigma(1)$,*

$$\pi_\sigma(x) = \sum_{i=0}^{n-1} \frac{x(\widehat{\sigma}_i)}{1 - x(\widehat{\sigma}_n)}\widehat{\sigma}_i \quad \textit{and} \quad d(x, \pi_\sigma(x)) = 2x(\widehat{\sigma})\,.$$

Proof. The first equality follows from

$$x = (1 - x(\widehat{\sigma}_n))\pi_\sigma(x) + x(\widehat{\sigma}_n)\widehat{\sigma}_n = \sum_{i=0}^{n} x(\widehat{\sigma}_i)\widehat{\sigma}_i\,.$$

Since $1 - x(\widehat{\sigma}_n) = \sum_{i=0}^{n-1} x(\widehat{\sigma}_i)$, we have

$$\begin{aligned} d(x, \pi_\sigma(x)) &= \sum_{i=0}^{n-1} \left(\frac{x(\widehat{\sigma}_i)}{1 - x(\widehat{\sigma}_n)} - x(\widehat{\sigma}_i)\right) + x(\widehat{\sigma}_n) \\ &= \frac{x(\widehat{\sigma}_n)}{1 - x(\widehat{\sigma}_n)} \sum_{i=0}^{n-1} x(\widehat{\sigma}_i) + x(\widehat{\sigma}_n) = 2x(\widehat{\sigma}_n) = 2x(\widehat{\sigma})\,. \ \blacksquare \end{aligned}$$

Let $L$ be a subcomplex of $K$. Then

$$W(L) = \{x \in |K| \mid \exists\sigma \in L \text{ such that } x(\widehat{\sigma}) > 0\}$$

is an open neighborhood of $|L|$ in $|K|$. For each $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, we write $K_L^{(n)} = L \cup K^{(n)}$, where $K^{(n)}$ denotes the $n$-skeleton of $K$. Let

$$W_n(L) = W(L) \cap |K_L^{(n)}| = \{x \in W(L) \mid x(\widehat{\sigma}) = 0,\ \forall\sigma \in K \setminus K_L^{(n)}\}\,.$$

Thus we have a tower $|L| = W_0(L) \subset W_1(L) \subset \ldots$ with $W(L) = \bigcup_{n\in\mathbb{Z}_+} W_n(L)$. Since $W_n(L) \setminus W_{n-1}(L)$ is covered by

$$S_n(L) = \{\sigma \in K_L^{(n)} \setminus K_L^{(n-1)} \mid \sigma \cap |L| \neq \emptyset\}\,,$$

we can define a retraction $p_n^L : W_n(L) \to W_{n-1}(L)$ by the radial projections, i.e., $p_n^L|\sigma \cap W_n(L) = \pi_\sigma|\sigma \cap W_n(L)$ for each $\sigma \in S_n(L)$. We define a retraction

$\pi^L : W(L) \to |L|$ by $\pi^L|W_n(L) = p_1^L \dots p_n^L$ for each $n \in \mathbb{N}$. Let

$$S(L) = \bigcup_{n\in\mathbb{N}} S_n(L) = \{\sigma \in K \setminus L \mid \sigma \cap |L| \neq \emptyset\}\,.$$

For each $\varepsilon \in (0,1)^{S(L)}$, we inductively define an open neighborhood $W(L,\varepsilon) = \bigcup_{n\in\mathbb{Z}_+} W_n(L,\varepsilon)$ of $|L|$ in $|K|$ as follows: $W_0(L,\varepsilon) = |L|$ and

$$\begin{aligned} W_n(L,\varepsilon) &= |L| \cup \bigcup\{\sigma(\varepsilon(\sigma)) \cap (p_n^L)^{-1}(W_{n-1}(L,\varepsilon)) \mid \sigma \in S_n(L)\} \\ &\Big(= |L| \cup \bigcup\{\sigma(\varepsilon(\sigma)) \cap \pi_\sigma^{-1}(W_{n-1}(L,\varepsilon)) \mid \sigma \in S_n(L)\}\Big)\,. \end{aligned}$$

For each $m \in \mathbb{N}$, let

$$\mathcal{E}_m^L = \{\varepsilon \in (0,1)^{S(L)} \mid \forall \sigma \in S(L),\ \varepsilon(\sigma) < 2^{-(m+\dim\sigma+1)}\}\,.$$

1.2. LEMMA. *Let $m \in \mathbb{N}$ and $\varepsilon \in \mathcal{E}_m^L$. Then $d_{\mathrm{H}}(A, \pi^L(A)) < 2^{-m}$ for any $A \in \mathfrak{K}(W(L,\varepsilon))$.*

Proof. From compactness of $A$, $A \subset W_n(L,\varepsilon)$ for some $n \in \mathbb{N}$. Each $x \in W_n(L,\varepsilon)$ is contained in $\sigma(\varepsilon(\sigma))$ for some $\sigma \in S_n(L)$, whence

$$d(x, p_n^L(x)) = d(x, \pi_\sigma(x)) = 2x(\widehat{\sigma}) < 2\varepsilon(\sigma) < 2^{-(m+n)}\,.$$

Then $d(x, p_n^L(x)) < 2^{-(m+n)}$ for each $x \in A$, whence $d_{\mathrm{H}}(A, p_n^L(A)) < 2^{-(m+n)} = 2^{-m}2^{-n}$. Note $p_n^L(A) \subset W_{n-1}(L,\varepsilon)$. By induction, we have

$$\begin{aligned} d_{\mathrm{H}}(p_1^L \dots p_n^L(A), A) &\le d_{\mathrm{H}}(p_1^L \dots p_{n-1}^L(p_n^L(A)), p_n^L(A)) + d_{\mathrm{H}}(p_n^L(A), A) \\ &< 2^{-m} \sum_{i=1}^{n} 2^{-i}\ (< 2^{-m})\,. \ \blacksquare \end{aligned}$$

Let $A \in \mathfrak{K}(K)$. By $L(A)$, we denote the smallest subcomplex of $K$ which contains $A$. Since $A$ is compact, $L(A)$ is a finite subcomplex of $K$. For each $\delta > 0$ and $\varepsilon \in (0,1)^{S(L(A))}$, we define

$$V(A,\delta,\varepsilon) = \{B \in \mathfrak{K}(W(L(A),\varepsilon)) \mid d_{\mathrm{H}}(\pi^{L(A)}(B), A) < \delta\}\,.$$

Since $\mathfrak{K}(W(L(A),\varepsilon))$ is an open neighborhood of $A$ in $\mathfrak{K}(K)$ and $\pi^{L(A)}$ induces a map from $\mathfrak{K}(W(L(A),\varepsilon))$ to $\mathfrak{K}(L(A))$, $V(A,\delta,\varepsilon)$ is an open neighborhood of $A$ in $\mathfrak{K}(K)$.

1.3. LEMMA. *For each $A \in \mathfrak{K}(K)$, $\{V(A,\delta,\varepsilon) \mid \delta > 0,\ \varepsilon \in (0,1)^{S(L(A))}\}$ is a neighborhood base of $A$ in $\mathfrak{K}(K)$.*

Proof. In the proof, we simply write $p_n = p_n^{L(A)}$. Let $\langle U_1, \dots, U_k\rangle$ be a basic neighborhood of $A$ in $\mathfrak{K}(K)$. For each $i = 1, \dots, k$, choose $x_i \in A \cap U_i$ and $\delta_i > 0$ so that $\operatorname{cl} N_d(x_i, \delta_i) \cap |L(A)| \subset U_i$. Let $\eta > 0$ be a Lebesgue number for the open cover $\{U_i \cap |L(A)| \mid i = 1, \dots, k\}$ of $A$ in $|L(A)|$, that is, each $B \subset |L(A)|$ is contained in some $U_i \cap |L(A)|$ if $\operatorname{diam}_d B < \eta$ and $B \cap A \neq \emptyset$. Let

$$\delta_0 = \min\{\eta/3, \delta_1, \dots, \delta_k\} > 0 .$$

By compactness, we can choose more points $x_j \in A$, $j = k+1, \dots, m$, so that

$$A \subset \bigcup_{j=1}^{m} N_d(x_j, \delta_0) \cap |L(A)| .$$

For each $j = 1, \dots, m$, let $V_0(x_j) = N_d(x_j, \delta_0) \cap |L(A)|$. Then for each $j \leq k$,

$$\operatorname{cl} V_0(x_j) \subset \operatorname{cl} N_d(x_j, \delta_j) \cap |L(A)| \subset U_j .$$

Let $i(j) = j$ for each $j \leq k$, while for each $j > k$, choose $i(j) \leq k$ so that

$$\operatorname{cl} V_0(x_j) \subset \operatorname{cl} N_d(x_j, \eta/3) \cap |L(A)| \subset U_{i(j)} .$$

By induction on dimension, we can choose $\varepsilon_j(\sigma) \in (0,1)$ for each $\sigma \in S(L(A))$ so that

$$C(j,\sigma) = \pi_\sigma^{-1}(C(j,\partial\sigma)) \cap \sigma[\varepsilon_j(\sigma)] \subset U_{i(j)} ,$$

where

$$C(j,\partial\sigma) = (\partial\sigma \cap \operatorname{cl} V_0(x_j)) \cup \bigcup\{C(j,\tau) \mid \tau \in S(L(A)),\ \tau < \sigma\} .$$

In the above, $C(j,\partial\sigma) = \partial\sigma \cap \operatorname{cl} V_0(x_j)$ if $\dim \sigma = 1$. Thus we have $\varepsilon_j \in (0,1)^{S(L(A))}$ for each $j = 1, \dots, m$. We define $\varepsilon \in (0,1)^{S(L(A))}$ by $\varepsilon(\sigma) = \min_{1 \leq j \leq m} \varepsilon_j(\sigma)$. We inductively define

$$\begin{aligned} V_n(x_j) &= \{y \in W_n(L(A),\varepsilon) \mid p_n(y) \in V_{n-1}(x_j)\} \\ &= W_n(L(A),\varepsilon) \cap p_n^{-1}(V_{n-1}(x_j)) . \end{aligned}$$

Then $V(x_j) = \bigcup_{n \in \mathbb{Z}_+} V_n(x_j)$ is an open neighborhood of $x_j$ in $|K|$. Since

$$V_n(x_j) \subset V_0(x_j) \cup \bigcup\{C(j,\sigma) \mid \sigma \in S(L(A))\} \subset U_{i(j)} ,$$

we have $V(x_j) \subset U_{i(j)}$ for each $i = 1, \dots, m$. Hence

$$A \in \langle V(x_1), \dots, V(x_m)\rangle \subset \langle U_1, \dots, U_k\rangle .$$

Let $\zeta > 0$ be a Lebesgue number for the open cover $\{V_0(x_i) \mid i = 1, \dots, m\}$ of $A$ in $|L(A)|$ and let $\delta = \min\{\delta_0, \zeta, 1\} > 0$. We show that

$$V(A,\delta,\varepsilon) \subset \langle V(x_1), \dots, V(x_m)\rangle \subset \langle U_1, \dots, U_k\rangle .$$

To this end, it suffices to show that for every $n \in \mathbb{Z}_+$,

$$(*_n) \qquad V(A,\delta,\varepsilon) \cap \mathfrak{K}(W_n(L(A),\varepsilon)) \subset \langle V(x_1), \dots, V(x_m)\rangle .$$

To see $(*_0)$, let $B \in V(A,\delta,\varepsilon) \cap \mathfrak{K}(W_0(L(A),\varepsilon))$. Then $B \subset W_0(L(A),\varepsilon) = |L(A)|$ and $d_{\mathrm{H}}(A,B) < \delta$. For any $y \in B$, we have $x \in A$ such that $d(x,y) < \delta \leq \zeta$, whence $\{x,y\} \subset V(x_i) \cap |L(A)|$ for some $i = 1, \dots, m$.

Therefore $B \subset \bigcup_{i=1}^m V(x_i)$. For each $i = 1, \ldots, m$, there is a $y \in B$ such that $d(x_i, y) < \delta \le \delta_0$. Then $y \in V_0(x_i) \subset V(x_i)$, whence $B \cap V(x_i) \neq \emptyset$. Therefore $B \in \langle V(x_1), \ldots, V(x_m)\rangle$. Thus we have $(*_0)$.

Next assume $(*_{n-1})$ and let $B \in V(A, \delta, \varepsilon) \cap \mathfrak{K}(W_n(L(A), \varepsilon))$. Since $B \subset W_n(L(A), \varepsilon)$, we have $p_n(B) \subset W_{n-1}(L(A), \varepsilon)$, whence

$$p_n(B) \in V(A, \delta, \varepsilon) \cap \mathfrak{K}(W_{n-1}(L(A), \varepsilon)) \subset \langle V(x_1), \ldots, V(x_m)\rangle .$$

For each $y \in B \subset W_n(L(A), \varepsilon)$, $p_n(y)$ is contained in some $V_{n-1}(x_i)$, since

$$p_n(B) \subset \bigcup_{i=1}^m V(x_i) \cap W_{n-1}(L(A), \varepsilon) = \bigcup_{i=1}^m V_{n-1}(x_i) .$$

Then it follows that $y \in V_n(x_i)$. Therefore $B \subset \bigcup_{i=1}^m V(x_i)$. For each $i = 1, \ldots, m$,

$$p_n(B) \cap V_{n-1}(x_i) = p_n(B) \cap V(x_i) \neq \emptyset ,$$

that is, $p_n(y) \in V_{n-1}(x_i)$ for some $y \in B \subset W_n(L(A), \varepsilon)$. Then $y \in V_n(x_i) \subset V(x_i)$, whence $B \cap V(x_i) \neq \emptyset$. Therefore $B \in \langle V(x_1), \ldots, V(x_m)\rangle$. By induction, $(*_n)$ holds for every $n \in \mathbb{Z}_+$. ■

1.4. Lemma. *Let* $A_0 \in \mathfrak{K}(K)$, $\delta > 0$ *and* $\varepsilon \in (0,1)^{S(L(A_0))}$.

(1) *If* $\mathcal{A} \subset V(A_0, \delta, \varepsilon)$ *and* $\mathcal{A}$ *is compact*, *then* $\bigcup \mathcal{A} \in V(A_0, \delta, \varepsilon)$.

(2) *For each* $A, B, C \in \mathfrak{K}(K)$, *if* $A \subset B \subset C$ *and* $A, C \in V(A_0, \delta, \varepsilon)$ *then* $B \in V(A_0, \delta, \varepsilon)$.

(3) *If* $\varepsilon \in \mathcal{E}_m^{L(A_0)}$ *then* $V(A_0, 2^{-m}, \varepsilon) \subset N_{d_{\mathrm{H}}}(A_0, 2^{-m+1})$.

Proof. Since $\bigcup \mathcal{A}$ is compact, (1) follows from the definition. By the definition, (2) is also easily observed. Finally, (3) follows from Lemma 1.2. ■

## 2. A stratification of $\mathfrak{K}(K)$.

Recall that a $T_1$-space $X$ is *stratifiable* if each open set $U$ in $X$ can be assigned a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets in $X$ so that

(a) $\operatorname{cl} U_n \subset U$,
(b) $U = \bigcup_{n\in\mathbb{N}} U_n$,
(c) $U \subset V$ implies $U_n \subset V_n$ for all $n \in \mathbb{N}$,

where $(U_n)_{n\in\mathbb{N}}$ is called a *stratification* of $U$ and the correspondence $U \to (U_n)_{n\in\mathbb{N}}$ is a *stratification* of $X$. In this section, we prove the following:

2.1. Theorem. *For any simplicial complex* $K$, $\mathfrak{K}(K)$ *is stratifiable.*

Proof. For any $A \in \mathfrak{K}(K)$ and $\sigma \in K$ with $\mathring{\sigma} \cap A \neq \emptyset$, let

$$\alpha(A, \sigma) = \sup\{x(\widehat{\sigma}) \mid x \in \sigma \cap A\} .$$

Then $\alpha(A,\sigma) = x(\hat{\sigma}) > 0$ for some $x \in \mathring{\sigma} \cap A$ because $\sigma \cap A$ is compact and $x(\hat{\sigma}) = 0$ for all $x \in \partial\sigma \cap A$. We define

$$\alpha(A) = \min\{\alpha(A,\sigma) \mid \sigma \in L(A),\ \mathring{\sigma} \cap A \neq \emptyset\} > 0\,.$$

For each open set $\mathcal{U}$ in $\mathfrak{K}(K)$ and $n \in \mathbb{N}$, let

$$\mathcal{U}'_n = \{A \in \mathcal{U} \mid \alpha(A) > 2^{-n},\ \exists \varepsilon \in \mathcal{E}_n^{L(A)} \text{ such that } V(A, 2^{-n}, \varepsilon) \subset \mathcal{U}\}\,.$$

By Lemma 1.3, $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}'_n$. For each $A \in \mathcal{U}'_n$, let

$$\mathcal{U}_n(A) = \bigcup\{V(A, 2^{-(n+2)}, 2^{-2}\varepsilon) \mid \varepsilon \in \mathcal{E}_n^{L(A)},\ V(A, 2^{-n}, \varepsilon) \subset \mathcal{U}\}\,.$$

Then $\mathcal{U}_n(A) \subset N_{d_{\mathrm{H}}}(A, 2^{-(n+1)})$ by Lemma 1.4(3). Thus we have open sets $\mathcal{U}_n = \bigcup_{A\in\mathcal{U}'_n} \mathcal{U}_n(A)$ in $\mathfrak{K}(K)$ and $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$. It follows from the definition that $\mathcal{U} \subset \mathcal{V}$ implies $\mathcal{U}_n \subset \mathcal{V}_n$ for any open sets $\mathcal{U}$ and $\mathcal{V}$ in $\mathfrak{K}(K)$. We will show that $\mathrm{cl}\,\mathcal{U}_n \subset \mathcal{U}$ for each $n \in \mathbb{N}$. Then $\mathcal{U} \to (\mathcal{U}_n)_{n\in\mathbb{N}}$ is a stratification.

Now let $n \in \mathbb{N}$ and $C \in \mathfrak{K}(K) \setminus \mathcal{U}$ be fixed. To see that $C \notin \mathrm{cl}\,\mathcal{U}_n$, it suffices to construct a neighborhood $\mathcal{V}_C$ of $C$ in $\mathfrak{K}(K)$ so that $\mathcal{V}_C \cap \mathcal{U}_n(A) = \emptyset$ for all $A \in \mathcal{U}'_n$. Let $\mathcal{L}_C$ be the collection of all subcomplexes $L$ of $L(C)$ such that $L \neq L(C)$ and $C \subset W(L)$. First, we show that if $A \in \mathcal{U}'_n$ and $L(A) \notin \mathcal{L}_C$ then

$$N_{d_{\mathrm{H}}}(C, 2^{-(n+1)}) \cap \mathcal{U}_n(A) = \emptyset\,.$$

To this end, since $\mathcal{U}_n(A) \subset N_{d_{\mathrm{H}}}(A, 2^{-(n+1)})$, it suffices to show that $d_{\mathrm{H}}(A,C) \geq 2^{-n}$. We consider three cases. In case $L(A) \not\subset L(C)$, we have $\sigma \in L(A) \setminus L(C)$ such that $\mathring{\sigma} \cap A \neq \emptyset$. Choose $x \in \mathring{\sigma} \cap A$ so that $x(\hat{\sigma}) = \alpha(A,\sigma)$. Since $\sigma \notin L(C)$, $y(\hat{\sigma}) = 0$ for all $y \in C$. Then

$$\begin{aligned} d_{\mathrm{H}}(A,C) &\geq \mathrm{dist}_d(x, C) = \inf\{d(x,y) \mid y \in C\} \\ &\geq x(\hat{\sigma}) = \alpha(A,\sigma) \geq \alpha(A) > 2^{-n}\,. \end{aligned}$$

In case $L(A) = L(C)$, since $C \subset |L(A)|$ and $C \notin V(A, 2^{-n}, \varepsilon)$ for $\varepsilon \in \mathcal{E}_n^{L(A)}$ such that $V(A, 2^{-n}, \varepsilon) \subset \mathcal{U}$, it follows that $d_{\mathrm{H}}(A,C) \geq 2^{-n}$. In case $L(A) \subsetneq L(C)$ and $C \not\subset W(L(A))$, we have $x \in C$ such that $x(\hat{\sigma}) = 0$ for all $\sigma \in L(A)$, whence

$$\mathrm{dist}_d(x, A) \geq \mathrm{dist}_d(x, |L(A)|) = 2\,,$$

which implies $d_{\mathrm{H}}(A,C) \geq 2$.

Next, we construct a neighborhood $\mathcal{V}_L$ of $C$ in $\mathfrak{K}(K)$ for each $L \in \mathcal{L}_C$ so that if $A \in \mathcal{U}'_n$ and $L(A) = L$ then $\mathcal{V}_L \cap \mathcal{U}_n(A) = \emptyset$. Then, since $\mathcal{L}_C$ is finite,

$$\mathcal{V}_C = \bigcap_{L\in\mathcal{L}_C} \mathcal{V}_L \cap N_{d_{\mathrm{H}}}(C, 2^{-(n+1)})$$

is the desired neighborhood. (In case $\mathcal{L}_C = \emptyset$, $\mathcal{V}_C = N_{d_{\mathrm{H}}}(C, 2^{-(n+1)})$.)

Now let $L \in \mathcal{L}_C$, that is, $L \subsetneq L(C)$ and $C \subset W(L)$. Since $\pi^L$ induces a map from $\mathfrak{K}(W(L))$ to $\mathfrak{K}(L)$, $C$ has a neighborhood $\mathcal{V}_0$ in $\mathfrak{K}(W(L))$ such that $\pi^L(B) \in N_{d_{\mathrm{H}}}(\pi^L(C), 2^{-(n+1)})$ for all $B \in \mathcal{V}_0$. For each $i \in \mathbb{N}$, define $\pi^L_i : W(L) \to W_i(L)$ by $\pi^L_i|W_j(L) = p^L_{i+1} \dots p^L_j$ for each $j > i$. Since $L \subsetneq L(C)$ and $C \subset W(L)$, we have $\sigma \in S(L)$ such that $\mathring{\sigma} \cap C \neq \emptyset$ and $\dim \sigma = \dim(L(C) \setminus L)$, whence $\pi^L_{\dim\sigma}(C) = C$. Let

$$\{\sigma_1, \dots, \sigma_m\} = \{\sigma \in S(L) \mid \mathring{\sigma} \cap \pi^L_{\dim\sigma}(C) \neq \emptyset\}.$$

For each $i = 1, \dots, m$, let $k_i = \dim \sigma_i$ and

$$t_i = \inf\{t > 0 \mid \pi^L_{k_i}(C) \cap \sigma_i \subset \sigma_i(t)\} > 0.$$

Then $\mathcal{V}'_i = \langle W_{k_i}(L), \sigma_i \setminus \sigma_i[\frac{1}{2}t_i]\rangle$ is a neighborhood of $\pi^L_{k_i}(C)$ in $\mathfrak{K}(W_{k_i}(L))$, whence $C$ has a neighborhood $\mathcal{V}_i$ in $\mathfrak{K}(W(L))$ such that for each $B \in \mathcal{V}_i$, $\pi^L_{k_i}(B) \in \mathcal{V}'_i$, that is, $\pi^L_{k_i}(B) \cap \sigma_i \not\subset \sigma_i[\frac{1}{2}t_i]$. Then $\mathcal{V}_L = \bigcap_{i=0}^m \mathcal{V}_i$ is the desired neighborhood of $C$.

In fact, let $A \in \mathcal{U}'_n$ with $L(A) = L$ and $\varepsilon \in \mathcal{E}^L_n$ such that $V(A, 2^{-n}, \varepsilon) \subset \mathcal{U}$. Then $C \notin V(A, 2^{-n}, \varepsilon)$ since $C \notin \mathcal{U}$. In case $d_{\mathrm{H}}(A, \pi^L(C)) \geq 2^{-n}$, it follows that for each $B \in \mathcal{V}_L \subset \mathcal{V}_0$,

$$\begin{aligned} d_{\mathrm{H}}(A, \pi^L(B)) &\geq d_{\mathrm{H}}(A, \pi^L(C)) - d_{\mathrm{H}}(\pi^L(B), \pi^L(C)) \\ &> 2^{-n} - 2^{-(n+1)} > 2^{-(n+2)}, \end{aligned}$$

which implies $B \notin V(A, 2^{-(n+2)}, 2^{-2}\varepsilon)$. In case $d_{\mathrm{H}}(A, \pi^L(C)) < 2^{-n}$, we have $C \not\subset W(L, \varepsilon)$, whence $\pi^L_i(C) \not\subset W_i(L, \varepsilon)$ for some $i \in \mathbb{N}$. Let

$$k = \min\{i \in \mathbb{N} \mid \pi^L_i(C) \not\subset W_i(L, \varepsilon)\}$$

and choose $\sigma \in S_k(L)$ so that $\pi^L_k(C) \cap \sigma \not\subset \sigma(\varepsilon(\sigma))$. Then $\sigma = \sigma_i$ for some $i = 1, \dots, m$, whence $k = k_i$ and $t_i \geq \varepsilon(\sigma_i)$. For each $B \in \mathcal{V}_L \subset \mathcal{V}_i$, $\pi^L_{k_i}(B) \cap \sigma_i \not\subset \sigma_i(\frac{1}{2}t_i)$, whence $\pi^L_{k_i}(B) \not\subset W_{k_i}(L, 2^{-1}\varepsilon)$, so $B \not\subset W(L, 2^{-1}\varepsilon)$. Then $B \notin V(A, 2^{-(n+2)}, 2^{-2}\varepsilon)$. Therefore $\mathcal{V}_L \cap \mathcal{U}_n(A) = \emptyset$. ∎

**3. CW-complexes have no $\sigma$-CF quasi-base.** In [Mi], Mizokami gave a condition for a stratifiable space $X$ so that $\mathfrak{K}(X)$ is stratifiable. In this section, we show that non-locally compact CW-complexes do not satisfy this condition. Let $\mathcal{A}$ be a family of subsets of a space $X$. Recall that $\mathcal{A}$ is *closure preserving* in $X$ if $\mathrm{cl} \bigcup \mathcal{B} = \bigcup\{\mathrm{cl}\, B \mid B \in \mathcal{B}\}$ for any subfamily $\mathcal{B} \subset \mathcal{A}$. Moreover, $\mathcal{A}$ is *$\sigma$-closure preserving* in $X$ if $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$, where each $\mathcal{A}_n$ is closure preserving in $X$ [Ce]. It was proved independently by Gruenhage [Gr] and Junnila [Ju] that a regular space $X$ is stratifiable if and only if $X$ has a $\sigma$-closure preserving quasi-base, where a *quasi-base* of $X$ is a family $\mathcal{B}$ of (not necessarily open) subsets of $X$ such that for each $x \in X$, $\{B \in \mathcal{B} \mid x \in \mathrm{int}\, B\}$ is a neighborhood base of $x$ in $X$. We say that $\mathcal{A}$ is *finite on compact sets* (CF) in $X$ if $\{A \cap C \mid A \in \mathcal{A}\}$ is finite for each compact set $C$ in $X$ [Mi]. Finally, $\mathcal{A}$ is *$\sigma$-CF* in $X$ if $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$, where each $\mathcal{A}_n$ is CF in $X$ [Mi].

The following is shown by Mizokami [Mi, Theorem 4.5]:

3.1. THEOREM. *Let $X$ be regular. Then $\mathfrak{K}(X)$ is stratifiable if $X$ has a quasi-base $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ consisting of closed sets such that each $\mathcal{B}_n$ is CF and closure preserving in $X$.*

Note that if $\mathcal{B}$ is a quasi-base for a regular space $X$ then $\{\mathrm{cl}\, B \mid B \in \mathcal{B}\}$ is also a quasi-base for $X$. We show the following:

3.2. PROPOSITION. *Any non-locally compact CW-complex $X$ has no $\sigma$-CF quasi-base consisting of closed sets.*

P r o o f. From non-local compactness, $X$ contains the cone $C$ of $\mathbb{N}$ with vertex $v$, i.e., $C = (\mathbb{N} \times \mathbf{I})/(\mathbb{N} \times \{0\})$ and $v = \mathbb{N} \times \{0\} \in C$. Thus it suffices to show that $C$ has no $\sigma$-CF quasi-base consisting of closed sets.

Assume that $C$ has a quasi-base $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ consisting of closed sets such that each $\mathcal{B}_n$ is CF in $C$. Let $q : \mathbb{N} \times \mathbf{I} \to C$ be the quotient map and set $\mathbf{I}_n = q(\{n\} \times \mathbf{I})$ for each $n \in \mathbb{N}$. Then each $\{B \cap \mathbf{I}_n \mid B \in \mathcal{B}_n\}$ is finite. Hence for each $n \in \mathbb{N}$, $v$ has a neighborhood $U_n$ in $\mathbf{I}_n$ such that $B \cap \mathbf{I}_n \not\subset U_n$ for all $B \in \mathcal{B}_n$ with $v \in \mathrm{int}\, B$. Then $U = \bigcup_{n\in\mathbb{N}} U_n$ is a neighborhood of $v$ in $C$. Since $\mathcal{B}$ is a quasi-base for $C$, there exists $B \in \mathcal{B}$ such that $v \in B \subset U$. Then $B \in \mathcal{B}_n$ for some $n \in \mathbb{N}$, whence $B \cap \mathbf{I}_n \subset U \cap \mathbf{I}_n = U_n$. This is a contradiction. ∎

## 4. Cauty's test space $Z(X)$.

Let $X$ be a stratifiable space. In [Ca$_4$], Cauty constructed a space $Z(X)$ and showed that $X$ is an AR($\mathcal{S}$) (resp. an ANR($\mathcal{S}$)) if and only if $X$ is a retract (resp. a neighborhood retract) of $Z(X)$. Let $F(X)$ denote the full simplicial complex with $X$ the set of vertices (i.e., $X = F(X)^{(0)}$). Recall that $|F(X)|$ has the *weak topology*. The space $Z(X)$ is defined as the space $|F(X)|$ with the topology generated by open sets $W$ in $|F(X)|$ such that

$$(*) \qquad W \cap X \text{ is open in } X \text{ and } |F(W \cap X)| \subset W\,.$$

The second condition of $(*)$ means that each $\tau \in F(X)$ is contained in $W$ if all vertices of $\tau$ are contained in $W \cap X$. For each $A \subset X$, $Z(A)$ is a subspace of $Z(X)$ and

$$Z(X) \setminus Z(A) = \{(1-t)x + ty \mid x \in Z(X \setminus A),\ y \in Z(A),\ 0 \le t < 1\}\,,$$

whence if $A$ is closed in $X$ then $Z(A)$ is closed in $Z(X)$. Each map $f : X \to Y$ induces a map $\widetilde{f} : Z(X) \to Z(Y)$ which is simplicial with respect to $F(X)$ and $F(Y)$. Observe that $\widetilde{f}(Z(X)) = Z(f(X))$, $\widetilde{f}(Z(X)\setminus X) \subset Z(Y)\setminus Y$. For each $n \in \mathbb{Z}_+$, let $Z_n(X) = |F(X)^{(n)}|$, a subspace of $Z(X)$. Then $Z_0(X) = X$ and $Z(X) = \bigcup_{n\in\mathbb{Z}_+} Z_n(X)$.

For each $A \subset X$, $F(A)$ is a subcomplex of $F(X)$. Here using different notations, we write $W(F(A)) = M(A)$, $S(F(A)) = T(A)$, $W_n(F(A)) =$

$M_n(A)$ and $S_n(F(A)) = T_n(A)$ for each $n \in \mathbb{Z}_+$:

$$M(A) = \{x \in Z(X) \mid \exists \tau \in F(A) \text{ such that } x(\widehat{\tau}) > 0\},$$
$$T(A) = \{\tau \in F(X) \setminus F(A) \mid \tau \cap A \neq \emptyset\},$$
$$M_n(A) = Z(A) \cup (M(A) \cap Z_n(X)),$$
$$T_n(A) = T(A) \cap F(X)^{(n)} \setminus F(X)^{(n-1)}.$$

For each $\varepsilon \in (0,1)^{T(A)}$, we write

$$M(A,\varepsilon) = \bigcup_{n \in \mathbb{Z}_+} M_n(A,\varepsilon),$$

where $M_0(A,\varepsilon) = Z(A) = |F(A)|$ and

$$M_n(A,\varepsilon) = Z(A) \cup \bigcup \{\tau(\varepsilon(\tau)) \cap \pi_\tau^{-1}(M_{n-1}(A,\varepsilon)) \mid \tau \in T_n(A)\}$$

for each $n \in \mathbb{N}$. Then $M(A,\varepsilon) \cap X = A$. For any open set $U$ in $X$, $M(U,\varepsilon)$ is an open set in $Z(X)$.

4.1. Lemma. *Let $\mathcal{N}(x)$ be an open neighborhood base of $x$ in $X$. Then*

$$\{M(U,\varepsilon) \mid U \in \mathcal{N}(x), \varepsilon \in (0,1)^{T(U)}\}$$

*is a neighborhood base of $x$ in $Z(X)$.*

Proof. As observed above, $M(U,\varepsilon)$ is an open neighborhood of $x$ in $Z(X)$ for each $U \in \mathcal{N}(x)$ and $\varepsilon \in (0,1)^{T(U)}$. Let $W$ be an open set in $|F(X)|$ satisfying $(*)$ and $x \in W$. Since $W \cap X$ is an open neighborhood of $x$ in $X$, $W \cap X$ contains some $U \in \mathcal{N}(x)$. Then $Z(U) \subset Z(W \cap X) \subset W$. Let $C_\tau = \tau$ for all $\tau \in F(U)$ and $C_\tau = \emptyset$ for all $\tau \in F(X \setminus U)$. By induction on dimension, we can choose $\varepsilon(\tau) \in (0,1)$ for each $\tau \in T(U)$ so that

$$C_\tau = \pi_\tau^{-1}(C_{\partial\tau}) \cap \tau[\varepsilon(\tau)] \subset W \cap \tau,$$

where $C_{\partial\tau} = \bigcup_{\tau' < \tau} C_{\tau'}$. Thus we have $\varepsilon \in (0,1)^{T(U)}$ such that

$$M(U,\varepsilon) \subset Z(U) \cup \bigcup_{\tau \in T(U)} C_\tau \subset W. \ \blacksquare$$

Let $p_n = p_n^A : M_n(A) \to M_{n-1}(A)$ be the retraction defined by the radial projections and $\pi^A : M(A) \to Z(A)$ the retraction defined by $\pi^A | M_n(A) = p_1 \ldots p_n$ for each $n \in \mathbb{N}$. Consider $M_n(A)$'s and $M(A)$ as subspaces of $Z(X)$. Then it is easy to see that the retractions are continuous.

**5. A retraction of $Z(\mathfrak{K}(K))$ onto $\mathfrak{K}(K)$.** The main theorem implies the following:

5.1. Theorem. *For any connected simplicial complex $K$, there exists a retraction $r : Z(\mathfrak{K}(K)) \to \mathfrak{K}(K)$ such that $r(Z(\mathfrak{C}(K))) = \mathfrak{C}(K)$. Hence $\mathfrak{K}(K)$ and $\mathfrak{C}(K)$ are $AR(\mathcal{S})$'s.*

In the proof, we first construct a retraction $r_1 : Z_1(\mathfrak{K}(K)) \to \mathfrak{K}(K)$ and then extend $r_1$ to $r : Z(\mathfrak{K}(K)) \to \mathfrak{K}(K)$. To construct $r_1$, we introduce several notations. Let

$$\mathcal{H} = \{\langle A, B\rangle \in F(\mathfrak{K}(K))^{(1)} \mid A \neq B,\ d_{\mathrm{H}}(A, B) < 1/2\}\,.$$

For each $\langle A, B\rangle \in \mathcal{H}$, we define $C\langle A, B\rangle \in \mathfrak{K}(K)$ so that $A \cup B \subset C\langle A, B\rangle$ and each component of $C\langle A, B\rangle$ meets both $A$ and $B$. Let $n = \dim L(A \cup B)$. By downward induction, we define $L_i$, $q_i^n$ $(i = 0, 1, \ldots, n)$ and $R_i$, $q_i$ $(i = 1, \ldots, n)$ as follows: $L_n = L(A \cup B)$, $q_n^n = \mathrm{id}$ and

$$R_i = \{\sigma \in L_i \mid \dim \sigma = i,\ \forall \tau \in L_i,\ \sigma \not< \tau,\ \sigma \cap q_i^n(A \cup B) \subset \sigma[2^{-(i+2)}]\}\,,$$
$$q_i : |L_i \setminus R_i| \cup \bigcup_{\sigma \in R_i} \sigma[2^{-(i+2)}] \to |L_i \setminus R_i|$$

is the retraction defined by

$$q_i|\sigma[2^{-(i+2)}] = \pi_\sigma|\sigma[2^{-(i+2)}] \quad \text{for each } \sigma \in R_i\,,$$
$$q_{i-1}^n = q_i \ldots q_n|A \cup B \quad \text{and} \quad L_{i-1} = L(q_{i-1}^n(A \cup B))\,.$$

Observe that $\mathring{\sigma} \cap q_i^n(A \cup B) \neq \emptyset$ for each $\sigma \in R_i$. Next we define $\eta : \bigcup_{i=1}^n R_i \to (0, 1)$ by

$$\eta(\sigma) = \inf\{t > 0 \mid q_i^n(A \cup B) \cap \sigma \subset \sigma(t)\} > 0 \quad \text{if } \sigma \in R_i\,.$$

Now we inductively define $N_i$ $(i = 0, 1, \ldots, n)$ as follows: $N_0 = |L_0|$ and

$$N_i = N_{i-1} \cup \bigcup_{\sigma \in R_i} (q_i^{-1}(N_{i-1}) \cap \sigma[\eta(\sigma)])\,.$$

Finally, we define

$$C\langle A, B\rangle = \bigcup\{C \in \mathfrak{C}(N_n) \mid \mathrm{diam}_d\, C \leq 2d_{\mathrm{H}}(A, B),\ C \cap (A \cup B) \neq \emptyset\}\,.$$

5.2. Lemma. *For each $\langle A, B\rangle \in \mathcal{H}$, each component $C$ of $C\langle A, B\rangle$ meets both $A$ and $B$.*

Proof. Since $C$ meets at least one of $A$ and $B$, we may assume that $A \cap C \neq \emptyset$ and show that $B \cap C \neq \emptyset$. Let $x \in A \cap C$. Then we have $y \in B$ such that $d(x, y) \leq d_{\mathrm{H}}(A, B) < 1/2$. Since $x, y \in |L_n| = |\mathrm{Sd}\, L_n|$, we have $\sigma_0 < \ldots < \sigma_m \in L_n$ and $\sigma'_0 < \ldots < \sigma'_{m'} \in L_n$ such that $\dim \sigma_i = i$, $\dim \sigma'_j = j$,

$$x \in \langle \widehat{\sigma}_0, \ldots, \widehat{\sigma}_m\rangle \quad \text{and} \quad y \in \langle \widehat{\sigma}'_0, \ldots, \widehat{\sigma}'_{m'}\rangle\,.$$

Let $k = \max\{i \mid \sigma_i \in L_0\} \geq 0$. Then $\langle \widehat{\sigma}_0, \ldots, \widehat{\sigma}_k\rangle \cap \langle \widehat{\sigma}'_0, \ldots, \widehat{\sigma}'_{m'}\rangle \neq \emptyset$. In fact, if $k = m$, this follows from $d(x, y) < 1/2$. If $k < m$, then $\sigma_m \in R_m$ and $x(\widehat{\sigma}_m) \leq \eta(\sigma_m)$ since $x \in N_n \setminus N_0$. For each $j = k + 1, \ldots, m$, by using

Lemma 1.1 inductively, we have

$$q_{j-1}^m(x) = q_j \dots q_m(x) = \sum_{i=0}^{j-1} \frac{x(\widehat{\sigma}_i)}{1-(x(\widehat{\sigma}_j)+\dots+x(\widehat{\sigma}_m))}\widehat{\sigma}_i\,.$$

Then for each $j = k+1, \dots, m-1$,

$$\frac{x(\widehat{\sigma}_j)}{1-(x(\widehat{\sigma}_{j+1})+\dots+x(\widehat{\sigma}_m))} \le \eta(\sigma_j) \le 2^{-(j+2)}$$

because $q_{j+1}\dots q_m(x) \in \sigma_j(\eta(\sigma_j))$. By Lemma 1.1,

$$\begin{aligned} d(x, q_k^m(x)) &= d(x, q_{k+1}\dots q_m(x)) \\ &\le d(x, q_m(x)) + \dots + d(q_{k+2}\dots q_m(x), q_{k+1}\dots q_m(x)) \\ &\le 2^{-(m+1)} + \dots + 2^{-(k+2)} < 2^{-(k+1)} \le 1/2\,. \end{aligned}$$

Since $q_k^m(x) \in \langle \widehat{\sigma}_0, \dots, \widehat{\sigma}_k\rangle \subset \sigma_k$ and

$$d(q_k^m(x), y) \le d(x,y) + d(x, q_k^m(x)) < 1/2 + 1/2 = 1\,,$$

we have $\langle \widehat{\sigma}_0, \dots, \widehat{\sigma}_k\rangle \cap \langle \widehat{\sigma}'_0, \dots, \widehat{\sigma}'_{m'}\rangle \neq \emptyset$.

Now we write

$$\langle \widehat{\tau}_0, \dots, \widehat{\tau}_l\rangle = \langle \widehat{\sigma}_0, \dots, \widehat{\sigma}_m\rangle \cap \langle \widehat{\sigma}'_0, \dots, \widehat{\sigma}'_{m'}\rangle\,,$$

where $\tau_0 < \dots < \tau_l$. Then $\tau_0 \in L_0$ since $\tau_0 \le \sigma_k$. We define $z \in \langle \widehat{\tau}_0, \dots, \widehat{\tau}_l\rangle$ by

$$\begin{aligned} &z(\widehat{\sigma}) = \min\{x(\widehat{\sigma}), y(\widehat{\sigma})\} \quad \text{for each } \sigma \in K \setminus \{\tau_0\}\,, \\ &z(\widehat{\tau}_0) = 1 - \sum_{i=1}^{l} z(\widehat{\tau}_i)\,. \end{aligned}$$

Then $\langle x, z\rangle \subset \langle \widehat{\sigma}_0, \dots, \widehat{\sigma}_m\rangle$ and $\mathrm{diam}_d\langle x, z\rangle \le 2d_{\mathrm{H}}(A,B)$ because

$$\begin{aligned} d(x,z) &= |x(\widehat{\tau}_0) - z(\widehat{\tau}_0)| + \sum_{i=1}^{l} |x(\widehat{\tau}_i) - z(\widehat{\tau}_i)| + \sum_{\sigma_i \neq \tau_0,\dots,\tau_l} x(\widehat{\sigma}_i) \\ &= \Big|\sum_{\sigma_i \neq \tau_0} x(\widehat{\sigma}_i) - \sum_{i=1}^{l} z(\widehat{\tau}_i)\Big| + \sum_{i=1}^{l} |x(\widehat{\tau}_i) - z(\widehat{\tau}_i)| + \sum_{\sigma_i \neq \tau_0,\dots,\tau_l} x(\widehat{\sigma}_i) \\ &\le 2\Big(\sum_{i=1}^{l} |x(\widehat{\tau}_i) - z(\widehat{\tau}_i)| + \sum_{\sigma_i \neq \tau_0,\dots,\tau_l} x(\widehat{\sigma}_i)\Big) \\ &\le 2\Big(\sum_{i=1}^{l} |x(\widehat{\tau}_i) - y(\widehat{\tau}_i)| + \sum_{\sigma_i \neq \tau_0,\dots,\tau_l} x(\widehat{\sigma}_i) + \sum_{\sigma'_i \neq \tau_0,\dots,\tau_l} y(\widehat{\sigma}'_i)\Big) \\ &= 2d(x,y) \le 2d_{\mathrm{H}}(A,B)\,. \end{aligned}$$

For each $t \in \mathbf{I}$, let $x_t = (1-t)x + tz$. Then

$$x_t(\widehat{\sigma}) = (1-t)x(\widehat{\sigma}) + tz(\widehat{\sigma}) \le x(\widehat{\sigma}) \quad \text{for each } \sigma \in K \setminus \{\tau_0\}\,.$$

Hence $x_t(\widehat{\sigma}_m) \le \eta(\sigma_m)$. For each $j = k+1, \ldots, m$, by using Lemma 1.1 inductively, we have

$$q^m_{j-1}(x_t) = \sum_{i=0}^{j-1} \frac{x_t(\widehat{\sigma}_i)}{1 - (x_t(\widehat{\sigma}_j) + \ldots + x_t(\widehat{\sigma}_m))}\widehat{\sigma}_i\,.$$

Then for each $j = k+1, \ldots, m-1$,

$$\frac{x_t(\widehat{\sigma}_j)}{1 - (x_t(\widehat{\sigma}_{j+1}) + \ldots + x_t(\widehat{\sigma}_m))} \le \frac{x(\widehat{\sigma}_j)}{1 - (x(\widehat{\sigma}_{j+1}) + \ldots + x(\widehat{\sigma}_m))} \le \eta(\sigma_j)\,,$$

whence $q^m_j(x_t) \in \sigma_j(\eta(\sigma_j))$. On the other hand, $q^m_k(x_t) \in \sigma_k \subset N_0 \subset N_k$. By induction, $q^m_j(x_t) \in N_j$ for each $j > k$, so $x_t \in N_m \subset N_n$. Thus $\langle x, z\rangle \subset N_n$.

Similarly we have $\operatorname{diam}_d\langle y, z\rangle \le 2d_{\mathrm{H}}(A, B)$ and $\langle y, z\rangle \subset N_n$. Therefore $\langle x, z\rangle \cup \langle y, z\rangle \subset C\langle A, B\rangle$, whence $\langle x, z\rangle \cup \langle y, z\rangle \subset C$, which implies $B \cap C \ne \emptyset$. ■

By the definition of $C\langle A, B\rangle$ and the above lemma, $d_{\mathrm{H}}(A, C\langle A, B\rangle) \le 4d_{\mathrm{H}}(A, B)$ and $d_{\mathrm{H}}(B, C\langle A, B\rangle) \le 4d_{\mathrm{H}}(A, B)$ for each $A, B \in \mathcal{H}$.

5.3. Lemma. *Let $A_0 \in \mathfrak{K}(K)$ and $\varepsilon \in \mathcal{E}^{L(A_0)}_m$. If $A, B \in V(A_0, 2^{-(m+5)}, 2^{-5}\varepsilon)$ and $A \ne B$, then $\langle A, B\rangle \in \mathcal{H}$ and $C\langle A, B\rangle \in V(A_0, 2^m, \varepsilon)$.*

Proof. Since $2^{-5}\varepsilon \in \mathcal{E}^{L(A_0)}_{m+5}$, we have $V(A_0, 2^{-(m+5)}, 2^{-5}\varepsilon) \subset N_{d_{\mathrm{H}}}(A_0, 2^{-(m+4)})$ by Lemma 1.4(3). Then

$$d_{\mathrm{H}}(A, B) \le d_{\mathrm{H}}(A, A_0) + d_{\mathrm{H}}(B, A_0) < 2^{-(m+3)} < 1/2\,,$$

whence $\langle A, B\rangle \in \mathcal{H}$ and $d_{\mathrm{H}}(A, C\langle A, B\rangle) \le 4d_{\mathrm{H}}(A, B) < 2^{-(m+1)}$.

Let $\dim L(A \cup B) = n$. We use the notations from the definition of $C\langle A, B\rangle$ and simply write $p^{L(A_0)}_i = p_i$ and $p^n_i = p_{i+1} \ldots p_n$ ($p^n_n = \mathrm{id}$). Then $L_n = L(A \cup B) \subset K^{(n)}_{L(A_0)}$ and $A, B \subset C\langle A, B\rangle \subset |L_n| \subset |K^{(n)}_{L(A_0)}|$. Since $A, B \in V(A_0, 2^{-(m+5)}, 2^{-5}\varepsilon)$, we have $p^n_i(A \cup B) \subset W_i(L(A_0), 2^{-5}\varepsilon)$. First note that

$$p^n_n|(A \cup B) \setminus |L(A_0)| = \mathrm{id} = q^n_n|(A \cup B) \setminus |L(A_0)|\,.$$

Moreover, $S_n(L(A_0)) = R_n \setminus L(A_0)$. In fact, for each $\sigma \in S_n(L(A_0))$, we have $\sigma \notin L(A_0)$ and

$$(A \cup B) \cap \sigma \subset \sigma(2^{-5}\varepsilon(\sigma)) \subset \sigma(2^{-(m+n+6)}) \subset \sigma[2^{-(n+2)}],$$

whence $\sigma \in R_n \setminus L(A_0)$. Conversely, for each $\sigma \in R_n \setminus L(A_0)$, we have $(A \cup B) \cap \mathring{\sigma} \ne \emptyset$ and $A \cup B \subset W(L(A_0))$, whence $\sigma \cap |L(A_0)| \ne \emptyset$, that is, $\sigma \in S_n(L(A_0))$.

Assume that

$$p_i^n|(A \cup B) \setminus |L(A_0)| = q_i^n|(A \cup B) \setminus |L(A_0)|$$

and $S_i(L(A_0)) = R_i \setminus L(A_0)$. Then

$$p_i^n((A \cup B) \setminus |L(A_0)|) = q_i^n((A \cup B) \setminus |L(A_0)|) \subset |L_i|\,,$$

whence it follows that

$$p_i|p_i^n((A \cup B) \setminus |L(A_0)|) = q_i|q_i^n((A \cup B) \setminus |L(A_0)|)\,.$$

Since $p_{i-1}^n = p_i p_i^n$ and $q_{i-1}^n = q_i q_i^n$, we have

$$p_{i-1}^n|(A \cup B) \setminus |L(A_0)| = q_{i-1}^n|(A \cup B) \setminus |L(A_0)|\,.$$

Since $q_{i-1}^n((A \cup B) \cap |L(A_0)|) \subset |L(A_0)|$, it follows that

$$q_{i-1}^n(A \cup B) \subset p_{i-1}^n(A \cup B) \cup |L(A_0)| \subset W_{i-1}(L(A_0), 2^{-5}\varepsilon)\,.$$

Then for each $\sigma \in S_{i-1}(L(A_0))$, we have $\sigma \notin L(A_0)$ and

$$q_{i-1}^n(A \cup B) \cap \sigma \subset \sigma(2^{-5}\varepsilon(\sigma)) \subset \sigma(2^{-(m+i+5)}) \subset \sigma[2^{-(i+1)}]\,,$$

whence $\sigma \in R_{i-1} \setminus L(A_0)$. Conversely, for each $\sigma \in R_{i-1} \setminus L(A_0)$, $q_{i-1}^n(A \cup B) \cap \mathring{\sigma} \neq \emptyset$ and $q_{i-1}^n(A \cup B) \subset W(L(A_0))$, whence $\sigma \cap |L(A_0)| \neq \emptyset$, that is, $\sigma \in S_{i-1}(L(A_0))$. Hence $S_{i-1}(L(A_0)) = R_{i-1} \setminus L(A_0)$.

By induction, we have

$$p_i^n|(A \cup B) \setminus |L(A_0)| = q_i^n|(A \cup B) \setminus |L(A_0)|$$

and $S_i(L(A_0)) = R_i \setminus L(A_0)$ for each $i = 1, \dots, n$. It follows that

$$L_0 = L_n \setminus \bigcup_{i=1}^n R_i \subset L_n \setminus \bigcup_{i=1}^n S_i(L(A_0)) = L(A_0)\,.$$

Moreover, for each $\sigma \in S_i(L(A_0)) = R_i \setminus L(A_0)$,

$$q_i^n(A \cup B) \cap \sigma = p_i^n(A \cup B) \cap \sigma \subset \sigma(2^{-5}\varepsilon(\sigma))\,,$$

which implies $\eta(\sigma) \le 2^{-5}\varepsilon(\sigma) < 2^{-4}\varepsilon(\sigma)$. It follows that

$$C\langle A, B\rangle \subset N_n \subset W(L(A_0), 2^{-4}\varepsilon),$$

that is, $C\langle A, B\rangle \in \mathfrak{K}(W(L(A_0), 2^{-4}\varepsilon)) \subset \mathfrak{K}(W(L(A_0), \varepsilon))$. Since $2^{-4}\varepsilon \in \mathcal{E}_{m+4}^{L(A_0)}$, we obtain

$$d_{\mathrm{H}}(\pi^{L(A_0)}(C\langle A, B\rangle), C\langle A, B\rangle) < 2^{-(m+4)}$$

by Lemma 1.2. Thus we have

$$\begin{aligned} d_{\mathrm{H}}(A_0, \pi^{L(A_0)}(C\langle A, B\rangle)) &\le d_{\mathrm{H}}(A_0, A) + d_{\mathrm{H}}(A, C\langle A, B\rangle) \\ &\quad + d_{\mathrm{H}}(C\langle A, B\rangle, \pi^{L(A_0)}(C\langle A, B\rangle)) \\ &\le 2^{-(m+4)} + 2^{-(m+1)} + 2^{-(m+4)} < 2^{-m}\,. \end{aligned}$$

Therefore $C\langle A, B\rangle \in V(A_0, 2^{-m}, \varepsilon)$. ∎

Since each compact set in $|K|$ is contained in a metrizable continuum, the following is a consequence of [Ke, Lemma 2.3].

5.4. Lemma. *Let $A, C \in \mathfrak{K}(K)$ and $A \subset C$. If each component of $C$ meets $A$, then there exists a map $\varphi_{A,C} : \mathbf{I} \to \mathfrak{K}(K)$ such that $\varphi_{A,C}(0) = A$, $\varphi_{A,C}(1) = C$ and for each $t \in \mathbf{I}$, $A \subset \varphi_{A,C}(t) \subset C$ and each component of $\varphi_{A,C}(t)$ meets $A$.* ■

5.5. Lemma. *Let $n > 1$ and $\tau$ be an $n$-simplex. Then each map $f : \partial\tau \to \mathfrak{K}(K)$ extends to a map $\widetilde{f} : \tau \to \mathfrak{K}(K)$ such that*

$$f(\pi_\tau(x)) \subset \widetilde{f}(x) \subset \widetilde{f}(\widehat{\tau}) = \bigcup f(\partial\tau) = \bigcup_{y \in \partial\tau} f(y)$$

*for each $x \in \tau(1) = \tau \setminus \{\widehat{\tau}\}$. Moreover, if $f(\partial\tau) \subset \mathfrak{C}(K)$ then $\widetilde{f}(\tau) \subset \mathfrak{C}(K)$.*

Proof. Let $X = f(\partial\tau) \subset \mathfrak{K}(K)$. Then $X$ is a Peano continuum and $\mathfrak{C}(X) \subset \mathfrak{C}(\mathfrak{K}(K)) \subset \mathfrak{K}(\mathfrak{K}(K))$. As is shown in the proof of [Ke, Theorem 3.3], $X$ has a homotopy $h : X \times \mathbf{I} \to \mathfrak{C}(X)$ such that

$$h_0(x) = \{x\} \subset h_t(x) \subset h_1(x) = X \quad \text{for each } x \in X \text{ and } t \in \mathbf{I}.$$

On the other hand, we have the map $\varsigma : \mathfrak{K}(\mathfrak{K}(K)) \to \mathfrak{K}(K)$ defined by $\varsigma(\mathcal{A}) = \bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$ (cf. [Ke]). Then $\widetilde{f} : \tau \to \mathfrak{K}(K)$ can be defined by

$$\widetilde{f}(x) = \begin{cases} \varsigma(f(\partial\tau)) & \text{if } x = \widehat{\tau}, \\ \varsigma \circ h(f(\pi_\tau(x)), x(\widehat{\tau})) & \text{otherwise.} \end{cases}$$

Since $\varsigma(\mathcal{A}) \in \mathfrak{C}(K)$ for any $\mathcal{A} \in \mathfrak{C}(\mathfrak{K}(K))$ with $\mathcal{A} \cap \mathfrak{C}(K) \neq \emptyset$ by [Ke, Lemma 1.2], we have the additional statement. ■

Now we prove Theorem 5.1.

Proof of Theorem 5.1. For simplicity, we write $Z(\mathfrak{K}(K)) = Z$, $Z_1(\mathfrak{K}(K)) = Z_1$ and $F(\mathfrak{K}(K)) = F$. First, we construct a retraction $r_1 : Z_1 \to \mathfrak{K}(K)$. For each $\langle A, B\rangle \in \mathcal{H}$, we have defined $C\langle A, B\rangle \in \mathfrak{K}(K)$. For each $\langle A, B\rangle \in F^{(1)} \setminus \mathcal{H}$, choose $C\langle A, B\rangle \in \mathfrak{C}(K)$ so that $A \cup B \subset C\langle A, B\rangle$. By using Lemma 5.4, we can define $r_1$ as follows: $r_1|\mathfrak{K}(K) = \mathrm{id}$ and

$$r_1((1-t)A + tB) = \begin{cases} A & \text{if } 0 \le t \le 1/4, \\ \varphi_{A,C\langle A,B\rangle}(4t-1) & \text{if } 1/4 \le t \le 1/2, \\ \varphi_{B,C\langle A,B\rangle}(3-4t) & \text{if } 1/2 \le t \le 3/4, \\ B & \text{if } 3/4 \le t \le 1, \end{cases}$$

for each 1-simplex $\langle A, B\rangle \in F$. If $A$ and $B$ are connected, each $r_1((1-t)A + tB)$ is also connected. Thus $r_1(Z_1(\mathfrak{C}(K))) = \mathfrak{C}(K)$.

We have to show that $r_1$ is continuous. Since $Z_1 \setminus \mathfrak{K}(K)$ is a subspace of $|F^{(1)}|$ and $r_1|\langle A, B\rangle$ is continuous for each $\langle A, B\rangle \in F^{(1)}$, $r_1|Z_1 \setminus \mathfrak{K}(K)$ is continuous. Since $Z_1 \setminus \mathfrak{K}(K)$ is open in $Z_1$, $r_1$ is continuous at each point of $Z_1 \setminus \mathfrak{K}(K)$.

To see the continuity of $r_1$ at each point $A_0 \in \mathfrak{K}(K)$, let $\mathcal{V}$ be a neighborhood of $A_0$ in $\mathfrak{K}(K)$. Choose $m \in \mathbb{N}$ and $\varepsilon \in \mathcal{E}_m^{L(A_0)}$ so that $V(A_0, 2^{-m}, \varepsilon) \subset \mathcal{V}$. Then

$$r_1(M_1(V(A_0, 2^{-(m+5)}, 2^{-5}\varepsilon), 1/2)) \subset \mathcal{V}.$$

In fact, let $\langle A, B\rangle \in F^{(1)}$, $A \neq B$ and $A \in V(A_0, 2^{-(m+5)}, 2^{-5}\varepsilon)$. In case $B \in V(A_0, 2^{-(m+5)}, 2^{-5}\varepsilon)$, $\langle A, B\rangle \in \mathcal{H}$ and $C\langle A, B\rangle \in V(A_0, 2^{-m}, \varepsilon)$ by Lemma 5.3. By Lemma 1.4(2), we have $\varphi_{A,C\langle A,B\rangle}(t) \in V(A_0, 2^{-m}, \varepsilon)$ and $\varphi_{B,C\langle A,B\rangle}(t) \in V(A_0, 2^{-m}, \varepsilon)$ for $t \in \mathbf{I}$. Then $r_1(\langle A, B\rangle) \subset V(A_0, 2^{-m}, \varepsilon)$. In case $B \notin V(A_0, 2^{-(m+5)}, 2^{-5}\varepsilon)$, let $\tau = \langle A, B\rangle$. Then $r_1((1-t)A + t\hat{\tau}) = r_1((1-t/2)A + (t/2)B) = A \in \mathcal{V}$ for each $t \in [0, 1/2]$.

Next, by the skeletonwise induction applying Lemma 5.5 at each step, we can extend $r_1$ to $r : Z \to \mathfrak{K}(K)$ such that

$$r(\pi_\tau(x)) \subset r(x) \subset r(\hat{\tau}) = \bigcup r(\partial\tau) = \bigcup_{y \in \partial\tau} r(y)$$

if $x \in \tau(1) \subset \tau \in F$. Since $r_1(Z_1(\mathfrak{C}(K))) = \mathfrak{C}(K)$, it follows that $r(Z(\mathfrak{C}(K))) = \mathfrak{C}(K)$.

We have to show that $r : Z \to \mathfrak{K}(K)$ is continuous. Since $Z \setminus \mathfrak{K}(K)$ is a subspace of $|F|$ and $r|\tau$ is continuous for each $\tau \in F$, $r|Z \setminus \mathfrak{K}(K)$ is continuous. Since $Z \setminus \mathfrak{K}(K)$ is open in $Z$, $r$ is continuous at each point of $Z \setminus \mathfrak{K}(K)$.

To see the continuity of $r$ at each $A \in \mathfrak{K}(K)$, let $\mathcal{V}$ be a neighborhood of $A$ in $\mathfrak{K}(K)$. We may assume that $\mathcal{V} = V(A, \delta, \varepsilon)$ for some $\delta > 0$ and $\varepsilon \in (0,1)^{S(L(A))}$. Then $r(\partial\tau) \subset \mathcal{V}$ implies $r(\tau) \subset \mathcal{V}$ for each $\tau \in F$ by Lemma 1.4. By the continuity of $r_1$, $r^{-1}(\mathcal{V}) \cap Z_1 = r_1^{-1}(\mathcal{V})$ is a neighborhood of $A$ in $Z_1$. By the topologization of $Z$, there is an open set $\mathcal{W}$ in $|F|$ such that $\mathcal{W} \cap \mathfrak{K}(K)$ is open in $\mathfrak{K}(K)$, $|F(\mathcal{W} \cap \mathfrak{K}(K))| \subset \mathcal{W}$ and $A \in \mathcal{W} \cap Z_1 \subset r^{-1}(\mathcal{V}) \cap Z_1$.

Let $\mathcal{U} = \mathcal{W} \cap \mathfrak{K}(K)$. Then $|F(\mathcal{U})| \subset r^{-1}(\mathcal{V})$, that is, $r(\tau) \subset \mathcal{V}$ for each $\tau \in F(\mathcal{U})$. In fact, this can be shown by induction on $\dim\tau$ since $r(\partial\tau) \subset \mathcal{V}$ implies $r(\tau) \subset \mathcal{V}$ and if $\dim\tau = 1$ then $\tau \subset \mathcal{W} \cap Z_1 \subset r^{-1}(\mathcal{V})$, i.e., $r(\tau) \subset \mathcal{V}$. On the other hand, $r^{-1}(\mathcal{V}) \cap \tau = (r|\tau)^{-1}(\mathcal{V})$ is open in $\tau$ for any $\tau \in F \setminus F^{(0)}_{F(\mathcal{U})}$. Let $V_\tau = \tau$ for all $\tau \in F(\mathcal{U})$ and $V_\tau = \emptyset$ for all $\tau \in F^{(0)} \setminus F(\mathcal{U})$. Similarly to the proof of Lemma 4.1, we can define $\eta \in (0,1)^{T(\mathcal{U})}$ so that

$$V_\tau = \pi_\tau^{-1}(V_{\partial\tau}) \cap \operatorname{cl}\tau(\eta(\tau)) \subset r^{-1}(\mathcal{V}) \cap \tau,$$

where $V_{\partial\tau} = \bigcup_{\tau' < \tau} V_{\tau'}$. Thus we have a neighborhood $M(\mathcal{U}, \eta)$ of $A$ in $Z$ such that

$$M(\mathcal{U}, \eta) \subset |F(\mathcal{U})| \cup \bigcup\{V_\tau \mid \tau \in T(\mathcal{U})\} \subset r^{-1}(\mathcal{V}).$$

Therefore $r : Z \to \mathfrak{K}(K)$ is continuous at $A \in \mathfrak{K}(K)$. ■

**Appendix.** Let $\mathcal{K}$ be the class of compact Hausdorff spaces. Here we show the following:

PROPOSITION. *For any connected CW-complex $X$, $\mathfrak{K}(X)$ and $\mathfrak{C}(X)$ are $AE(\mathcal{K})$'s. Hence for any CW-complex $X$, $\mathfrak{K}(X)$ and $\mathfrak{C}(X)$ are $ANE(\mathcal{K})$'s.*

Proof. Let $Z \in \mathcal{K}$ and $f : A \to \mathfrak{K}(X)$ a map from a closed set $A$ in $Z$. Since $\varsigma(f(A)) = \bigcup f(A)$ is a compact set in $X$, we have $f(A) \subset \mathfrak{K}(Y)$ for some compact connected subcomplex $Y$ of $X$, whence $Y$ is a Peano continuum. Since $\mathfrak{K}(Y)$ is an AE for normal spaces (in fact, $\mathfrak{K}(Y)$ is homeomorphic to the Hilbert cube ([CS$_1$] or [CS$_2$])), $f$ extends to a map $\widetilde{f} : Z \to \mathfrak{K}(Y) \subset \mathfrak{K}(X)$. Hence $\mathfrak{K}(X)$ is an AE($\mathcal{K}$). ■

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