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# Relatively recursive expansions II

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**Abstract.** In [AK], we asked when a recursive structure $\mathcal{A}$ and a sentence $\varphi$, with a new relation symbol, have the following property: for each $\mathcal{B} \cong \mathcal{A}$ there is a relation $S$ such that $S$ is recursive relative to $\mathcal{B}$ and $(\mathcal{B}, S) \models \varphi$. Here we consider several related properties, in which there is a uniform procedure for determining $S$ from $\mathcal{B} \cong \mathcal{A}$, or from $(\mathcal{B}, \overline{b}) \cong (\mathcal{A}, \overline{a})$, for some fixed sequence of parameters $\overline{a}$ from $\mathcal{A}$; or in which $\mathcal{B}$ and $S$ are required to be recursive. We investigate relationships between these properties, showing that for certain kinds of sentences $\varphi$, some of these properties do or do not imply others. Many questions are left open.

We consider only those structures having universe $\omega$. In [AK] we considered the question, for a recursive $L$-structure $\mathcal{A}$ and a recursive infinitary sentence $\varphi(P)$ of $L \cup \{P\}$, whether every isomorphic copy $\mathcal{B}$ of $\mathcal{A}$ allows a relation $S$ on $\mathcal{B}$, recursive relative to (the diagram of) $\mathcal{B}$, for which $\varphi(P)$ is true in $(\mathcal{B}, S)$. We showed that, for a recursive $\Pi_2$ sentence $\varphi(P)$, this was true if and only if a corresponding recursive-syntactical condition held for $\varphi(P)$ and $\mathcal{A}$. It was observed by Slaman that, in this case, it follows that there is a *uniform* procedure which obtains some such $S$ recursively from $\mathcal{B}$.

Having recognised this, we see that there are several senses in which one may consider whether "every copy of $\mathcal{A}$ has a relatively recursive $S$ satisfying $\varphi(P)$".

DEFINITIONS. Let $\mathcal{A}$ be a recursive structure with universe $\omega$ and let $\varphi(P)$ be an infinitary sentence. We say that $\varphi(P)$ has the property A (respectively, UA, UAP, R, UR, URP) on $\mathcal{A}$ if the following holds:

(A) On every isomorphic copy of $\mathcal{A}$ there exists a relatively recursive relation satisfying $\varphi(P)$.

(UA) There is a uniform recursive procedure which yields from each isomorphic copy of $\mathcal{A}$ a relatively recursive relation satisfying $\varphi(P)$.

(That is, there is an index $e$ such that, for each $\mathcal{B} \cong \mathcal{A}$, $\varphi_e^{\mathcal{B}}$ is the characteristic function of a relation $S$ for which $(\mathcal{B}, S) \models \varphi(P)$.)

(UAP) For some finite sequence $\overline{a}$ from $\mathcal{A}$, $\varphi(P)$ has the property UA on $(\mathcal{A}, \overline{a})$.

(R) On every *recursive* copy of $\mathcal{A}$ there exists a *recursive* relation satisfying $\varphi(P)$.

(UR) There is a uniform recursive procedure which yields, from each *recursive* isomorphic copy of $\mathcal{A}$, a *recursive* relation satisfying $\varphi(P)$.

(URP) For some finite sequence $\overline{a}$ from $\mathcal{A}$, $\varphi(P)$ has the property UR on $(\mathcal{A}, \overline{a})$.

In this paper we consider which implications between these properties do or do not hold, depending on the complexity of the sentence $\varphi(P)$. The class $\Sigma_0 = \Pi_0$ consists of precisely the open finitary formulas. The class $\Sigma_{n+1}$ consists of formulas $\bigvee\!\!\!\bigvee_i (\exists \overline{y}_i)\varphi_i$ where each $\varphi_i$ is $\Pi_n$, and the class $\Pi_{n+1}$ consists of formulas $\bigwedge\!\!\!\bigwedge_i (\forall \overline{y}_i)\varphi_i$ where each $\varphi_i$ is $\Sigma_n$. A formula is a *recursive* $\Sigma_n$ (similarly $\Pi_n$) formula if it is in the $\Sigma_n$ ($\Pi_n$) form and each disjunction or conjunction appearing is recursive, using a system of Gödel numbers, defined simultaneously.

We give, in §2, some implications we have found and, in §3, some examples. In §4, we tabulate these results by considering all Boolean combinations of the conditions A, UA, UAP, R, UR, URP. The gaps in this table represent unanswered questions.

**1. Theorems.** In this section we prove four positive results:

THEOREM 1. *If $\varphi(P)$ is a recursive $\Sigma_1$ sentence and $\mathcal{A}$ has some expansion satisfying $\varphi(P)$, then $\varphi(P)$ has property* UA *on $\mathcal{A}$.*

THEOREM 2. *If $\varphi(P)$ is a recursive $\Sigma_2$ sentence having property* UR *on $\mathcal{A}$, then $\varphi(P)$ also has property* UAP *on $\mathcal{A}$.*

THEOREM 3. *If $\varphi(P)$ is a recursive $\Pi_2$ sentence having property* UR *on $\mathcal{A}$, then $\varphi(P)$ also has property* UA *on $\mathcal{A}$.*

THEOREM 4. *If $\varphi(P)$ is a recursive $\Sigma_3$ sentence having property* A *on $\mathcal{A}$, then $\varphi(P)$ also has property* UAP *on $\mathcal{A}$.*

We begin by proving Theorem 1.

THEOREM 1. *If $\varphi(P)$ is a recursive $\Sigma_1$ sentence satisfiable on a recursive structure $\mathcal{A}$, then $\varphi(P)$ has property* UA *on $\mathcal{A}$.*

Proof. It is possible to express $\varphi(P)$ in the form

$$\bigvee\!\!\!\bigvee_n (\exists \overline{x}_n)(\psi_n(\overline{x}_n) \,\&\, \Theta_n(\overline{x}_n))$$

where each $\psi_n(\overline{x}_n)$ is a conjunction of negated or unnegated atomic formulas of $L$ and each $\Theta_n(\overline{x}_n)$ is a conjunction of negated or unnegated instances of $P$ or $\neg P$.

Since $\varphi(P)$ is satisfiable on $\mathcal{A}$, so is $(\exists\overline{x}_n)(\psi_n(\overline{x}_n)\,\&\,\Theta_n(\overline{x}_n))$ for some $n$. Then for any $\mathcal{B}\cong\mathcal{A}$, we may find a suitable relation on $\mathcal{B}$ by searching for elements $\bar{b}$ such that $\mathcal{B}\models\psi_n(\bar{b})$ and defining $S$ so that $\mathcal{B}\models\Theta_n(\bar{b})$. Clearly this procedure is uniform in (the diagram of) $\mathcal{B}$.

Comment. We assume, here and elsewhere, that $\varphi(P)$ is rewritten so that function and constant symbols are treated as relation symbols.

In the next few results, we shall use a kind of forcing which produces generic copies of a given $L$-structure $\mathcal{A}$. This kind of forcing was developed in [AKMS] and applied in [AK]. The forcing language includes the symbols of $L$, constant symbols naming the natural numbers ($\omega$ is the universe of all the structures considered) and a function symbol $f$, standing for a permutation of $\omega$ which will be the isomorphism from $\mathcal{B}$ to $\mathcal{A}$.

It is possible to use the forcing language to make various statements about the copy $\mathcal{B}$. There are sentences with the following intended meanings:

(1) $\mathcal{B}\models\alpha(\bar{b})$, where $\alpha(\overline{x})$ is a quantifier-free formula in the language $L$,
(2) $\varphi_e^{\mathcal{B}}(k)=j$,
(3) $\varphi_e^{\mathcal{B}}(k)\downarrow$,
(4) $\varphi_e^{\mathcal{B}}$ is total $0,1$-valued.

The first sentence, saying $\mathcal{B}\models\alpha(\bar{b})$, is just $\alpha(f(\bar{b}))$. The second sentence, saying $\varphi_e^{\mathcal{B}}(k)=j$, is a recursive disjunction of sentences like the first. There is one disjunct for each computation that starts with input $k$ and halts with output $j$, having asked questions about atomic sentences in the language of $D(\mathcal{B})$. The disjunct carries the answers to the questions. Having indicated how the symbol $f$ is used in writing sentence (1) formally, we shall not need to use $f$ further.

We shall write simply $\alpha(\bar{b})$, $\varphi_e^D(k)=j$, $\varphi_e^D(k)\downarrow$, and "$\varphi_e^D$ is total $0,1$-valued" for the sentences (1), (2), (3), and (4).

Let $P$ be a relation symbol not in the language $L$, and let $\psi$ be a sentence involving $P$. We want to be able to say in our language that $\varphi_e^{\mathcal{B}}$ is the characteristic function of a relation $P$ such that $(\mathcal{B},P)\models\psi$. Let $\psi^e$ be the result of replacing all positive occurrences of $P\bar{b}$ in $\psi$ by $\varphi_e^D(\bar{b})=1$, and all negative occurrences by $\varphi_e^D(\bar{b})=0$.

The forcing conditions are finite partial permutations of $\omega$, thought of as having domain in the generic copy $\mathcal{B}$ and range in the given structure $\mathcal{A}$. We write $p,q$, etc. for forcing conditions, and we write $p\Vdash^{\mathcal{A}}\varphi$ to indicate that $p$ *forces* $\varphi$, and $\mathcal{A}$ is the structure being copied. Forcing is defined in a fairly standard way except for sentences of the first kind. If $\alpha(\overline{x})$ is a quantifier-free formula in the language $L$, then $p \Vdash^{\mathcal{A}} \alpha(\bar{b})$ iff $\bar{b}$ is included in $\mathrm{dom}(p)$ and $\mathcal{A} \models \alpha(p(\bar{b}))$.

We shall prove several lemmas.

LEMMA A. *Let $\mathcal{A}$ be a recursive structure, and suppose that for all recursive $\mathcal{B} \cong \mathcal{A}$, $\varphi_e^{\mathcal{B}}$ is total $0,1$-valued. Then $\phi \Vdash^{\mathcal{A}} \varphi_e^D$ is total $0,1$-valued.*

Proof. We must show that for all forcing conditions $p$ and all $k$, there exists $q \supseteq p$ such that $q \Vdash^{\mathcal{A}} \varphi_e^D(k) = 0$ or $q \Vdash^{\mathcal{A}} \varphi_e^D(k) = 1$. There is a recursive copy $\mathcal{B}$ of $\mathcal{A}$ with an isomorphism $f$ from $\mathcal{B}$ to $\mathcal{A}$ such that $f \supseteq p$. Since $\varphi_e^{\mathcal{B}}(k)$ converges to 0 or 1 for this $\mathcal{B}$, there is some $q$ such that $p \subseteq q \subseteq f$ and $q \Vdash^{\mathcal{A}} \varphi_e^D(k) = 0$ or $q \Vdash^{\mathcal{A}} \varphi_e^D(k) = 1$. (We put into $\mathrm{dom}(q)$ all elements mentioned in the part of the diagram needed for the computation.)

LEMMA B. *Let $\mathcal{A}$ be recursive. Let $P$ be a new relation symbol, and let $\psi$ be a (recursive) $\Pi_2$ sentence involving $P$. Suppose that $\psi$ has property* UR *on $\mathcal{A}$, where the uniform procedure has index $e$. Then $\phi \Vdash^{\mathcal{A}} \psi^e$. Also, $\phi \Vdash^{\mathcal{A}} \varphi_e^D$ is total $0,1$-valued.*

Proof. The idea is like that in Lemma A.

Let $\psi = \bigwedge\!\!\!\bigwedge_i (\forall \overline{x}_i) \bigvee\!\!\!\bigvee_j (\exists \overline{y}_j) \delta_{ij}(\overline{x}_i, \overline{y}_j)$. To show that $\phi \Vdash^{\mathcal{A}} \psi^e$, we must show that for any $p, i$, and $\bar{b}$ (appropriate to substitute for $\overline{x}_i$), there exist $q \supseteq p, j$, and $\bar{d}$ such that $q \Vdash^{\mathcal{A}} \delta_{ij}(\overline{c}, \bar{d})^e$. We take $q$ such that $p \subseteq q \subseteq f$ and $q$ carries the information needed to compute $\varphi_e^{\mathcal{B}}(\overline{u})$ for $\overline{u}$ in $\overline{c}^\wedge \bar{d}$.

When we say that $\psi$ has property UR, as witnessed by $e$, this implies that for all recursive $\mathcal{B} \cong \mathcal{A}$, $\varphi_e^{\mathcal{B}}$ is total $0,1$-valued, so by Lemma A, $\phi \Vdash^{\mathcal{A}} \varphi_e^D$ is total $0,1$-valued.

In Lemmas A and B, we had in mind building a generic copy of a fixed structure $\mathcal{A}$, and the forcing conditions had range in $\mathcal{A}$. The statement $p \Vdash^{\mathcal{A}} \varphi$ may be thought of as saying that $\overline{a} = \mathrm{ran}(p)$ has a certain property in $\mathcal{A}$.

We could substitute another $L$-structure $\mathcal{C}$ for $\mathcal{A}$ (still with universe $\omega$). We may regard $p$ as a forcing condition for building generic copies of $\mathcal{C}$. We write $p \Vdash^{\mathcal{C}} \varphi$ when we are thinking of $p$ in this way and we are saying that $\overline{a} = \mathrm{ran}(p)$ has a certain property in $\mathcal{C}$. Then the following fact is obvious.

LEMMA C. *If $\phi \Vdash^{\mathcal{A}} \varphi$, then for all $\mathcal{C} \cong \mathcal{A}$, $\phi \Vdash^{\mathcal{C}} \varphi$.*

LEMMA D. *Let $\mathcal{A}$ be a fixed structure, let $P$ be a new relation symbol, and let $\psi$ be a recursive $\Pi_2$ sentence involving $P$. Suppose $\phi \Vdash^{\mathcal{A}} \varphi_e^D$ is total $0,1$-valued, and $\phi \Vdash^{\mathcal{A}} \psi^e$. Then there is a uniform procedure which, when applied to any $\mathcal{C} \cong \mathcal{A}$, yields a function $f$ and a structure $\mathcal{B}$ such that*

(1) *$f \leq_T \mathcal{C}$ and $\mathcal{B} \leq_T \mathcal{C}$,*
(2) *$\mathcal{B} \cong_f \mathcal{C}$,*
(3) *$\varphi_e^{\mathcal{B}}$ is the characteristic function of a relation $P$ such that $(\mathcal{B}, P) \models \psi$.*

P r o o f. By Lemma C, $\phi \Vdash^{\mathcal{C}} \varphi_e^D$ is total 0, 1-valued, and $\phi \Vdash^{\mathcal{C}} \psi^e$. Let $\psi = \bigwedge\!\!\!\bigwedge_i (\forall \overline{x}_i) \bigvee\!\!\!\bigvee_j (\exists \overline{y}_j) \delta_{ij}(\overline{x}_i, \overline{y}_j)$. The universe of $\mathcal{C}$ and $\mathcal{B}$ will be $\omega$, but we write $c_0, c_1, c_2, \ldots$ for the elements of $\mathcal{C}$, and we think of $b_0, b_1, b_2, \ldots$ as constants naming the elements of $\mathcal{B}$.

Let $(i_n, \bar{b}_n)_{n\in\omega}$ be a recursive list of the pairs $(i, \bar{b})$ such that $\bar{b}$ is appropriate to substitute for $\overline{x}_i$. Focusing on the first pair $(i_0, \bar{b}_0)$, we enumerate $D(\mathcal{C})$, searching for $j_0, \bar{d}_0$, and $p_0$ such that $p_0 \Vdash^{\mathcal{C}} \delta_{i_0 j_0}(\bar{b}_0, \bar{d}_0)^e$. We extend $p_0$, if necessary, so that the elements of $\bar{b}_0$ and $\bar{d}_0$ are in $\mathrm{dom}(p_0)$ and $c_0 \in \mathrm{ran}(p_0)$. Given $p_n$, we consider $(i_{n+1}, \bar{b}_{n+1})$ and search for $j_{n+1}, \bar{d}_{n+1}$, and $p_{n+1}$ such that $p_{n+1} \supseteq p_n$ and $p_{n+1} \Vdash^{\mathcal{C}} \delta_{i_{n+1} j_{n+1}}(\bar{b}_{n+1}, \bar{d}_{n+1})^e$. We extend $p_{n+1}$ so that the elements of $\bar{b}_{n+1}, \bar{d}_{n+1}$ are in $\mathrm{dom}(p_{n+1})$, and $c_{n+1} \in \mathrm{ran}(p_{n+1})$. In addition, we make sure that

$$p_{n+1} \Vdash^{\mathcal{C}} \varphi_e^D(n) = 0 \quad \text{ or } \quad p_{n+1} \Vdash^{\mathcal{C}} \varphi_e^D(n) = 1\,.$$

Now, let $f = \bigcup_{n\in\omega} p_n$, and let $\mathcal{B}$ be the structure such that $\mathcal{B} \cong_f \mathcal{C}$. We have $f$ and $\mathcal{B}$ recursive in $\mathcal{C}$. We made sure that $\varphi_e^{\mathcal{B}}$ is total 0, 1-valued, and is the characteristic function of a relation $P$ such that $(\mathcal{B}, P) \models \delta_{i_n j_n}(\bar{b}_n, \bar{d}_n)$ for all $n$. Therefore, $(\mathcal{B}, P) \models \psi$. The procedure described above for obtaining $f$ and $\mathcal{B}$ is clearly uniform in $\mathcal{C}$.

We now have all of the lemmas we shall need for the next two theorems.

THEOREM 2. *Let $\mathcal{A}$ be recursive. If $\varphi(P)$ is a recursive $\Sigma_2$ sentence having property* UR *on $\mathcal{A}$, then $\varphi(P)$ also has property* UAP *on $\mathcal{A}$.*

P r o o f. Let $e$ witness the fact that $\varphi(P)$ has property UR on $\mathcal{A}$. By Lemma A, $\phi \Vdash^{\mathcal{A}} \varphi_e^D$ is total 0, 1-valued.

CLAIM. *For some forcing condition $p$, $p \Vdash^{\mathcal{A}} \varphi(P)^e$.*

Suppose that the claim is true. Let $\varphi(P) = \bigvee\!\!\!\bigvee_i (\exists \overline{x}_i) \bigwedge\!\!\!\bigwedge_j (\forall \overline{y}_j) \delta_{ij}(\overline{x}_i, \overline{y}_j)$. Then for some $i$ and $\bar{b}$, we have $p \Vdash^{\mathcal{A}} \psi_i(\bar{b})^e$, where $\psi_i(\bar{b})$ is the $\Pi_1$ sentence $\bigwedge\!\!\!\bigwedge_j (\forall \overline{y}_j) \delta_{ij}(\bar{b}, \overline{y}_j)$. Let $\bar{b}_1, \overline{a}_1$ be the domain and range of $p$. We may assume that $\bar{b} \subseteq \bar{b}_1$, and let $\overline{a} = p(\bar{b})$.

At this point, we appeal to the proof of Theorem 1.1 of [AK]. This proof describes a uniform procedure which, when applied to any $(\mathcal{C}, \overline{c}_1) \cong (\mathcal{A}, \overline{a}_1)$, yields a relation $P \leq_T \mathcal{C}$ such that if $\overline{c}$ is the part of $\overline{c}_1$ corresponding to $\overline{a}$, then $(\mathcal{C}, P) \models \psi_i(\overline{c})$, so $(\mathcal{C}, P) \models \varphi(P)$.

All that remains is to prove the claim.

P r o o f o f C l a i m. Suppose there is no $p$ such that $p \Vdash^{\mathcal{A}} \varphi(P)^e$. Then for each $p, i$, and $\bar{b}$, there exist $q \supseteq p$, $j$, and $\bar{d}$ such that for no $r \supseteq q$ do we have $r \Vdash^{\mathcal{A}} \delta_{ij}(\overline{c}, \bar{d})^e$. Since $\phi \Vdash^{\mathcal{A}} \varphi_e^D$ is total 0, 1-valued, we can take $q$ such that $q \Vdash^{\mathcal{A}} \neg\delta_{ij}(\overline{c}, \bar{d})^e$. The negation of $\varphi(P)$ is logically equivalent to the recursive $\Pi_2$ sentence $\varphi^*(P) = \bigwedge\!\!\!\bigwedge_i(\forall\overline{x}_i)\bigvee\!\!\!\bigvee_j(\exists\overline{y}_j)\neg\delta_{ij}(\overline{x}_i,\overline{y}_j)$, and we have $\phi \Vdash^{\mathcal{A}} \varphi^*(P)$.

We apply Lemma D to the structure $\mathcal{A}$ itself to get a recursive copy $\mathcal{B}$ of $\mathcal{A}$ and a recursive $f$ such that $\mathcal{B} \cong_f \mathcal{A}$ and $\varphi_e^{\mathcal{B}}$ is the characteristic function of a relation $P$ for which we have $(\mathcal{B}, P) \models \varphi^*(P)$. This contradicts the assumption that $\varphi(P)$ has property UR on $\mathcal{A}$ as witnessed by $e$.

THEOREM 3. *If $\varphi(P)$ is a recursive $\Pi_2$ sentence having property* UR *on $\mathcal{A}$, then $\varphi(P)$ also has property* UA *on $\mathcal{A}$.*

Proof. Let $e$ witness the fact that $\varphi(P)$ has property UR on $\mathcal{A}$. By Lemmas A and B, $\phi \Vdash^{\mathcal{A}} \varphi_e^D$ is total 0, 1-valued, and $\phi \Vdash^{\mathcal{A}} \varphi(P)^e$. We apply the procedure from Lemma D to an arbitrary $\mathcal{C} \cong \mathcal{A}$ to get $\mathcal{B}$ and $f$, both recursive in $\mathcal{C}$, such that $\mathcal{B} \cong_f \mathcal{C}$ and $\varphi_e^{\mathcal{B}}$ is the characteristic function of a relation $P$ such that $(\mathcal{B}, P) \models \varphi(P)$. If $Q = f(P)$, then $(\mathcal{C}, Q) \models \varphi(P)$. The procedure for obtaining $Q$ is clearly uniform in $\mathcal{C}$.

THEOREM 4. *If $\varphi(P)$ is a recursive $\Sigma_3$ sentence with property* A *on $\mathcal{A}$, then $\varphi(P)$ also has property* UAP *on $\mathcal{A}$.*

Proof. Assume $\varphi(P)$ is $\Sigma_3$ and has property A on $\mathcal{A}$. So for each $\mathcal{B} \cong \mathcal{A}$ there exists $S \leq_T \mathcal{B}$ with $(\mathcal{B}, S) \models \varphi(P)$. Then, in particular, if $\mathcal{B}$ is a generic copy of $\mathcal{A}$ then some $p \Vdash^{\mathcal{A}} \varphi(P)^e$. Say that $\varphi(P)$ is $\bigvee\!\!\!\bigvee_i(\exists\overline{x}_i)\psi_i(\overline{x}_i)$ where each $\psi_i$ is $\Pi_2$. Then $p \Vdash^{\mathcal{A}} \psi_i(\bar{b}^*)^e$ for some $i$ and some $\bar{b}^* \in \mathcal{B}$. We may suppose that $\bar{b}^* \in \mathrm{dom}(p)$, say $p(\bar{b}^*) = \overline{a}^*$. Let $\overline{a} = \mathrm{ran}(p)$. Now, applying the method of Theorem 1.1 of [AK] to the structure $(\mathcal{A}, \overline{a})$ and the $\Pi_2$ sentence $\psi_i(\overline{a}^*)$, we get a formally $\Sigma_1^0$ expansion family for $\psi_i(\overline{a}^*)$ on $(\mathcal{A}, \overline{a})$ involving parameters $\overline{a}$. Then there is a uniform procedure which, when applied to an arbitrary $(\mathcal{B}, \overline{c}) \cong (\mathcal{A}, \overline{a})$, yields $S \leq_T \mathcal{B}$ with $(\mathcal{B}, S) \models \psi_i(\overline{c}^*)$, where $\overline{c}^*$ is the part of $\overline{c}$ corresponding to $\overline{a}^*$. Therefore, $(\mathcal{B}, S) \models \varphi(P)$.

**2. Examples.** In this section we show that there exist recursive sentences $\varphi(P)$ and recursive structures $\mathcal{A}$, with $\varphi(P)$ satisfiable by a recursive relation on $\mathcal{A}$, providing examples as indicated:

EXAMPLE 1. $\varphi(P)$ is $\Pi_1$ and does not have property R on $\mathcal{A}$.

EXAMPLE 2. $\varphi(P)$ is $\Pi_2$ and has property UAP but not UR on $\mathcal{A}$.

EXAMPLE 3. $\varphi(P)$ is $\Sigma_2$ and has property UAP but not UR on $\mathcal{A}$.

EXAMPLE 4. $\varphi(P)$ is $\Pi_2$ and has property R but not A on $\mathcal{A}$.

EXAMPLE 5. $\varphi(P)$ is $\Sigma_3$ and has property UR but not A on $\mathcal{A}$.

EXAMPLE 6. $\varphi(P)$ is $\Sigma_2$ and has property UR but not UA on $\mathcal{A}$.

EXAMPLE 7. $\varphi(P)$ is $\Pi_3$ and has property UR but not A on $\mathcal{A}$.

We produce the examples, in order.

EXAMPLE 1. *There exist a recursive $\Pi_1$ sentence $\varphi(P)$ and a recursive structure $\mathcal{A}$ such that $\varphi(P)$ is satisfiable by a recursive relation on $\mathcal{A}$ but $\varphi(P)$ does not have property* R *on $\mathcal{A}$.*

Proof. Take $\mathcal{A}$ to be a graph $(A, N)$ consisting of infinitely many disconnected copies of $\mathbb{Z}$, where vertices are adjacent when one is the successor of the other in the same copy of $\mathbb{Z}$.

Take $\varphi(P)$ to be the $\Pi_1$ sentence

$$(\forall x)(\forall y)(N(x,y) \to ((P(x) \,\&\, \neg P(y)) \vee (\neg P(x) \,\&\, P(y))))$$

so that the relations making $\varphi(P)$ true correspond to 2-colourings of the graph $\mathcal{A}$.

Clearly, $\mathcal{A}$ can be chosen so that it has a recursive 2-colouring. We may, however, construct a recursive copy $\mathcal{B}$ of $\mathcal{A}$ which does not, as follows.

Construct $\mathcal{B}$, in steps, to consist of copies $0, 1, 2, \ldots$ of $\mathbb{Z}$, computing values of the $\varphi_e$ simultaneously. If and when $\varphi_e$ gives $\varphi_e(a) = 0$ and $\varphi_e(b) = 1$ where $a$ is in the $(2e)$th copy of $\mathbb{Z}$ and $b$ is in the $(2e+1)$th copy, merge the partly constructed $(2e)$th and $(2e+1)$th copies into a single copy, making $a$ and $b$ an even number of edges apart.

If $\mathcal{A}'$ is the variant of $\mathcal{A}$ with just two copies of $\mathbb{Z}$, then $\varphi(P)$ has property UAP but not property R on $\mathcal{A}'$.

EXAMPLE 2. *There exist a recursive $\Pi_2$ sentence $\varphi(P)$ and a recursive structure $\mathcal{A}$ such that $\varphi(P)$ has property* UAP *but does not have property* UR *on $\mathcal{A}$.*

Proof. Take $\mathcal{A}$ to be the partially ordered set consisting of unrelated chains, one of order type $\omega$ and one of each finite length.

Take $\varphi(P)$ to be a recursive $\Pi_2$ sentence equivalent to the conjunction of:

$$(\exists x)(P(x))$$
$$(\forall x)(\forall y)((P(x) \,\&\, P(y)) \to (x \le y \vee y \le x))$$
$$(\forall x)(P(x) \to (\exists y)(P(y) \,\&\, x < y))$$

so that the relations on $\mathcal{A}$ making $\varphi(P)$ true correspond to chains in the p.o. set $\mathcal{A}$ having no largest element.

Clearly, $\varphi(P)$ has property UAP since we need only take a parameter from the chain of type $\omega$. We show that $\varphi(P)$ does not have property UR. For a contradiction, suppose that, for each recursive $\mathcal{B} \cong \mathcal{A}$, $\varphi_e^{\mathcal{B}}$ is the characteristic function of a chain in $\mathcal{B}$ having no largest element. Construct a recursive copy $\mathcal{B}$ of $\mathcal{A}$ in steps, evaluating $\varphi_e^{\mathcal{B}}$ simultaneously until $\varphi_e^{\mathcal{B}}(b) = 1$ for some $b$. At this stage, finitely many finite chains have been constructed, and the one containing this element $b$ can be kept finite, contradicting the assumption on $\varphi_e^{\mathcal{B}}$.

EXAMPLE 3. *There exist a recursive $\Sigma_2$ sentence $\varphi(P)$ and a recursive structure $\mathcal{A}$ such that $\varphi(P)$ has property* UAP *on $\mathcal{A}$ but does not have property* UR *on $\mathcal{A}$.*

Proof. Take $\mathcal{A}$ as in Example 2. Take $\varphi(P)$ to be a recursive $\Sigma_2$ sentence equivalent to the conjunction of:

$$(\exists x)P(x)$$
$$(\forall x)(\forall y)((P(x) \mathbin{\&} P(y)) \to (x \le y \vee y \le x))$$
$$(\exists x)(P(x) \mathbin{\&} (\forall y)(\neg x < y))\,.$$

The relations satisfying $\varphi(P)$ are the chains in $\mathcal{A}$ containing a maximal element of $\mathcal{A}$.

Clearly, $\varphi(P)$ has property UAP, since we need only take a parameter from one of the finite chains. We show that $\varphi(P)$ does not have property UR on $\mathcal{A}$. For a contradiction, suppose there exists $e$ such that for each recursive $\mathcal{B} \cong \mathcal{A}$, $\varphi_e^{\mathcal{B}}$ is the characteristic function of a chain in $\mathcal{A}$ containing a maximal element of $\mathcal{A}$. Construct a recursive copy $\mathcal{B}$ of $\mathcal{A}$ in stages, evaluating $\varphi_e^{\mathcal{B}}$ simultaneously until $\varphi_e^{\mathcal{B}}(b) = 1$ for some $b$. At this stage, only finitely many finite chains have been constructed, and the one containing the element $b$ can be made into the copy of $\omega$, contradicting the assumption on $\varphi_e^{\mathcal{B}}$.

EXAMPLE 4. *There exist a recursive $\Pi_2$ sentence $\varphi(P)$ and a recursive structure $\mathcal{A}$ such that $\varphi(P)$ has property* R *on $\mathcal{A}$ but does not have property* A *on $\mathcal{A}$.*

Proof. The structure $\mathcal{A}$ is of the form $(A,\ S,\ R,\ G,\ \{P_e\}_{e<\omega})$ where $(A, S)$ is a forest, $S$ being the successor relation on the component trees, and $R$, $G$ and all the $P_e$ being unary relations on $A$.

First, for every infinite sequence, $s$, of symbols $R$ and $G$, we define an augmented tree $T_s^\infty = (T_s^\infty, S, R, G, \{\tau_n\}_{n<\omega}, \tau_\infty)$ in which $(T_s^\infty, S)$ is a tree, and the remaining constituents are unary relations on $T_s^\infty$ such that for each node exactly one of $\tau_0, \tau_1, \ldots, \tau_\infty$ holds, for each node other than the root, exactly one of $R$ and $G$ holds, and for the root neither $R$ nor $G$ holds.

We first define $(T_s^\infty, S, \{\tau_n\}_{n<\omega}, \tau_\infty)$, independently of $s$, to be the unique countable structure of this kind, such that

(i) $\tau_\infty$ holds for the root,

(ii) for each node satisfying $\tau_\infty$ there are infinitely many successors satisfying each of $\tau_0, \tau_1, \ldots, \tau_\infty$,

(iii) for each node satisfying $\tau_n$, for $n < \omega$, each successor satisfies $\tau_k$ for some $k < n$ and there are infinitely many of these successors for each $k < n$.

Thus, an element satisfying $\tau_0$ has no successors, and, more generally, the $\{\tau_n\}_{n<\omega}$ assign to each node a rank, namely the least $n < \omega$ for which each successor node has rank $< n$, while unranked nodes are assigned the symbol $\infty$.

Now we define $R$ and $G$ on $T_s^\infty$ as follows. The root satisfies neither $R$ nor $G$. A node of level $k = 1, 2, 3, \ldots$ and satisfying $\tau_n$ for $n = 1, 2, \ldots, \infty$ satisfies $R$ or $G$ to *agree* with the $k$th entry of the sequence $s$. A node of level $k \geq 1$ and satisfying $\tau_0$ satisfies $R$ or $G$ to *disagree* with the $k$th entry of $s$.

Suppose now that we have a recursive procedure which evaluates each term in a sequence $s$ of $R$'s and $G$'s in turn, so that the computation at some stage may not converge, resulting in only a finite sequence. Then there is a uniform procedure which enumerates either, when $s$ is infinite, a copy of $T_s^\infty$ or, when $s$ is finite, a structure $T_s^{\mathrm{fin}}$ defined as follows.

Let $k$ be the length of $s$. Then define $T_s^{\mathrm{fin}}$ to consist of the first $k$ levels of $T_{s'}^\infty$ for any infinite $s'$ extending $s$ plus a $(k+1)$th level. This level consists, for exactly those nodes at level $k$ whose $R, G$ sequences are $s$, of infinitely many successors satisfying each of $R$ and $G$, and not satisfying any of the $\tau_n$ or $\tau_\infty$.

For each such $s$, let $T_s^\infty(-)$ and $T_s^{\mathrm{fin}}(-)$ denote the reducts which ignore the relations $\tau_n$. To obtain the desired structure $\mathcal{A}$, we shall define a certain uniform sequence $s_0, s_1, s_2, \ldots$ of such recursive procedures. Then we take $\mathcal{A}$ to be a labelled forest $(A, S, R, G, \{P_e\}_{e<\omega})$ consisting of unrelated trees $T_0, T_1, T_2, \ldots$, where each $T_e$ is either a copy of $T_{s_e}^\infty(-)$, if $s_e$ is infinite, or a copy of $T_{s_e}^{\mathrm{fin}}(-)$, if $s_e$ is finite. For each $e$, we let the unary relation $P_e$ hold for the root of $T_e$ and for no other element.

The $\{s_e\}_{e<\omega}$ chosen to construct $\mathcal{A}$ are obtained as follows. Let $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots$ be a uniformly recursive enumeration of the partial recursive structures of type $(F, S, R, G, \{P_e\}_{e<\omega})$. Now, uniformly in $e$, let $p_e$ be the sequence of elements of $\mathcal{B}_e$ obtained by beginning with the first element discovered in $\mathcal{B}_e$ satisfying $P_e^{\mathcal{B}_e}$ and continuing at each later stage by taking the first element discovered in $\mathcal{B}_e$ satisfying either $R^{\mathcal{B}_e}$ or $G^{\mathcal{B}_e}$ and in the relation $S^{\mathcal{B}_e}$ with the last previously chosen element.

From each $p_e$, we uniformly obtain the corresponding sequence of $R$'s and $G$'s which we take to be $s_e$ in the description of $\mathcal{A}$ above.

For this $\mathcal{A}$, we take the sentence $\varphi(P)$ to be a $\Pi_2$ sentence equivalent to

$$\begin{aligned}
&(\exists x)(P(x))\\
&\ \& \ (\forall x)(P(x) \to (\exists y)(P(y) \ \& \ S(x,y)))\\
&\ \& \ (\forall x)(\forall y)(P(x) \ \& \ P(y) \to x = y \vee (\exists z)(P(z) \ \& \ (S(z,x) \vee S(z,y))))\\
&\ \& \ (\forall x)(\forall y)(\forall z)(P(x) \ \& \ P(y) \ \& \ P(z) \ \& \ S(x,y) \ \& \ S(x,z) \to y = z)
\end{aligned}$$

so that the sets satisfying $\varphi(P)$ in $\mathcal{A}$ correspond to infinite branches in the forest $\mathcal{A}$.

We see that $\varphi(P)$ does have property R on $\mathcal{A}$, as follows. Each recursive $\mathcal{B} \cong \mathcal{A}$ is one of the $\mathcal{B}_e$. Suppose, for a contradiction, that the corresponding $p_e$ is finite. Then, by construction, $T_e = T^{\mathrm{fin}}_{s_e}(-)$ has no terminal node whose corresponding $R, G$ sequence is $s_e$, while in $\mathcal{B}_e$, by choice of $p_e$, there is such a terminal node above the element satisfying $P_e$. This contradicts $\mathcal{B}_e \cong \mathcal{A}$. Hence, if $\mathcal{B} = \mathcal{B}_e \cong \mathcal{A}$, then $p_e$ is infinite, is recursive and satisfies $\varphi(P)$ on $\mathcal{A}$.

Now we wish to show that $\varphi(P)$ does not have property A on $\mathcal{A}$. Suppose, for a contradiction, that it does. Then we may appeal to the results of [AK] (obtained by considering a generic copy of $\mathcal{A}$). By Theorem 1.1 of [AK] there exists a formally r.e. expansion family for $\varphi(P)$ on $\mathcal{A}$. This consists of a set $\mathcal{S}$ of consistent finite sets $\sigma(a_1, \ldots, a_k)$ of sentences $P(a_i)$ or $\neg P(a_i)$ such that, by Lemma 1.2 of [AK], for each $\sigma$ in $\mathcal{S}$ there is a chain $\sigma = \sigma_0 \subseteq \sigma_1 \subseteq \ldots$ determining a relation satisfying $\varphi(P)$ in $\mathcal{A}$ and also satisfying each $\sigma_i$. It follows both that there are infinitely many $a \in \mathcal{A}$ for which $P(a)$ occurs positively in some $\sigma \in \mathcal{S}$ and that each such $a \in \mathcal{A}$ lies on an infinite path in $\mathcal{A}$.

Since the expansion family is formally r.e., the set of all such $a \in \mathcal{A}$ is defined by a recursive disjunction of existential formulas with some single finite set of parameters. The contradiction is therefore obtained if we show that $\mathcal{A}$ has no infinite set of elements, all lying on infinite paths, which is definable by a recursive $\Sigma_1$ formula with finitely many parameters.

However, this fact is reasonably obvious. For a hypothetical such formula having parameters $\overline{c}$, we may consider an element, $a$, of the set which is not above any of $\overline{c}$ (considering the trees to grow downwards). Let $\bar{b}$ be witnesses for the fact that $a$ satisfies the $\Sigma_1$ formula. Then, because of the homogenous nature of $\mathcal{A}$, we may find $a'$ satisfying $\tau_n$ for some $n < \omega$ and $\bar{b}'$ such that $\overline{c}, a', \bar{b}'$ satisfy the same quantifier-free formulas as do $\overline{c}, a, \bar{b}$.

Example 5. *There exist a recursive $\Sigma_3$ sentence $\varphi(P)$ and a recursive structure $\mathcal{A}$ such that $\varphi(P)$ has property* UR *on $\mathcal{A}$ but does not have property* A *on $\mathcal{A}$.*

Proof. Take $\mathcal{A}$ to be a recursive structure such as that of Example 4 for which some formula $\psi(Q)$ has property R but not A, on $\mathcal{A}$. Let $\mathcal{B} \cong \mathcal{A}$ be such that no $S$ recursive in $\mathcal{B}$ has $(\mathcal{B}, S) \models \psi(Q)$.

Then we note that there can be no isomorphism $f$ from $\mathcal{B}$ to a recursive structure $\mathcal{B}'$ for which $f$ is recursive in $\mathcal{B}$. If there were such $f, \mathcal{B}'$, then by assumption on $\psi(Q)$ there would exist $S'$ on $\mathcal{B}'$ for which $(\mathcal{B}', S') \models \psi(Q)$, and then we could take $S$ to be $f^{-1}(S')$ and have $(\mathcal{B}, S) \models \psi(Q)$ where $S$ is recursive in $\mathcal{B}$, in contradiction to the choice of $\mathcal{B}$.

Thus, for our present purpose, it is sufficient to give $\varphi(P)$ for which the $S$ for which $(\mathcal{B}, S) \models \varphi(P)$ correspond to isomorphisms from $\mathcal{B}$ to some recursive structure, since certainly there is a uniform procedure for finding such a recursive isomorphism when $\mathcal{B}$ is recursive, by taking the identity function.

To consider isomorphisms from $\mathcal{B}$ to recursive structures, we may discuss bijections $f : \mathcal{B} \to \omega$ by working instead with the binary relation $S(x, y)$ for which $S(b_1, b_2)$ if and only if $f(b_2) = f(b_1) + 1$ or $f(b_2) = f(b_1) = 0$. In terms of the corresponding relation symbol $P(x, y)$, there is a formula $P_k(x)$ true in $(\mathcal{B}, S)$ for the unique $x$ such that $f(x) = k$. We may take $P_k(x)$ to be the formula

$$(\exists y_0) \ldots (\exists y_k)(P(y_0, y_0) \,\&\, y_0 \neq y_1 \,\&\, P(y_0, y_1) \,\&\, \ldots \,\&\, P(y_{k-1}, y_k) \,\&\, y_k = x)\,.$$

Now let $\Theta_e(P)$ be a recursive $\Pi_2$ formula equivalent to the conjunction of the following, and let $\varphi(P)$ be $\bigvee\!\!\!\bigvee_e \Theta_e(P)$.

$$(\forall x) \bigvee\!\!\!\bigvee_k P_k(x)$$

$$(\forall x) \bigwedge\!\!\!\bigwedge_{i \neq j} \neg(P_i(x) \,\&\, P_j(x))$$

"$\varphi_e$ is total and 0, 1-valued"

$$\bigwedge\!\!\!\bigwedge_{\substack{i, n_1, \ldots, n_k \\ \varphi_e(\langle i, n_1, \ldots, n_k\rangle) = 1}} (\forall x_1) \ldots (\forall x_k)(P_{n_1}(x_1) \,\&\, \ldots \,\&\, P_{n_k}(x_k) \to R_i(x_1, \ldots, x_k))$$

$$\bigwedge\!\!\!\bigwedge_{\substack{i, n_1, \ldots, n_k \\ \varphi_e(\langle i, n_1, \ldots, n_k\rangle) = 0}} (\forall x_1) \ldots (\forall x_k)(P_{n_1}(x_1) \,\&\, \ldots \,\&\, P_{n_k}(x_k) \to \neg R_i(x_1, \ldots, x_k))\,.$$

In the last two sentences, the conjunction is taken over all relation symbols, $R_i$, of the language of $\mathcal{A}$.

EXAMPLE 6. *There exist a recursive $\Sigma_2$ sentence $\varphi(P)$ and a recursive structure $\mathcal{A}$ such that $\varphi(P)$ has property* UR *but does not have property* UA *on $\mathcal{A}$.*

Proof. First let $T$ be an infinite r.e. subtree of $2^{<\omega}$ having no infinite $\Delta^0_2$ branch. (It is fairly easy to construct such a tree.)

Let $\mathcal{A}_1$ be the infinitely branching recursive labelled tree $(T_1, c, S, R, G)$, having root $c$, defined by the following properties:

(1) The root, $c$, satisfies neither $R$ nor $G$.

(2) Every element other than $c$ satisfies $R$ or $G$ but not both.

(3) Every element having a successor satisfying $R$ (respectively $G$) has infinitely many successors satisfying $R$ (or $G$).

For each element of $T_1$, the sequence of elements from the root determines a finite sequence of 0's and 1's, taking 0 for elements satisfying $R$ and 1 for elements satisfying $G$.

(4) The sequences of 0's and 1's corresponding to elements of $T_1$ are exactly the elements of $T$.

We note that no recursive copy of $\mathcal{A}_1$ has a $\Delta_2^0$ infinite branch, because this would give a $\Delta_2^0$ branch in $T$.

Now we obtain the desired structure $\mathcal{A} = (A, c, S, \equiv, \leq)$ from $\mathcal{A}_1 = (T_1, c, S, R, G)$ as follows. Let $A$ be obtained from $T_1$ by replacing each $u \in T_1$ other than $c$ by an infinite set $E_u$. Let $\equiv$ be the equivalence relation whose equivalence classes are the $E_u$ and $\{c\}$. Let $S'$ be the binary relation relating all pairs from $E_u$ and $E_v$ respectively for which $S(u, v)$ and relating $c$ to each element of $E_u$ for which $S(c, u)$. Let $\leq$ be a pre-order relating only elements of the same $E_u$ and having the following properties. The restriction of $\leq$ to each $E_u$ is a pre-order of order type $\omega^*$ in which each equivalence class except possibly the last has two elements. The last equivalence class of this restriction has one element if $u$ satisfies $R$ in $\mathcal{A}_1$ and two elements if $u$ satisfies $G$ in $\mathcal{A}_1$.

We take $\varphi(P)$ to be a $\Sigma_2$ sentence equivalent to the conjunction of:

$$P(c)$$
$$(\forall x)(\forall y)(P(x) \,\&\, x \equiv y \to P(y))$$
$$(\forall x)(P(x) \,\&\, x \neq c \to (\exists y)(P(y) \,\&\, S(y, x)))$$
$$(\exists z)(P(z) \,\&\, (\forall x)(P(x) \to \neg S(z, x)))$$

so that the sets satisfying $\varphi(P)$ in $\mathcal{A}$ are unions of whole equivalence classes corresponding to maximal branches of $T_1$ which are also finite.

It is clear that there is a $\Delta_2^0$ procedure which, when applied to an element of $E_u$, determines whether $u$ satisfied $R$ or $G$ in $\mathcal{A}_1$, and likewise for each recursive copy of $\mathcal{A}$.

There is also a uniformly recursive procedure for obtaining in each such recursive copy of $\mathcal{A}$ a subset which is the union of whole equivalence classes corresponding to maximal branches of $T_1$, namely taking the first successors which appear and closing under $\equiv$. (The resulting set is recursive since, for any $a$, either $a$ itself appears on the path or, for some $k$ and for some $b$ on the path, the predecessors of $b$ are the $(k+1)$th predecessors of $a$ but $b$ is not a $k$th predecessor of $a$.) If the branch in $T_1$ resulting from this procedure were infinite, then we should have an infinite $\Delta_2^0$ branch in $T$, contradicting the choice of $T$. We have thus shown that $\varphi(P)$ has property UR on $\mathcal{A}$.

It remains to show that $\varphi(P)$ does not have property UA on $\mathcal{A}$. Suppose, for a contradiction, that for each $\mathcal{B} \cong \mathcal{A}$, $\varphi_e^{\mathcal{B}}$ is the characteristic function of a relation on $\mathcal{B}$ satisfying $\varphi(P)$. Then we may construct (non-recursively) as follows $\mathcal{C} \cong \mathcal{A}$ such that $\varphi_e^{\mathcal{C}}$ is the characteristic function of a subset of $\mathcal{C}$ having members of infinitely many equivalence classes.

While defining an enumeration of the diagram of $\mathcal{C}$, we can change an equivalence class of one type to the other type by inserting new elements, without affecting any desired finite part of the diagram or finite number of computations. We may thus begin by enumerating as much of a copy of any $\mathcal{B}_0 \cong \mathcal{A}$ as is needed to give representatives of a maximal finite branch. By the choice of $T_1$, changing the types of these equivalence classes results in a copy, $\mathcal{B}_1 \cong \mathcal{A}$, containing the finite part already enumerated and the finitely many computations for the representatives so far, in which the corresponding path is not maximal. Now we may change to enumerating $\mathcal{B}_1$ and repeat the process. If we also ensure that the type of each equivalence class is changed only finitely often and that each permitted initial segment of types of equivalence classes occurs infinitely often, then the union, $\mathcal{C}$, of these $\mathcal{B}_n$ will be an isomorphic copy of $\mathcal{A}$ while $\varphi_e^{\mathcal{C}}$ will correspond to an infinite branch.

EXAMPLE 7. *There is a recursive $\Pi_3$ sentence $\varphi'(P)$ and a recursive structure $\mathcal{A}'$ such that $\varphi'(P)$ has property* UR *but not property* A *on $\mathcal{A}'$.*

Proof. Let $\mathcal{A}'$ be a structure of type $(A, \sim, c, S, \equiv, \leq)$ consisting of an equivalence relation $\sim$ having infinitely many equivalence classes, each class containing a copy of the structure $\mathcal{A}$ of Example 6. Let $\varphi(P, x)$ be the sentence of Example 6 relativized to the equivalence class of the variable $x$, and let $\varphi'(P)$ be $(\forall x)\varphi(P, x)$.

Thus, relations satisfying $\varphi'(P)$ on $\mathcal{A}'$ correspond to simultaneous choices for each equivalence class of maximal finite branches in each of the copies of $\mathcal{A}$.

By the same argument as in Example 6, $\varphi'(P)$ has property UR on $\mathcal{A}'$. It remains to see that $\varphi'(P)$ does not have property A on $\mathcal{A}'$.

In Example 6, we showed that, for each $e$, there is a copy $\mathcal{C}$ of $\mathcal{A}$ in which $\varphi_e^{\mathcal{C}}$ is not the characteristic function of a subset of $\mathcal{C}$ which corresponds to a finite branch. (If $\varphi_e^{\mathcal{C}}$ is not sufficiently total, then the result of the construction was that $\mathcal{C}$ was equal to some $\mathcal{B}_n$.)

Similarly, we may define a copy $\mathcal{C}'$ of $\mathcal{A}'$ in which $\varphi_e^{\mathcal{C}'}$ is not the characteristic function of a subset of $\mathcal{C}'$ which corresponds to a finite maximal branch of the $e$th copy of $\mathcal{A}$ in $\mathcal{C}'$. Thus, no $\varphi_e^{\mathcal{C}'}$ is the characteristic function of a subset of $\mathcal{C}'$ satisfying $\varphi'(P)$.

**3. Systematic treatment.** We may systematize our investigation by asking which Boolean combinations of the properties A, UA, UAP, R, UR, URP are possible and, if so, which forms $\Sigma_1, \Pi_1, \Sigma_2, \ldots$ of sentences $\varphi(P)$ may be used.

Clearly, from the definitions, we have the implications

$$\mathrm{UR} \Rightarrow \mathrm{URP} \Rightarrow \mathrm{R}, \quad \mathrm{UA} \Rightarrow \mathrm{UAP} \Rightarrow \mathrm{A}.$$

This leaves Boolean combinations from the table below:

| | | No | Yes |
|---|---|---|---|
| 1 | ¬R | $\Sigma_1$ | $\Pi_1$ |
| 2 | R & ¬A & ¬URP | $\Sigma_1$ | $\Pi_2$ |
| 3 | URP & ¬A & ¬UR | $\Sigma_2, \Pi_2$ | |
| 4 | A & ¬URP | $\Sigma_3$ | |
| 5 | UR & ¬A | $\Sigma_2, \Pi_2$ | $\Sigma_3, \Pi_3$ |
| 6 | URP & A & ¬UR & ¬UAP | $\Sigma_3$ | |
| 7 | UR & A & ¬UAP | $\Sigma_3$ | |
| 8 | UAP & ¬UR | $\Sigma_1$ | $\Pi_1$ |
| 9 | UR & UAP & ¬UA | $\Pi_2$ | $\Sigma_2$ |
| 10 | UA | | $\Sigma_1, \Pi_1$ |

In this table, for example, the appearance of $\Sigma_2$ in the "No" column of line 5 records that UR⇒A for $\Sigma_2$ sentences, while $\Sigma_3$ in the "Yes" column records that this implication fails for $\Sigma_3$ sentences. The table is not complete; for example line 4 shows that A⇒URP for $\Sigma_3$ sentences, but we do not presently know whether the implications holds more generally.

*Justification of the table*

Line 1: Theorem 1 justifies the "No", since UA⇒R, and Example 1 justifies the "Yes".

Line 2: Theorem 1 justifies the "No", since UA⇒A. Example 4 and Theorem 3 justify the "Yes". Since UR⇒UA for $\Pi_2$ sentences, we have URP⇒UAP, and so, because UAP⇒A, we also have ¬A⇒ ¬URP for $\Pi_2$ sentences.

Line 3: Theorem 3 justifies the "No" for $\Pi_2$ sentences, since, as above, we have ¬A⇒ ¬URP for $\Pi_2$ sentences. Theorem 2 justifies the "No" for $\Sigma_2$ sentences. We have UR⇒UAP, and therefore URP⇒UAP, and, since UAP⇒A, also URP⇒A for $\Sigma_2$ sentences.

Line 4: Theorem 4 justifies the "No". We have A⇒UAP for $\Sigma_3$ sentences, and of course UAP⇒URP.

Line 5: Theorem 2 justifies the "No" for $\Sigma_2$ sentences. We have UR⇒UAP and UAP⇒A. Theorem 3 justifies the "No" for $\Pi_2$ sentences. We have UR⇒UA and UA⇒A. Example 5 justifies the "Yes" for $\Sigma_3$ sentences, and Example 7 justifies the "Yes" for $\Pi_3$ sentences.

Line 6: Theorem 4 justifies the "No" for $\Sigma_3$ sentences, since it gives A⇒ UAP.

Line 7: Theorem 4 justifies the "No" for $\Sigma_3$ sentences.

Line 8: Theorem 1 justifies the "No" for $\Sigma_1$ sentences; these have property UA, and UA$\Rightarrow$UR. The variant of Example 1 justifies the "Yes" for $\Pi_1$ sentences.

Line 9: Theorem 3 justifies the "No" for $\Pi_2$ sentences. Example 6 and Theorem 2 justify the "Yes" for $\Sigma_2$ sentences. The example satisfies UR & $\neg$UA, where by the theorem, UR$\Rightarrow$UAP.

Line 10: We have "Yes" for $\Sigma_1$ or $\Pi_1$ sentences, by taking $\varphi(P)$ to be trivially true.

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