FUNDAMENTA
MATHEMATICAE
141 (1992)

Weak variants of Martin's Axiom

by

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Abstract. A hierarchy of weak variants of Martin's Axiom is extended and shown to be strict.

1. Introduction. Variants of MA_κ in which the ccc condition on the partial order $\mathcal{P}$ is replaced by some other condition Φ have been well studied. In particular, denoting the variant thus obtained by $\mathrm{MA}_\kappa(\Phi)$ $[\mathrm{MA}_\kappa = \mathrm{MA}_\kappa(\mathrm{ccc})]$, the implication diagram below is obtained.

In $\mathcal{M}[c]$, c Cohen over $\mathcal{M}$, none of the $\mathrm{MA}_{\aleph_1}(\Phi)$ in the diagram hold except for the bottom one, MA(σ-centered). Using the way in which that is proved, and some results of Devlin, Shelah and Todorčević, we complete the known facts about the diagram by proving that no implications exist other than the ones shown: that is, if $\mathrm{MA}(\Phi_1)$ is above $\mathrm{MA}(\Phi_2)$ in the hierarchy, then $\neg\mathrm{CH} + \mathrm{MA}(\Phi_1) + \neg\mathrm{MA}_{\aleph_1}(\Phi_2)$ is consistent. We first recall the definitions of the conditions Φ under consideration.

Definition 1.1. Let $\mathcal{P}$ be a partial order and $n \in \omega$. Then $A \subseteq \mathcal{P}$ is *n-linked* if and only if for every $p_1, \ldots, p_n \in A$, there is some $p \in \mathcal{P}$ with $p \le p_1, \ldots, p_n$. A is *centered* if and only if A is n-linked for all $n \in \omega$.

Definition 1.2. Let $\mathcal{P}$ be a partial order.

1. Given $n \in \omega$, $\mathcal{P}$ has *property* K_n iff every $A \in [\mathcal{P}]^{\aleph_1}$ contains an uncountable n-linked subset.

2. Given $n \in \omega$, $\mathcal{P}$ is *σ-n-linked* iff $\mathcal{P} = \bigcup_{k<\omega} \mathcal{P}_k$, where each $\mathcal{P}_k$ is n-linked.

3. $\mathcal{P}$ is $\bigwedge_{n\in\omega}$ *σ-n-linked* iff $\mathcal{P}$ is σ-n-linked for all $n \in \omega$.

4. $\mathcal{P}$ has *pre-caliber* ω_1 (*pc* ω_1) iff every $A \in [\mathcal{P}]^{\aleph_1}$ contains an uncountable centered subset.

5. $\mathcal{P}$ is *σ-centered* iff $\mathcal{P} = \bigcup_{k<\omega} \mathcal{P}_k$, where each $\mathcal{P}_k$ is centered.

It is clear from the definitions that the following hierarchy holds.

$$\begin{tikzcd}
 & \mathrm{MA} \arrow[d] \\
 & \mathrm{MA}(\mathrm{K}_2) \arrow[dl] \arrow[d] \\
\mathrm{MA}(\sigma\text{-2-linked}) \arrow[d] & \mathrm{MA}(\mathrm{K}_3) \arrow[dl] \arrow[d] \\
\mathrm{MA}(\sigma\text{-3-linked}) \arrow[d] & \vdots \\
\vdots & \mathrm{MA}(\mathrm{K}_n) \arrow[dl] \\
\mathrm{MA}(\sigma\text{-}n\text{-linked}) & \vdots \arrow[d] \\
\vdots \arrow[d] & {} \\
\mathrm{MA}\Big(\bigwedge\!\!\!\bigwedge_{n\in\omega}\sigma\text{-}n\text{-linked}\Big) \arrow[d] & \mathrm{MA}(\mathrm{pc}\ \omega_1) \arrow[dl] \\
\mathrm{MA}(\sigma\text{-centered}) &
\end{tikzcd}$$

The consistency of $\mathrm{MA}(\mathrm{K}_2) + \neg\mathrm{MA}_{\aleph_1} + \neg\mathrm{CH}$ is shown in [KT] by extending a model $\mathcal{M}$ in which there is a Suslin tree to a model $\mathcal{M}[G_{\mathcal{P}}]$ of $\neg\mathrm{CH} + \mathrm{MA}(\mathrm{K}_2)$ via iterated forcing. Since the partial order $\mathcal{P}$ used to obtain $\mathcal{M}(G_{\mathcal{P}})$ has property K_2 (see Lemma 1.4), one need then only check that no K_2 partial order destroys a Suslin tree to conclude that $\mathcal{M}[G_{\mathcal{P}}] \models [2^{\aleph_0} > \aleph_1 + \mathrm{MA}(\mathrm{K}_2) + \neg\mathrm{MA}_{\aleph_1}]$. Similarly, Herink [He] uses a counterexample to the statement "The measure algebra has pc ω_1", which is a consequence of $\mathrm{MA}_{\aleph_1}(\mathrm{K}_n)$, to show the consistency of $\neg\mathrm{CH}+\mathrm{MA}(\mathrm{pc}\ \omega_1)$ with $\forall n \in \omega\, \neg\mathrm{MA}_{\aleph_1}(\mathrm{K}_n)$. Since the measure algebra is $\bigwedge\!\!\!\bigwedge_{n\in\omega}\sigma$-$n$-linked, $\mathrm{MA}_{\aleph_1}(\bigwedge\!\!\!\bigwedge_{n\in\omega}\sigma\text{-}n\text{-linked})$ also implies the measure algebra has pc ω_1. Thus, Herink's proof in fact yields $\mathrm{Con}(\neg\mathrm{CH} + \mathrm{MA}(\mathrm{pc}\ \omega_1) + \neg\mathrm{MA}_{\aleph_1}(\bigwedge\!\!\!\bigwedge_{n\in\omega}\sigma\text{-}n\text{-linked}))$. We remark that this result also follows from Pawlikowski's proof in [Pa] that $\mathrm{MA}(\mathrm{pc}\ \omega_1)$ is consistent with

(†) There exists a covering of the real line by ω_1 measure zero sets

since $\mathrm{MA}_{\aleph_1}(\bigwedge\!\!\!\bigwedge_{n\in\omega}\sigma\text{-}n\text{-linked})$ implies that there is no such covering.

Herink also shows in [He] that $\neg\mathrm{CH}+\mathrm{MA}(\sigma\text{-}n\text{-linked})$ is consistent with $\neg\mathrm{MA}_{\aleph_2}(\mathrm{pc}\,\omega_1)$ by showing that an (ω_2, ω_2^*) gap cannot be filled by a σ-2-linked partial order. This left open whether $\mathrm{MA}_{\aleph_2}(\mathrm{pc}\,\omega_1)$ could be improved to $\mathrm{MA}_{\aleph_1}(\mathrm{pc}\,\omega_1)$. In Section 2 we show that this is the case using the notion of a uniformization of a ladder system. In order to complete the diagram, we thus need only show the following two results:

- $\mathrm{Con}(\neg\mathrm{CH} + \mathrm{MA}(\mathrm{K}_{n+1}) + \neg\mathrm{MA}_{\aleph_1}(\sigma\text{-}n\text{-linked}))$,
- $\mathrm{Con}\Big(\neg\mathrm{CH} + \mathrm{MA}(\sigma\text{-centered}) + \neg\mathrm{MA}_{\aleph_1}\Big(\bigwedge\limits_{n\in\omega}\sigma\text{-}n\text{-linked}\Big)\Big)$.

The first we establish in Section 3 using the notion of a $\le n$-ary set of reals. The second result follows from Theorem 1.3 below and the fact that $\mathcal{M}[c] \models (\dagger)$ (see [CP]), where $(\dagger)$ is as above. In Section 4 we give an alternate proof of this consistency result using a generalization of the notion of a $\le n$-ary set of reals.

For all of these results, counterexamples to the various Martin's Axiom statements are constructed in the Cohen extension $\mathcal{M}[c]$. To establish the consistency of $\neg\mathrm{CH}+\mathrm{MA}(\sigma\text{-centered})+\neg\mathrm{MA}_{\aleph_1}(\Phi)$, we then need only apply the following well-known theorem (see [$\mathrm{Ro}_{1,2}$], [IS]).

THEOREM 1.3. $\mathcal{M} \models \mathrm{MA}_{\aleph_1}(\sigma\textit{-centered}) \Rightarrow \mathcal{M}[c] \models \mathrm{MA}_{\aleph_1}(\sigma\textit{-centered})$.

To establish the consistency of $\neg\mathrm{CH}+\mathrm{MA}(\Phi')+\neg\mathrm{MA}_{\aleph_1}(\Phi)$ for conditions Φ' other than σ-centered, we use the fact that, for the conditions Φ' under consideration here, the partial order used to force $\mathrm{MA}(\Phi')$ itself satisfies the condition Φ'. This follows from the following well-known lemmas, whose proofs we include for completeness (see also [Ba]).

LEMMA 1.4. *Suppose $\langle\langle\mathcal{P}_\xi\rangle_{\xi\le\alpha}, \langle\Pi_\xi\rangle_{\xi<\alpha}\rangle$ is an α-stage finite support iteration such that*

$$\forall\xi<\alpha \qquad \Vdash_{\mathcal{P}_\xi} [\Pi_\xi \textit{ has property } \Phi]\,,$$

where Φ is either K_n or pc ω_1. Then $\mathcal{P}_\alpha$ also has property Φ.

Proof. We give the proof in the case where Φ is property K_n; the proof for pc ω_1 is similar. We proceed by induction on α.

Case 1: Suppose $\alpha=\beta+1$. Then $\mathcal{P}_\alpha=\mathcal{P}_{\beta+1} = \mathcal{P}_\beta * \Pi_\beta$, where we assume $\mathcal{P}_\beta$ has property K_n and $\Vdash_{\mathcal{P}_\beta} [\Pi_\beta$ has property $\mathrm{K}_n]$. Let $\langle\langle p_\gamma\rangle, \langle\pi_\gamma\rangle\rangle_{\gamma\in\omega_1} \subseteq \mathcal{P}_\alpha$. Since $\mathcal{P}_\beta$ is ccc, there is some $p\in\mathcal{P}_\beta$ such that $p \Vdash |\{\gamma\in\omega_1 : p_\gamma \in \dot{G}_{\mathcal{P}_\beta}\}| = \aleph_1$. Let $\dot{A}$ be a $\mathcal{P}_\beta$ term for this set and assume without loss of generality that

$$p \Vdash_{\mathcal{P}_\beta} [\{\pi_\gamma : \gamma\in\dot{A}\} \text{ is } n\text{-linked}]\,.$$

Then there exists $B\in[\omega_1]^{\aleph_1}$ such that for all $\gamma\in B$, there is some $p'_\gamma\in\mathcal{P}_\beta$ such that

$$p'_\gamma \le p_\gamma\,, \quad p'_\gamma \le p \quad \text{ and } \quad p'_\gamma \Vdash \gamma\in\dot{A}\,.$$

By the induction hypothesis, $\mathcal{P}_\beta$ has property K_n, so that $\{p'_\gamma : \gamma\in B'\}$ is n-linked for some $B'\in[B]^{\aleph_1}$. Then $\{\langle p_\gamma, \pi_\gamma\rangle : \gamma\in B'\}$ is the desired n-linked subset of $\langle\langle p_\gamma\rangle, \langle\pi_\gamma\rangle\rangle_{\gamma\in\omega_1}$.

Case 2: Suppose α is a limit ordinal. Then $\mathcal{P}_\alpha = *_{\xi<\alpha}\mathcal{P}_\xi$, where we assume that each $\mathcal{P}_\xi$, $\xi < \alpha$, has property K_n. Let $\langle p_\gamma\rangle_{\gamma\in\omega_1} \subseteq \mathcal{P}_\alpha$ and $S_\gamma = \mathrm{supp}(p_\gamma)$ for all $\gamma \in \omega_1$. We may assume by the Δ-system lemma that $S_\gamma \cap S_{\gamma'} = S \subseteq \beta < \omega_1$ for all $\gamma < \gamma' < \omega_1$. Since $q_\gamma = p_\gamma \restriction (\beta+1) \in \mathcal{P}_\beta$ for all $\gamma \in \omega_1$ and $\mathcal{P}_\beta$ has property K_n by the induction hypothesis, there is some $B \in [\omega_1]^{\aleph_1}$ such that $\{q_\gamma : \gamma \in B\}$ is n-linked. Then $\{p_\gamma \in \mathcal{P}_\alpha : \gamma \in B\}$ is the desired n-linked subset of $\langle p_\gamma\rangle_{\gamma\in\omega_1}$. ■

LEMMA 1.5. *Suppose* $\alpha \le 2^{\aleph_0}$ *and* $\langle\langle\mathcal{P}_\xi\rangle_{\xi\le\alpha}, \langle\Pi_\xi\rangle_{\xi<\alpha}\rangle$ *is an* α*-stage finite support iteration such that*

$$\forall\xi<\alpha \qquad \Vdash_{\mathcal{P}_\xi} [\Pi_\xi \text{ has property } \Phi],$$

where Φ *is either* σ*-centered or* σ*-*n*-linked*, $n\in\omega$. *Then* $\mathcal{P}_\alpha$ *also has property* Φ.

Proof. We give the proof for σ-n-linked; the proof for σ-centered is similar. Let $\langle\langle\mathcal{P}_\xi\rangle_{\xi\le\alpha}, \langle\Pi_\xi\rangle_{\xi<\alpha}\rangle$ be a finite support iteration, $\alpha \le 2^{\aleph_0}$, such that for all $\xi < \alpha$,

$$\Vdash_{\mathcal{P}_\xi} \Big[\Pi_\xi = \bigcup_{k\in\omega} \Pi_k^\xi \text{ is an } n\text{-linked decomposition}\Big].$$

We show that $\mathcal{P}_\alpha$ is also σ-n-linked.

Let us say a condition $p \in \mathcal{P}$ is *determined* just in case for all $\xi \in \mathrm{supp}(p)$, there is some $k_\xi \in \omega$ satisfying $p \restriction \xi \Vdash p(\xi) \in \Pi_{k_\xi}^\xi$. Let $\overline{\mathcal{P}}_\alpha$ be the set of all determined conditions. By induction on α, $\overline{\mathcal{P}}_\alpha$ is dense in the finite support iteration $\mathcal{P}_\alpha$. Thus, we need only show $\overline{\mathcal{P}}_\alpha$ is σ-n-linked to complete the proof.

To this end, let $g : 2^{\aleph_0} \mapsto 2^\omega$ be 1-1 and onto, so that g labels the branches of 2^ω by ordinals $\beta \le 2^{\aleph_0}$. For all $N \in \omega$, let T_N be the binary tree of height N and $\mathcal{T} = \{\langle T_N, h\rangle : N \in \omega \text{ and } h : A \mapsto \omega,\ A \text{ an antichain in } T_N\}$. Note that for any $p \in \mathcal{P}_\alpha$, the set $\{g(\xi) : \xi \in \mathrm{supp}(p)\}$ diverges in 2^ω below some level $N \in \omega$. Thus, the set $A = \{g(\xi) \restriction N : \xi \in \mathrm{supp}(p)\}$ is an antichain in T_N. If additionally $p \in \overline{\mathcal{P}}_\alpha$ and $z_\xi = g(\xi) \restriction N$, then we can define a map h on A by $h(z_\xi) = k_\xi$ if and only if $p \restriction \xi \Vdash p(\xi) \in \Pi_{k_\xi}^\xi$.

For all $\langle T_N, h\rangle \in \mathcal{T}$, let $\mathcal{P}_{\langle T_N,h\rangle}$ be the set of all $p \in \overline{\mathcal{P}}_\alpha$ satisfying

$$\forall\xi \in \mathrm{supp}(p)\,\exists z \in \mathrm{dom}(h)\ g(\xi) \text{ extends } z \ \wedge\ p \restriction \xi \Vdash p(\xi) \in \Pi_{h(z)}^\xi .$$

By the above comments, $\overline{\mathcal{P}}_\alpha = \bigcup\{\mathcal{P}_{\langle T_N,h\rangle} : \langle T_N, h\rangle \in \mathcal{T}\}$. Furthermore, since $\Vdash_{\mathcal{P}_\xi} [\Pi_k^\xi \text{ is } n\text{-linked}]$ for all $\xi < \alpha$, the cell $\mathcal{P}_{\langle T_N,h\rangle}$ is n-linked for all $\langle T_N, h\rangle \in \mathcal{T}$. Since $\mathcal{T}$ is countable, this completes the proof. ■

The author wishes to acknowledge Rich Laver for sharing his insight into these questions and the referee for bringing the results of [CP] and [Pa] to our attention.

2. MA(pc ω_1) **and ladder systems.** In this section we show that $\mathrm{MA}_{\aleph_1}(\mathrm{pc}\ \omega_1)$ does not hold in $\mathcal{M}[c]$ by constructing a non-uniformizable ladder system coloring. We first recall the basic definitions and theorem, which appear in [DS].

DEFINITION 2.1. Let Ω denote the set of limit ordinals below ω_1. Given $\alpha \in \Omega$, a *ladder* d_α on α is a strictly increasing ω-sequence $\langle d_m^\alpha \rangle_{m\in\omega}$ cofinal in α. $d = \langle d_\alpha \rangle_{\alpha\in\Omega}$ is a *ladder system* on Ω if and only if d_α is a ladder on α for all $\alpha \in \Omega$.

DEFINITION 2.2. Let d be a ladder system on Ω. A *coloring* on d is an Ω-sequence $k = \langle k_\alpha \rangle_{\alpha\in\Omega}$ with $k_\alpha \in 2^\omega$ for all $\alpha \in \Omega$. We say that the coloring system $\langle d, k\rangle$ on Ω is *uniformizable* if and only if there is a function $g : \Omega \mapsto \omega$ for which $h_g = \{\langle d_m^\alpha, k_\alpha(m)\rangle : \alpha \in \Omega, m \ge g(\alpha)\}$ is a function.

THEOREM 2.3 (Devlin, Shelah). $\mathrm{MA}_{\aleph_1}(pc\ \omega_1)$ *implies that every coloring system* $\langle d, k\rangle$ *on* Ω *is uniformizable.*

THEOREM 2.4. *Let* $\mathcal{C}$ *be the poset* $(2)^{<\omega}$ *adding a Cohen real. There is a* $\mathcal{C}$*-term* $\dot{k} = \langle \dot{k}_\alpha \rangle_{\alpha\in\Omega}$ *for a ladder system coloring such that for every* $d \in \mathcal{M}$, *if* d *is a ladder system on* ω_1, *then*

$$\mathcal{M}[c] \models [\textit{If } S \textit{ stationary, then } \langle \check{d}, \dot{k}\rangle \textit{ is non-uniformizable on } S]\,.$$

Proof. Let $\langle A_\alpha \rangle_{\alpha\in\Omega}$ be a family of almost disjoint subsets of ω in $\mathcal{M}$, $A_\alpha = \{a_{\alpha m}\}_{m\in\omega}$ for all $\alpha \in \Omega$. Let c, c' be two mutually generic Cohen reals with $c : \omega \mapsto \omega$, $c' : \omega \mapsto 2$. Define $k_\alpha : \omega \mapsto 2$ in $\mathcal{M}[c]$ by $k_\alpha(m) = c'(a_{\alpha c(m)})$ and let $\dot{k} = \langle \dot{k}_\alpha \rangle_{\alpha\in\omega_1}$. Let $d = \langle d_\alpha \rangle_{\alpha\in\Omega}$ be a ladder system in $\mathcal{M}$ and suppose $S \subseteq \Omega$ is stationary in $\mathcal{M}$. We show $\Vdash_{\mathcal{C}} [\langle \check{d}, \dot{k}\rangle$ non-uniformizable on $\check{S}]$. Then since every $\mathcal{M}[c]$ stationary set contains an $\mathcal{M}$ stationary subset of Ω, we are done.

Suppose $\langle p, p'\rangle \Vdash [\dot{g} : \check{S} \mapsto \omega]$. We show there is $\langle q, q'\rangle \le \langle p, p'\rangle$ such that $\langle q, q'\rangle \Vdash [\dot{g}$ does not uniformize $\langle \check{d}, \dot{k}\rangle$ on $\check{S}]$. By extending if necessary, we may assume that for all $\alpha \in S$, $\langle p, p'\rangle \Vdash \dot{g}(\alpha) = m_\alpha$ for some $m_\alpha \in \omega$. Further, assume that $m_\alpha = m$ for all $\alpha \in S$. Choose $l > m$ such that $l \notin \mathrm{dom}(p)$. Since $d_l^\alpha < \alpha$ for all $\alpha \in S$, Fodor's lemma gives us a stationary set $S' \subseteq S$ and some $\nu \in \omega_1$ such that $d_l^\alpha = \nu$ for all $\alpha \in S'$. Fix $\alpha, \beta \in S'$. Since $|A_\alpha \cap A_\beta| < \aleph_0$, there is $n \in \omega$ with $a_{\alpha n} \ne a_{\beta n}$ and $a_{\alpha n}, a_{\beta n} \notin \mathrm{dom}(p')$. Extend $\langle p, p'\rangle$ to $\langle q, q'\rangle$ in such a way that $q(l) = n$ and $q'(a_{\alpha n}) \ne q'(a_{\beta n})$. Thus, $\langle q, q'\rangle \Vdash [(\dot{g}(\alpha), \dot{g}(\beta) < l) \wedge (\dot{k}_\alpha(l) \ne \dot{k}_\beta(l))]$. Since $d_l^\alpha = d_l^\beta$, this gives $\langle q, q'\rangle \Vdash [\dot{g}$ does not uniformize $\langle \check{d}, \dot{k}\rangle$ on $\check{S}]$, as desired. ∎

COROLLARY 2.5. $\mathcal{M}[c] \models \neg\mathrm{MA}_{\aleph_1}(pc\ \omega_1)$.

Combining this with Theorem 1.3, we have the following consistency result.

COROLLARY 2.6. *Assume* Con(ZFC). *Then*

$$\mathrm{Con}(\mathrm{ZFC} + 2^{\aleph_0} > \aleph_1 + \mathrm{MA}(\sigma\text{-}centered) + \neg\,\mathrm{MA}_{\aleph_1}(pc\ \omega_1))\,.$$

LEMMA 2.7. *Let* $\langle d,k\rangle \in \mathcal{M}$ *be a coloring system on* ω_1 *which is not uniformizable on any stationary set* $S \in \mathcal{M}$. *Then for any* σ*-2-linked partial order* $\mathcal{P}$ *in* $\mathcal{M}$,

$$\mathcal{M}[G_{\mathcal{P}}] \vDash [\langle \check{d},\check{k}\rangle \text{ is not uniformizable}]\,.$$

Proof. Working in $\mathcal{M}$, let $\mathcal{P} = \bigcup_{i\in\omega}\mathcal{P}_i$ be a 2-linked decomposition of $\mathcal{P}$. Suppose $\dot{g}$ is a $\mathcal{P}$-term and $p \in \mathcal{P}$ such that $p \Vdash [\dot{g} : \Omega \mapsto \omega]$. We show that for some $q \le p$, $q \Vdash [\dot{g}$ does not uniformize$\langle \check{d},\check{k}\rangle]$. Since $p \Vdash [\dot{g} : \Omega \mapsto \omega]$, there is a stationary set $S \subseteq \Omega$ in $\mathcal{M}$ and some $l \in \omega$ such that for all $\alpha \in S$, there is $q_\alpha \le p$ satisfying $q_\alpha \Vdash \dot{g}(\alpha) = l$. Furthermore, we may assume that for some $i \in \omega$, $q_\alpha \in \mathcal{P}_i$ for all $\alpha \in S$. Since $\langle d,k\rangle$ is not uniformizable on S, there are some $\alpha,\beta \in S$ and $m,m' > l$ such that $d^\alpha_m = d^\beta_{m'}$ and $k_\alpha(m) \ne k_\beta(m')$. Using 2-linkedness of $\mathcal{P}_i$, let $q \le q_\alpha, q_\beta$. Then $q \Vdash [\dot{g}(\alpha) < m$ and $\dot{g}(\beta) < m']$ and we have $q \Vdash [\dot{g}$ does not uniformize $\langle \check{d},\check{k}\rangle]$, as desired. ∎

THEOREM 2.8. *Assume* Con(ZFC). *Then*

$$\mathrm{Con}(\mathrm{ZFC} + 2^{\aleph_0} > \aleph_1 + \mathrm{MA}(\sigma\text{-2-}linked) + \neg\,\mathrm{MA}_{\aleph_1}(pc\ \omega_1))\,.$$

Proof. Assume $\mathcal{M} \vDash [2^{\aleph_0} = 2^{\aleph_1} = \aleph_2]$. Using standard methods and Lemma 1.5, we obtain a σ-2-linked partial order $\mathcal{P}$ in $\mathcal{M}[c]$ forcing MA(σ-2-linked) in the extension. By Lemma 2.7, $\mathcal{M}[c][G_{\mathcal{P}}] \vDash [\langle \check{d},\dot{k}\rangle$ is not uniformizable], where $\dot{k}$ is the coloring of Theorem 2.4 and d is any ladder system in $\mathcal{M}$. By Theorem 2.3, we have $\mathcal{M}[c][G_{\mathcal{P}}] \vDash [2^{\aleph_0} = \aleph_2 +$ MA(σ-2-linked)$+\neg\,\mathrm{MA}_{\aleph_1}$(pc ω_1)], as desired. ∎

3. MA(σ-n-linked) **and** $\le n$**-ary sets of reals.** In this section we use the notion of a $\le n$-ary set of reals, which is due to Todorčević [To$_1$], to obtain results concerning $\neg\,$MA(σ-n-linked). In particular, we present his proof that $\mathcal{M}[c] \vDash \forall n \in \omega\, \neg\,\mathrm{MA}_{\aleph_1}(\sigma\text{-}n\text{-linked})$ in Theorem 3.3 [To$_2$].

DEFINITION 3.1. Given $f,g \in \omega^\omega$, let $\Delta(f,g) = \min\{i \in \omega : f(i) \ne g(i)\}$. A set $A \subseteq \omega^\omega$ is *co-divergent at level* $m \in \omega$ if and only if $\Delta(f,g) = m$ for all $f,g \in A$. Given $n < \omega$, A is $\le n$*-ary* if and only if A contains no co-divergent subsets of size $n+1$. A is *finitary* if and only if A contains no infinite co-divergent subsets.

If $f \in \omega^\omega$ and $A \subseteq \omega^\omega$, we will use $\Delta(f,A)$ to denote the minimum of $\Delta(f,g)$ for $g \in A$.

THEOREM 3.2 (Todorčević). *For all* $n \in \omega$, $\mathrm{MA}_{\aleph_1}$(σ-n-*linked*) *implies that every uncountable set of reals contains an uncountable* $\le n$*-ary subset.*

P r o o f. Assume $\mathrm{MA}_{\aleph_1}(\sigma$-$n$-linked) and let $A = \{f_\eta\}_{\eta\in\omega_1} \subseteq \omega^\omega$. We define a σ-n-linked partial order $\mathcal{P}$ which adds an uncountable $\le n$-ary subset of A as follows: $a \in \mathcal{P}$ if and only if

(1) a is a finite $\le n$-ary subset of A,
(2) $|\{f \in A : \Delta(f,a) > \max\{\Delta(g,g') : g,g' \in a\}\}| = \aleph_1$;

and $a \le a'$ if and only if

(1) $a \supseteq a'$,
(2) for all $f \in a \setminus a'$, $\Delta(f,a') > \max\{\Delta(g,g') : g,g' \in a'\}$.

Note that condition (2) of the definition of $a \in \mathcal{P}$ implies that for all $\eta \in \omega_1$, the set $D_\eta = \{a \in \mathcal{P} : f_\beta \in a \text{ for some } \beta > \eta\}$ is dense in $\mathcal{P}$.

Given $m \in \omega$ and a finite set $t \subseteq \omega^m$ containing no co-divergent subsets of size $n+1$, let

$$\mathcal{P}_t = \{a \in \mathcal{P} : a \restriction m = t \text{ and } \max\{\Delta(g,g') : g,g' \in a\} < m\}\,.$$

Clearly, $\mathcal{P}$ equals the union of all such $\mathcal{P}_t$. Also, if $a_0, a_1, \ldots, a_{n-1} \in \mathcal{P}_t$, then $a = \bigcup_{i\in n} a_i \in \mathcal{P}$ extending each a_i, so that $\mathcal{P}$ is σ-n-linked. Applying $\mathrm{MA}_{\aleph_1}(\sigma$-$n$-linked), let G be generic for $\langle D_\eta\rangle_{\eta\in\omega_1}$. Then $A' = \bigcup_{a\in G} a \subseteq A$ is $\le n$-ary, while $|A'| = \aleph_1$ by density of the D_η's. Thus, A' is the desired $\le n$-ary subset of A. ∎

Theorem 3.3 (Todorčević).

$$\mathcal{M}[c] \models [\exists X \in [\omega^\omega]^{2^{\aleph_0}}\ \forall n \in \omega\ X \textit{ contains no uncountable } {\le} n\textit{-ary subsets}]\,.$$

P r o o f. Let $c : \omega^{<\omega} \mapsto \omega$. For all $h \in \omega^\omega \cap \mathcal{M}$, define in $\mathcal{M}[c]$ a function $h^c : \omega \mapsto \omega$ by $h^c(n) = c(h \restriction n)$ and let $X = \{h^c : h \in \omega^\omega \cap \mathcal{M}\} \in \mathcal{M}[c]$. Fix $n \in \omega$ and suppose $p \Vdash [\{\dot{x}_\eta\}_{\eta\in\omega_1}$ enumerates $\dot{Y} \subseteq \dot{X}]$. We show there is $p' \le p$ such that $p' \Vdash [Y$ is not $\le n$-ary$]$.

Assume without loss of generality that for all $\eta \in \omega_1$, $h_\eta \in \omega^\omega \cap \mathcal{M}$ is such that $p \Vdash [h^c_\eta = \dot{x}_\eta]$. Thus, $\{h_\eta : \eta \in \omega_1\}$ is an uncountable set in $\mathcal{M}$. Let $m \in \omega$ be such that $\mathrm{dom}(z) \subseteq m$ for all $z \in \mathrm{dom}(p)$. Then for some $S' \in [S]^{\aleph_1}$ and $h \in \omega^m$, we have $h_\eta \restriction m = h$ for all $\eta \in S'$. Choose $\eta_0, \eta_1, \ldots, \eta_n \in S'$ and let $m' = \max\{\Delta(h_{\eta_i}, h_{\eta_j}) : i,j \le n\}$. Extend p to p' in such a way that

(1) $p'(h_{\eta_i} \restriction l) = p'(h_{\eta_j} \restriction l)$ for all $i,j \le n$ and all $l < m'$; note that p already satisfies $p(h_{\eta_i} \restriction l) = p(h \restriction l) = p(h_{\eta_j} \restriction l)$ for all $l < m$ with $h \restriction l \in \mathrm{dom}(p)$,
(2) $p'(h_{\eta_i} \restriction m') \ne p'(h_{\eta_j} \restriction m')$ for all $i,j \le n$.

Then $p' \Vdash [\{h^c_{\eta_i}\}_{i\le n} \subseteq \dot{Y}$ co-diverges at level $m']$, as desired. ∎

Corollary 3.4. $\mathcal{M}[c] \models [\forall n \in \omega\ \neg\mathrm{MA}_{\aleph_1}(\sigma\textit{-}n\textit{-linked})]$.

Again combining this corollary with Theorem 1.3, we have the following consistency result.

COROLLARY 3.5. *Assume* Con(ZFC). *Then*

$$\mathrm{Con}(\mathrm{ZFC} + 2^{\aleph_0} > \aleph_1 + \mathrm{MA}(\sigma\text{-}\mathit{centered}) + \forall n \in \omega\, \neg\mathrm{MA}_{\aleph_1}(\sigma\text{-}n\text{-}\mathit{linked}))\,.$$

In Section 4, we will show that we cannot replace the "$\le n$-ary" of Theorem 3.3 by "finitary" (see Lemma 4.1).

LEMMA 3.6. *Suppose $n \in \omega$ and $X \in \mathcal{M}$ is an uncountable set of reals containing no $\mathcal{M}$-uncountable $\le n$-ary subsets. Then for every property* K_{n+1} *partial order $\mathcal{P}$ in $\mathcal{M}$*,

$$\mathcal{M}[G_{\mathcal{P}}] \models [\check{X}\, \mathit{contains\ no\ uncountable}\ \le n\text{-}\mathit{ary\ subsets}]\,.$$

Proof. Working in $\mathcal{M}$, suppose $p \Vdash [\{\dot{x}_\eta\}_{\eta\in\omega_1}$ enumerates $\dot{Y} \subseteq \check{X}]$. Then for each $\eta \in \omega_1$, there are some $p'_\eta \le p$ and some $h_\eta \in X$ such that $p'_\eta \Vdash \dot{x}_\eta = h_\eta$. Using property K_{n+1}, we may assume that $\{p'_\eta : \eta \in \omega_1\}$ is $(n+1)$-linked. Since $\{h_\eta : \eta \in \omega_1\}$ is an uncountable subset of X, there are $\eta_0, \eta_1, \ldots, \eta_n \in \omega_1$ such that the set $\{h_{\eta_i}\}_{i\le n}$ is co-divergent. Taking $p' \in \mathcal{P}$ with $p' \le p'_{\eta_i}$ for all $i \le n$, we have $p' \Vdash [\{h_{\eta_i}\}_{i\le n} \subseteq \dot{Y}$ is co-divergent], as desired. ■

THEOREM 3.7. *Assume* Con(ZFC). *Then for all $n \in \omega$*,

$$\mathrm{Con}(\mathrm{ZFC} + 2^{\aleph_0} > \aleph_1 + \mathrm{MA}(\mathrm{K}_{n+1}) + \neg\mathrm{MA}_{\aleph_1}(\sigma\text{-}n\text{-}\mathit{linked}))\,.$$

Proof. Similar to Theorem 2.8, using Lemmas 1.4 and 3.6. ■

4. f-Thin sets. In view of Theorem 3.3, a natural question to ask is whether there exists an uncountable $X \subseteq \omega^\omega$ in $\mathcal{M}[c]$ which contains no uncountable finitary subsets. The following theorem shows this is not necessarily the case.

THEOREM 4.1. $\mathrm{MA}_{\aleph_1}$*(σ-centered) implies that every uncountable $A \subseteq \omega^\omega$ contains an uncountable finitary subset.*

Proof. Assume $\mathrm{MA}_{\aleph_1}$(σ-centered) and define a partial order $\mathcal{P}$ as in Theorem 3.2 with condition (1) changed to

(1′) a is a finite subset of A.

Define the cells $\mathcal{P}_t$ for finite $t \subseteq \omega^{<\omega}$ as before. Then each cell is centered, so that $\mathcal{P}$ is σ-centered. The rest of the proof is the same. ■

COROLLARY 4.2. *If $\mathcal{M} \models \mathrm{MA}_{\aleph_1}$($\sigma$-centered), then*

$$\mathcal{M}[c] \models [\mathit{Every}\ X \in [\omega^\omega]^{\aleph_1}\ \mathit{has\ an\ uncountable\ finitary\ subset}]\,.$$

Although $\mathrm{MA}_{\aleph_1}$(σ-centered) guarantees the existence of an uncountable finitary subset for any uncountable $X \subseteq \omega^\omega$, it says nothing about how this set branches. In particular, we will show that $\mathrm{MA}_{\aleph_1}(\sigma\text{-centered})$ does not ensure the existence of a finitary subset which is thin in the following sense.

DEFINITION 4.3. Let $f \in \omega^\omega$. A set $A \subseteq \omega^\omega$ is *f-thin* if and only if every $B \subseteq A$ which is co-divergent at level m has cardinality $\le f(m)$.

Remark that for the constant function c_n defined by $c_n(j) = n$ for all $j \in \omega$, A is a c_n-thin set if and only if A is $\le n$-ary. In fact, by Corollary 4.8 below, for any function $f \in \omega^\omega$ with $\lim_{n\in\omega} f(n) \ne \omega$, the addition of an uncountable f-thin set is equivalent to the addition of an uncountable $\le n$-ary set, n being the least integer for which $|\{j \in \omega : f(j) = n\}| = \aleph_0$; in this case, the existence of an uncountable f-thin set is thus guaranteed by $\mathrm{MA}_{\aleph_1}(\sigma\text{-}n\text{-linked})$.

For $f \in \omega^\omega$ with $\lim_{n\in\omega} f(n) = \omega$, the existence of uncountable f-thin subsets follows from $\mathrm{MA}(\bigwedge\!\!\!\bigwedge_{n\in\omega} \sigma\text{-}n\text{-linked})$ (see Theorem 4.9 below). By Corollary 4.6, this case reduces to showing that $\mathrm{MA}(\bigwedge\!\!\!\bigwedge_{n\in\omega} \sigma\text{-}n\text{-linked})$ implies the existence of an uncountable i-thin set, where $\mathrm{i} \in \omega^\omega$ is the identity function.

DEFINITION 4.4. Let $f, g \in \omega^\omega$. Then f *eventually bounds g everywhere*, denoted by $g \preceq f$, if and only if $|\{j \in \omega : f(j) < g(m)\}| < \aleph_0$ for all $m \in \omega$. If there is $M \in \omega$ such that $|\{j \in \omega : f(j) < g(m)\}| < \aleph_0$ for all $m \ge M$, then f *eventually bounds g almost everywhere*, denoted by $g \preceq^* f$.

THEOREM 4.5. *Let $f, g \in \omega^\omega$ with $g \preceq^* f$. Suppose that every $A \subseteq \omega^\omega$ has an uncountable g-thin subset. Then every $A \subseteq \omega^\omega$ has an uncountable f-thin subset.*

Proof. Let $A \subseteq \omega^\omega$; we show A has an uncountable f-thin subset. To this end, let $M \in \omega$ with $|\{j \in \omega : f(j) < g(m)\}| < \aleph_0$ for all $m \ge M$. Then there is some $j_M \ge M$ such that $g(M) \le f(j)$ for all $j > j_M$. Without loss of generality, assume $s \restriction j_M = s' \restriction j_M$ for all $s, s' \in A$. Define by induction an increasing sequence of integers $\{j_m\}_{m \ge M}$ such that for all $j \in [j_m, j_{m+1})$ we have $g(m) \le f(j)$. Let $C_m = [j_m, j_{m+1})$ and $k_m = |C_m|$ for all $m \ge M$. Fix a 1-1 onto map $l_m : [\omega]^{k_m} \mapsto \omega$ for each $m \ge M$ and define a function $h_s \in \omega^\omega$ by $h_s(m) = l_m(s \restriction C_m)$ for each $s \in A$. Thus, h_s codes up segments of s onto single integers in such a way that for all $s, s' \in A$, $\Delta(h_s, h_{s'}) = m$ if and only if $\Delta(s, s') = j$ for some $j \in C_m$.

Let B' be an uncountable g-thin subset of $B = \{h_s : s \in A\}$. We show the set $A' = \{s \in A : h_s \in B'\}$ is f-thin; then since $|A'| = |B'| = \aleph_1$, we are done. So, suppose that for some $j \in \omega$ and $l > g(j)$, the set $\{s_i\}_{i\in l} \subseteq A'$ is co-divergent at level j. Note that $j \ge j_M$ since the tree $\{s \restriction k : k < j_M\}$ is 1-branching. Thus, $j \in C_m$ for some $m \ge M$. But then $\{h_{s_i}\}_{i\in l} \subseteq B'$ is co-divergent at level m and $g(m) \le f(j) < l$, contrary to B' being g-thin. ■

COROLLARY 4.6. *Let $f \in \omega^\omega$ with $\lim_{n\in\omega} f(n) = \omega$. Then every uncountable $A \subseteq \omega^\omega$ contains an uncountable f-thin set if and only if every uncountable $A \subseteq \omega^\omega$ contains an uncountable* i*-thin subset*, *where* $\mathrm{i} \in \omega^\omega$ *is the identity map.*

Proof. Since $\mathrm{i} \preceq^* f$ and $f \preceq^* \mathrm{i}$, both directions follow from Theorem 4.5. ∎

THEOREM 4.7. *Let $n \in \omega$ and $f \in \omega^\omega$ such that $|\{j \in \omega : f(j) = n\}| = \aleph_0$. Suppose that every uncountable $A \subseteq \omega^\omega$ contains an uncountable f-thin subset. Then every uncountable $A \subseteq \omega^\omega$ has an uncountable $\le n$-ary subset.*

Proof. Let $A \subseteq \omega^\omega$; we show A has an uncountable $\le n$-ary subset. To this end, let $M \in \omega$ with $\{j \in \omega : f(j) < n\} \subseteq M$. Without loss of generality, assume $s \restriction M = s' \restriction M$ for all $s, s' \in A$. Let $l : \omega \mapsto f^{-1}(\{n\})$ be an increasing enumeration of $f^{-1}(\{n\})$ and define for each $s \in A$ a function $h_s \in \omega^\omega$ as follows:

$$h_s(m) = \begin{cases} s(l(m)) & \text{if } m \in f^{-1}(\{n\}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\Delta(h_s, h_{s'}) = m$ if and only if $\Delta(s, s') \in f^{-1}(\{n\})$. Thus, if B' is an uncountable f-thin subset of $B = \{h_s : s \in A\}$, then $A' = \{s \in A : h_s \in B'\}$ is $\le n$-ary; since $|A'| = |B'| = \aleph_1$, A' is the desired subset. ∎

COROLLARY 4.8. *Let $f \in \omega^\omega$ with $\lim_{n\in\omega} f(n) \ne \omega$. Then every uncountable $A \subseteq \omega^\omega$ contains an uncountable f-thin set if and only if every uncountable $A \subseteq \omega^\omega$ contains an uncountable $\le n$-ary subset*, *where $n \in \omega$ is the least integer for which $|\{j \in \omega : f(j) = n\}| = \aleph_0$.*

Proof. Since $c_n \preceq^* f$ by minimality of n, the "if" direction follows from Theorem 4.5. The "only if" direction is Theorem 4.7. ∎

THEOREM 4.9. *For all $f \in \omega^\omega$ with $\lim_{n\in\omega} f(n) = \omega$,* $\mathrm{MA}_{\aleph_1}(\bigwedge\!\!\!\bigwedge_{n\in\omega} \sigma\text{-}n\text{-linked})$ *implies that every uncountable $A \subseteq \omega^\omega$ contains an uncountable f-thin subset.*

Proof. By Corollary 4.6, we need only show that $\mathrm{MA}_{\aleph_1}(\bigwedge\!\!\!\bigwedge_{n\in\omega} \sigma\text{-}n\text{-linked})$ yields an uncountable i-thin set, where i is the identity function. To this end, proceed as in Theorem 3.2, replacing condition (1) of the definition of $\mathcal{P}$ by

(1′) a is a finite i-thin subset of A.

Then the cells $\mathcal{P}_t$ are $\mathrm{i}(n) = n$-linked for all i-thin $t \in [\omega^n]^{<\omega}$. Furthermore, given $n < n' < \omega$, if $a \in \mathcal{P}_t$ for some i-thin $t \in [\omega^n]^{<\omega}$, then $a \in \mathcal{P}_{t'}$ where $t' = a \restriction n'$ is an i-thin subset of $[\omega^{n'}]^{<\omega}$. Thus, for each $n \in \omega$, $\mathcal{P} = \bigcup_{\mathrm{ht}(t)\ge n} \mathcal{P}_t$ is the desired n-linked decomposition of $\mathcal{P}$. The rest of the proof is the same. ∎

We now construct the Cohen counterexample to $\mathrm{MA}_{\aleph_1}(\bigwedge\!\!\!\bigwedge_{n\in\omega}\sigma\text{-}n\text{-linked})$.

THEOREM 4.10. $\mathcal{M}[c] \models [\exists X \in [\omega^\omega]^{2^{\aleph_0}}\ \forall f \in \omega^\omega \cap \mathcal{M}\ X$ *contains no uncountable f-thin subsets*$]$.

Proof. We modify the definition in Theorem 3.3 of a set containing no uncountable $\le n$-ary subsets. To this end, let c and c' be mutually generic Cohen reals with $c : \omega^{<\omega} \mapsto \omega$ and $c' : \omega \mapsto \omega$. Given $h \in \omega^\omega \cap \mathcal{M}$, define $\overline{h} \in \omega^\omega \cap \mathcal{M}[c]$ by $\overline{h}(n) = c(h \restriction c'(n))$ and let X be the set in $\mathcal{M}[c]$ of all such $\overline{h}$. Fix $f \in \omega^\omega \cap \mathcal{M}$ and suppose $\langle p, p'\rangle \Vdash [\{\dot{x}_\alpha\}_{\alpha\in\omega_1}$ enumerates $\dot{Y} \subseteq \dot{X}]$. We show there exists $\langle q, q'\rangle \le \langle p, p'\rangle$ such that $\langle q, q'\rangle \Vdash [\dot{Y}$ not $\check{f}$-thin$]$.

Without loss of generality, suppose for all $\eta \in \omega_1$, $h_\eta \in \omega^\omega \cap \mathcal{M}$ is such that $\langle p, p'\rangle \Vdash \dot{x}_\eta = \overline{h}_\eta$. Let $l, m \in \omega$ be such that

(1) for all $z \in \mathrm{dom}(p)$, $\mathrm{dom}(z) \subseteq l$,
(2) $\mathrm{ran}(p') \subseteq l$,
(3) $\mathrm{dom}(p') \subseteq m$.

Then there are $S \in [\omega_1]^{\aleph_1} \cap \mathcal{M}$ and $h \in \omega^l$ such that $h_\eta \restriction l = h$ for all $\eta \in S$. Let $m' > m$ and set $M = f(m')$. Choose $\eta_0, \eta_1, \ldots, \eta_M \in S$ and let $L > l$ be such that $\Delta(h_{\eta_i}, h_{\eta_j}) \le L$ for all $i, j \le M$. Extend $\langle p, p'\rangle$ to $\langle q, q'\rangle$ in such a way that

(1) for all $k < m'$, $q'(k) < l$; note that p' already satisfies $p'(k) < l$ for $k \in \mathrm{dom}(p')$,
(2) $q'(m') = L$,
(3) for all $l' < l$, $h \restriction l' \in \mathrm{dom}(q)$,
(4) for all $i, j \le M$, $q(h_{\eta_i} \restriction L) \ne q(h_{\eta_j} \restriction L)$.

Then by conditions (1) and (3) and the assumption $h_{\eta_i} \restriction l = h$ for all $i \le M$, we have $q(h_{\eta_i} \restriction q'(k)) = q(h_{\eta_j} \restriction q'(k))$ for all $k < m'$ and $i, j \le M$, while conditions (2) and (3) yield $q(h_{\eta_i} \restriction q'(m')) \ne q(h_{\eta_j} \restriction q'(m'))$ for all $i, j \le M$. Thus, $f(m') = M$ and $\langle q, q'\rangle \Vdash [\{\overline{h}_{\eta_i}\}_{i\le M} \subseteq \dot{Y}$ is co-divergent at level $m']$. It follows that $\langle q, q'\rangle \Vdash [\dot{Y}$ is not $\check{f}$-thin$]$, as desired. ∎

COROLLARY 4.11. $\mathcal{M}[c] \models \neg\mathrm{MA}_{\aleph_1}(\bigwedge\!\!\!\bigwedge_{n\in\omega}\sigma\text{-}n\text{-}linked)$.

Again, using the fact that $\mathcal{M}[c]$ preserves MA(σ-centered), we now have the following consistency result, which completes the MA hierarchy diagram under discussion.

COROLLARY 4.12. *Assume* Con(ZFC). *Then*

$$\mathrm{Con}\Big(\mathrm{ZFC} + 2^{\aleph_0} > \aleph_1 + \mathrm{MA}(\sigma\text{-}centered) + \neg\mathrm{MA}_{\aleph_1}\Big(\bigwedge\!\!\!\bigwedge_{n\in\omega}\sigma\text{-}n\text{-}linked\Big)\Big).$$

We close with a final result concerning the addition of f-thin subsets. By Theorem 4.5, for functions f, g satisfying $g \preceq^* f$, any Martin's Axiom variant which adds a g-thin subset through *every* uncountable set of reals necessarily adds a f-thin subset through every uncountable set of reals, even though g may grow at a much faster rate than f. However, the next theorem shows that for $g \preceq^* f$, the addition of an uncountable g-thin subset through a given set A of reals need not add an uncountable f-thin subset through A. Note that if $g \leq^* f$, where we say $g \leq^* f$ if and only if $|\{j \in \omega : f(j) < g(j)\}| < \omega$, then any uncountable g-thin subset of reals does contain an uncountable f-thin subset.

THEOREM 4.13. *Suppose $f, g \in \omega^\omega \cap \mathcal{M}$ such that $\neg(g \leq^* f)$. Let $\dot{X}$ be the $\mathcal{C}$-term of Theorem* 4.10 *for which*

$$\Vdash_{\mathcal{C}} [\dot{X} \textit{ has no uncountable } \check{f}\textit{-thin subsets}],$$

and Π a $\mathcal{C}$-term for the partial order which adds an uncountable g-thin subset of $\dot{X}$. Then

$$\mathcal{M}[c][G_\Pi] \vDash [\dot{X} \textit{ has no uncountable } \check{f}\textit{-thin subsets}].$$

Proof. Note that if $\langle p, p' \rangle \Vdash \pi \in \Pi$, then for some finite g-thin $a \subseteq \omega^\omega$ in $\mathcal{M}$ and some $\langle q, q' \rangle \leq \langle p, p' \rangle$, we have $\langle q, q' \rangle \Vdash \pi = \overline{a} = \{\overline{h} : h \in a\}$. Furthermore, we can assume that $\langle q, q' \rangle \Vdash [\overline{a} \in \Pi_t]$ for some finite g-thin $t \subseteq \omega^{<\omega}$, where Π_t is defined in $\mathcal{M}[c]$ as in Theorem 3.2.

Working in $\mathcal{M}$, suppose $\langle \langle p, p' \rangle, \pi \rangle \Vdash [\{\dot{x}_\eta\}_{\eta \in \omega_1}$ enumerates $\dot{Y} \subseteq \dot{X}]$. As before, we may assume $h_\eta \in \omega^\omega \cap \mathcal{M}$ and $\langle \langle p_1, p_1' \rangle, \pi_\eta' \rangle \leq \langle \langle p, p' \rangle, \pi \rangle$ satisfy $\langle \langle p_1, p_1' \rangle, \pi_\eta' \rangle \Vdash \overline{h}_\eta = \dot{x}_\eta$ for all $\eta \in \omega_1$. By the above comment, we may also assume that $\langle p_1, p_1' \rangle \Vdash \pi_\eta' = \overline{a}_\eta \in \Pi_{t_\eta}$ for all $\eta \in \omega_1$, where t_η is a finite g-thin subset of ω^{r_η} for some $r_\eta \in \omega$ and $a_\eta \in [\omega^\omega]^{<\omega}$. Finally, assume $t_\eta = t \in \omega^r$ and $|a_\eta| = |a_\beta|$ for all $\eta, \beta \in \omega_1$. Let $a_\eta = \{s_0^\eta, s_1^\eta, \ldots, s_N^\eta\}$ and take $l, m \in \omega$ such that

(1) for all $z \in \mathrm{dom}(p_1)$, $\mathrm{dom}(z) \subseteq l$,
(2) $\mathrm{ran}(p_1') \subseteq l$,
(3) $\mathrm{dom}(p_1') \subseteq m$,
(4) $r < m$.

Assume $h_\eta \restriction l = h_\beta \restriction l = h \in \omega^l$ and $s_n^\eta \restriction l = s_n^\beta \restriction l = s_n \in \omega^l$ for all $n \in N$ and all $\eta, \beta \in \omega_1$. Note that since $t \subseteq \omega^r$ and $\langle p_1, p_1' \rangle \Vdash [\overline{a}_\eta \in \Pi_t]$ for all $\eta \in \omega_1$, condition (4) implies $\langle p_1, p_1' \rangle \Vdash [\Delta(\overline{s}_n^\eta, \overline{s}_{n'}^\beta) = \Delta(\overline{s}_n^\eta, \overline{s}_{n'}^\eta) < m]$ for all $n \neq n'$ and $\eta, \beta \in \omega_1$. Since $\neg(g \leq^* f)$, there is some $m' > m$ such that $f(m') < g(m')$. Let $M = g(m')$ and choose $\eta_0, \eta_1, \ldots, \eta_{M-1} \in \omega_1$. Take $L > l$ such that for all $i, j < M$ and $n \in N$, $\Delta(h_{\eta_i}, h_{\eta_j}) \leq L$ and $\Delta(s_n^{\eta_i}, s_n^{\eta_j}) \leq L$. Extend $\langle p_1, p_1' \rangle$ to $\langle p_2, p_2' \rangle$ such that

(1) for all $k < m'$, $p_2'(k) < l$; again, we already have $p_1'(k) < l$ for $k \in \mathrm{dom}(p_1)$,
(2) $p_2'(m') = L$,

(3) for all $l' < l$ and $n \in N$, $h \restriction l', s_n \restriction l' \in \operatorname{dom}(p_2)$,
(4) for all $i, j < M$, $p_2(h_{\eta_i} \restriction L) \neq p_2(h_{\eta_j} \restriction L)$,
(5) for all $i, j < M$ and $n \in N$, $p_2(s_n^{\eta_i} \restriction L) \neq p_2(s_n^{\eta_j} \restriction L)$.

As before, this gives $\langle p_2, p_2' \rangle \Vdash [\Delta(\overline{h}_{\eta_i}, \overline{h}_{\eta_j}) = m'$ for all $i, j < M]$. Using conditions (1) and (3) and the fact that $s_\eta = s_n^\eta \restriction l$ for all $\eta \in \omega_1$ and $n \in N$, we also have $\langle p_2, p_2' \rangle \Vdash$[For all $n \in N$ and $i, j < M$, $\Delta(\overline{s}_n^{\eta_i}, \overline{s}_n^{\eta_j}) = m'$]. Since $M = g(m')$ and $\langle p_1, p_1' \rangle \Vdash [\Delta(s_n^{\eta_i}, s_{n'}^{\eta_j}) < m < m'$ for all $n, n' \in N$ and $i, j < M]$, we have $\langle p_2, p_2' \rangle \Vdash [\overline{a} = \bigcup_{i<M} \overline{a}_{\eta_i} \in \Pi]$. In particular, $\langle p_2, p_2' \rangle \Vdash [\overline{a} \leq \pi'_{\eta_i}]$ for all $i < M$. Thus, $f(m') < M$ and $\langle \langle p_2, p_2' \rangle, \overline{a} \rangle \Vdash [\{\overline{h}_{\eta_i}\}_{i<M} \subseteq \dot{Y}$ is co-divergent at level $m']$. It follows that $\langle \langle p_2, p_2' \rangle, \overline{a} \rangle \Vdash [\dot{Y}$ is not $\check{f}$-thin], as desired. ■

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Received 6 June 1991;
in revised form 28 January 1992