Expansive homeomorphisms and indecomposability

by

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Abstract. Suppose that $\mathbf{F}$ is a finite collection of graphs and a continuum X is $\mathbf{F}$-like. If X admits an expansive homeomorphism, then X contains an indecomposable (nondegenerate) subcontinuum.

1. Introduction. A compact connected (nondegenerate) metric space is called a *continuum*. A homeomorphism $f\colon X \to X$ of a compact metric space X is called *expansive* if there exists a constant $c > 0$ (called an *expansive constant* for f) such that if $x, y \in X$ and $x \neq y$, then there is an integer $n = n(x, y) \in \mathbf{Z}$ such that

$$d\big(f^n(x), f^n(y)\big) > c,$$

where d is a metric of X. Expansiveness does not depend on the choice of metric of X. We are interested in the following problem [3]: What kinds of continua admit expansive homeomorphisms? We know that if a continuum X is one-dimensional and admits an expansive homeomorphism, then X is considerably complicated. In fact, we know that all known one-dimensional continua admitting expansive homeomorphisms contain indecomposable subcontinua which play important parts in the dynamics of the expansive homeomorphisms. For instance, Williams' examples are solenoids [26] and Plykin's examples are lakes of Wada [22] and [23], which are well-known indecomposable continua. Naturally, we are interested in the following problem (A): Is it true that if a one-dimensional continuum X admits an expansive homeomorphism, then X contains an indecomposable (nondegenerate) subcontinuum? Note that if X is a continuum with $\dim X \geqslant 2$, then X always contains a hereditarily indecomposable subcontinuum Y with $\dim Y = \dim X - 1$.

In this paper, we give a partial answer to problem (A). More precisely, the following theorem is proved: Suppose that $\mathbf{F}$ is a finite collection of graphs and a continuum X is $\mathbf{F}$-like. If X admits an expansive homeomorphism, then X contains an indecomposable

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subcontinuum. Note that for any continuum X, $\dim X \leqslant 1$ if and only if there is a countable collection $\mathbf{C}$ of graphs such that X is $\mathbf{C}$-like. As a corollary, if $f\colon G \to G$ is an onto map of a graph G such that the shift homeomorphism $\tilde{f}$ is expansive, then the inverse limit (G, f) of f contains an indecomposable subcontinuum.

We refer the readers to [1] for the general properties of expansive homeomorphisms.

2. Definitions and preliminaries. A compact connected polyhedron P is called a *graph* if $\dim P = 1$. Let $\mathbf{P}$ be a collection of graphs. A continuum X is $\mathbf{P}$-*like* if for any $\varepsilon > 0$ there is a map f from X to some member P of $\mathbf{P}$ such that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for any $y \in f(X)$ (f is called an ε-*map*). In this paper, "ε-map" does not mean onto map. A compact metric space X is *tree-like* if X is a one-point set or X is a $\mathbf{T}$-like continuum, where $\mathbf{T} = \{\text{all trees } (= \text{ all graphs without simple closed curves})\}$. Note that a continuum X is $\mathbf{F}$-like for some finite collection $\mathbf{F}$ of graphs if and only if X is $\{G\}$-like for some graph G contained in $\mathbf{F}$.

An onto map $f\colon X \to Y$ is *monotone* if for any $y \in Y$, $f^{-1}(y)$ is connected. A continuum X is called a Θ_n-*continuum* if for any subcontinuum Y of X the complement $X - Y$ of Y in X has at most n components. We can easily see that if a continuum X is $\mathbf{F}$-like where $\mathbf{F}$ is a finite collection of graphs, then there is a natural number n such that if A and $B_1, \ldots, B_m$ are any subcontinua of X satisfying

(1) $A \cap B_i \neq \emptyset$ for $i = 1, \ldots, m$,

(2) B_i is not contained in $A \cup \big(\bigcup_{j=1, j\neq i}^{m} B_j\big)$ for $i = 1, \ldots, m$,

then $m \leqslant n$. By this fact, the following proposition is easily proved.

(2.1) PROPOSITION. *Let $\mathbf{F}$ be a finite collection of graphs. If a continuum X is $\mathbf{F}$-like, then there is a natural number n such that X is a Θ_n-continuum.*

A continuum X is *decomposable* if X is the union of two subcontinua different from X. A continuum X is *indecomposable* if X is not decomposable. A continuum X is *hereditarily decomposable* (resp. *hereditarily indecomposable*) if each nondegenerate subcontinuum of X is decomposable (resp. indecomposable).

We refer the readers to [17] for the properties of decomposable and indecomposable continua.

The following theorem obtained by Grace and Vought is very useful ([5, Theorem 2] and [25, Theorem 7]).

(2.2) THEOREM. *Let X be a hereditarily decomposable Θ_n-continuum. Then X admits an upper semicontinuous monotone decomposition $\mathcal{D}$ such that $X/\mathcal{D}$ is a (nondegenerate) graph which is a Θ_n-continuum. Furthermore, $\mathcal{D} = \{T^{2n}(x) \mid x \in X\}$, where for any subcontinuum A of X, $T(A) = A \cup \{x \in X - A \mid \text{there is no subcontinuum } H \text{ of } X \text{ such that } x \in \operatorname{Int}(H) \subset H \subset X - A\}$, and $T^0(A) = A$, $T^k(A) = T(T^{k-1}(A))$ for $k \geqslant 1$.*

Note that each homeomorphism $f\colon X \to X$ satisfies $f(T(A)) = T(f(A))$ for any subcontinuum A of X.

3. Main theorem. In this section, we prove the following main theorem of this paper.

(3.1) THEOREM. *Suppose that $\mathbf{F}$ is a finite collection of graphs and a continuum X is $\mathbf{F}$-like. If X admits an expansive homeomorphism, then X contains an indecomposable (nondegenerate) subcontinuum.*

To prove (3.1), we need the following results and notations.

(3.2) PROPOSITION ([24, (2.2)]). *If $f\colon X \to X$ is an expansive homeomorphism of a compact metric space X, then for any integer $n \in \mathbf{Z}$ $(n \neq 0)$, $f^n\colon X \to X$ is also expansive.*

(3.3) LEMMA ([12, Lemma 2.2]). *Let $f\colon X \to X$ be an expansive homeomorphism of a compact metric space X. Then there exists $\delta > 0$ such that for each nondegenerate subcontinuum A of X, there exists an integer $n_0 > 0$ satisfying one of the following conditions:*

(*) $\operatorname{diam} f^n(A) \geqslant \delta$ *for all* $n \geqslant n_0$, *or*

(**) $\operatorname{diam} f^{-n}(A) \geqslant \delta$ *for all* $n \geqslant n_0$.

In particular, if A is a nondegenerate subcontinuum of X such that $f^N(A) = A$ for some integer $N \neq 0$, then $\operatorname{diam} A \geqslant \delta$.

By a *refinement* of a finite collection $\mathcal{U}$ of subsets of a space X we mean, as usual, any finite collection of subsets of X whose elements are contained in elements of $\mathcal{U}$. Let $C_1, \ldots, C_m$ be a sequence of subsets of a space X. It is said to be a *chain*, and is denoted by $[C_1, \ldots, C_m]$, provided that $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$ for each $1 \leqslant i, j \leqslant m$. A chain $[C_1, \ldots, C_m]$ is said to be an η-*chain* $(\eta > 0)$ if $\operatorname{diam} C_i < \eta$ for each i. Let $[V_1, \ldots, V_m]$ be a chain such that $\mathcal{V} = \{V_i \mid 1 \leqslant i \leqslant m\}$ is a refinement of a finite open cover $\mathcal{U}$ of a space X. Let $U_1, U_2 \in \mathcal{U}$. Then the chain $[V_1, \ldots, V_m]$ is said to be *crooked between U_1 and U_2* if there are $1 \leqslant i(1) < i(2) < i(3) < i(4) \leqslant m$ such that $V_{i(1)} \subset U_1$, $V_{i(2)} \subset U_2$, $V_{i(3)} \subset U_1$ and $V_{i(4)} \subset U_2$. A chain $[V_1, \ldots, V_m]$ is said to be a *chain from x to y* $(x, y \in X)$ if $x \in V_1$ and $y \in V_m$.

(3.4) LEMMA ([13, (4.6)]). *Let $f\colon X \to X$ be an expansive homeomorphism of a continuum X. Then there exists $\delta > 0$ such that if $x, y \in X$, $x \neq y$, and $\mathcal{U}$ is any finite open cover of X, then there is an integer $N > 0$ and $\eta > 0$ such that if $[V_1, \ldots, V_m]$ is any η-chain from x to y, then either the chain $[f^N(V_1), \ldots, f^N(V_m)]$ or the chain $[f^{-N}(V_1), \ldots, f^{-N}(V_m)]$ is a refinement of $\mathcal{U}$, and is crooked between U_s and U_t, where $U_s, U_t \in \mathcal{U}$ and $d(U_s, U_t) \geqslant \delta - 2\operatorname{mesh}(\mathcal{U})$.*

(3.5) LEMMA (cf. [13, (5.1)]). *Suppose that $f\colon X \to X$ is a homeomorphism of a continuum X and $h\colon Z \to Z$ is a homeomorphism of a continuum Z. Also, suppose that $F\colon X \to Z$ is an onto map from X onto Z such that $F^{-1}(z)$ is hereditarily decomposable and tree-like for any $z \in Z$, and for some $z \in Z$, $F^{-1}(z)$ is nondegenerate (i.e., F is not a homeomorphism). If the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ Z & \xrightarrow[h]{} & Z \end{array}$$

is commutative, then f is not expansive.

Proof. Consider the set

$$H = \{(x, y) \in X \times X \mid F(x) = F(y)\}.$$

For any cover $\mathscr{U}$ of X, put $\bar{\mathscr{U}} = \{\mathrm{Cl}(U) \mid U \in \mathscr{U}\}$. For any subset M of H, define $M^f = \{(x, y) \in H \mid$ for any $\gamma > 0$ and any finite open cover $\mathscr{U}$ of X, there exists $(x', y') \in M$ such that $x' \neq y'$ and there exists a finite open cover $\mathscr{V}$ of X with $\mathrm{mesh}(\mathscr{V}) < \gamma$ such that $\bar{\mathscr{V}}$ is a refinement of $\mathscr{U}$, $\bar{\mathscr{V}}(F(x')) = \{\mathrm{Cl}(V) \mid V \in \mathscr{V}, V \cap F^{-1}(F(x')) \neq \emptyset\}$ is a refinement of $\mathscr{U}(F(x)) = \{U \in \mathscr{U} \mid U \cap F^{-1}(F(x)) \neq \emptyset\}$, the nerve $N(\mathscr{V}(F(x')))$ is a tree, and a chain $[V_1, \ldots, V_m]$ from x' to y' in $\mathscr{V}(F(x'))$ is crooked between U_x, U_y, where U_x and U_y are elements of $\mathscr{U}$ such that $x \in U_x$ and $y \in U_y\}$.

Then M^f is closed in H and $M^f \supset (M^f)^f$. For any ordinal numbers, define $M_1 = M^f$, $M_{\alpha+1} = (M_\alpha)^f$, and $M_\lambda = \bigcap_{\alpha<\lambda} M_\alpha$, where λ is a limit ordinal.

First, we shall prove the following.

Claim (I). *There is a countable ordinal α such that $M_\alpha = \emptyset$.*

Proof. Note that H is separable. Since M_α is closed in H and $M_\alpha \supset M_\beta$ for $\alpha \leqslant \beta$, there is a countable ordinal α such that $M_\alpha = M_\beta$ for all $\alpha \leqslant \beta$. In particular, $(M_\alpha)^f = M_\alpha$. We shall show that $M_\alpha = \emptyset$. Suppose, on the contrary, that $M_\alpha \neq \emptyset$. Choose $(x_1, y_1) \in M_\alpha$. By the definition of M^f, we may assume that $x_1 \neq y_1$. Since $F^{-1}(z)$ is tree-like for any $z \in Z$, there is a finite open cover $\mathscr{U}_1$ of X such that

a(1) $\mathrm{mesh}(\mathscr{U}_1) < \min\{1/2, d(x_1, y_1)/3\}$, and

b(1) the nerve $N(\mathscr{U}_1(F(x_1)))$ is a tree, where $\mathscr{U}_1(F(x_1)) = \{U \in \mathscr{U}_1 \mid U \cap F^{-1}(F(x_1)) \neq \emptyset\}$.

By induction, we can choose $(x_i, y_i) \in (M_\alpha)^f = M_\alpha$ $(i = 1, 2, \ldots)$ and a finite open cover $\mathscr{U}_i$ of X such that

a(i) $\mathrm{mesh}(\mathscr{U}_i) < 1/2^i$, $\bar{\mathscr{U}}_{i+1}$ is a refinement of $\mathscr{U}_i$ and $\bar{\mathscr{U}}_{i+1}(F(x_{i+1}))$ is a refinement of $\mathscr{U}_i(F(x_i))$,

b(i) the nerve $N(\mathscr{U}_i(F(x_i)))$ is a tree, and

c(i) a chain $[U_1^{i+1}, \ldots, U_{m(i+1)}^{i+1}]$ from x_{i+1} to y_{i+1} in $\mathscr{U}_{i+1}(F(x_{i+1}))$ is crooked between $U_{x_i}^i$ and $U_{y_i}^i$, where $x_i \in U_{x_i}^i \in \mathscr{U}_i$ and $y_i \in U_{y_i}^i \in \mathscr{U}_i$.

By b(i) and c(i), for each $i = 1, 2, \ldots$, we can choose a subchain $[U_{x_1}^i, \ldots, U_{y_1}^i]$ of $[U_1^i, \ldots, U_{m(i)}^i]$ such that

(1) $U_{x_1}^1 \supset \mathrm{Cl}(U_{x_1}^2) \supset U_{x_1}^2 \supset \mathrm{Cl}(U_{x_1}^3) \supset \ldots$,

(2) $U_{y_1}^1 \supset \mathrm{Cl}(U_{y_1}^2) \supset U_{y_1}^2 \supset \mathrm{Cl}(U_{y_1}^3) \supset \ldots$, and

(3) $[U_{x_1}^{i+1}, \ldots, U_{y_1}^{i+1}]$ is crooked between $U_{x_1}^i$ and $U_{y_1}^i$.

By a(i), $\lim_{i\to\infty} \mathrm{Cl}(U_{x_1}^i) = x$ and $\lim_{i\to\infty} \mathrm{Cl}(U_{y_1}^i) = y$. Note that $x \neq y$ and $F(x) = F(y)$. Also, $F^{-1}(z)$ is tree-like for each $z \in Z$. Put $Y = \bigcap_{i=1}^{\infty} G_i$, where $G_i = \bigcup \{\mathrm{Cl}(U) \mid U \in \mathscr{U}_i(F(x_i))\}$. By a($i$) and b($i$), Y is a continuum such that $Y \subset F^{-1}(F(x))$ and $x, y \in Y$. Let K be the irreducible subcontinuum between x and y in $F^{-1}(F(x))$. Since $F^{-1}(F(x))$ is tree-like, we see that $K \subset Y$. Thus K is nondegenerate.

We shall show that K is indecomposable. Suppose, on the contrary, that there are two proper subcontinua A and B of K such that $K = A \cup B$. Since K is irreducible between x and y in Y, we may assume that $x \in A - B$ and $y \in B - A$. By a(i), we can choose i such that $\mathrm{Cl}(U_{x_1}^i) \cap B = \emptyset$ and $\mathrm{Cl}(U_{y_1}^i) \cap A = \emptyset$. Consider the cover $\mathscr{U}_{i+1}(F(x_{i+1}))$ of $F^{-1}(F(x_{i+1}))$. Note that the nerve $N(\mathscr{U}_{i+1}(F(x_{i+1})))$ is a tree and

$\mathscr{U}_{i+1}(F(x_{i+1}))$ is a cover of K. By (3), there are two subchains $[U_{x_1}^{i+1}, \ldots, U_p^{i+1}]$ and $[U_q^{i+1}, \ldots, U_{y_1}^{i+1}]$ of $[U_{x_1}^{i+1}, \ldots, U_{y_1}^{i+1}]$ such that $p < q$ and $U_p^{i+1} \subset U_{y_1}^i$, $U_q^{i+1} \subset U_{x_1}^i$. Then $A \cap U_{x_1}^{i+1} \neq \emptyset$, $B \cap U_{y_1}^{i+1} \neq \emptyset$, $A \cap U_p^{i+1} = \emptyset$ and $B \cap U_q^{i+1} = \emptyset$. Since $N(\mathscr{U}_{i+1}(F(x_{i+1})))$ is a tree, we see that $A \cap B = \emptyset$. This is a contradiction.

Hence K is indecomposable. Since $F^{-1}(F(x))$ is hereditarily decomposable, this is a contradiction. Hence, Claim (I) is true.

Next, we shall prove the following.

Claim (II). *If $f\colon X \to X$ is an expansive homeomorphism, then $M_\alpha \neq \emptyset$ for any countable ordinal α, where $M = H$.*

Proof. Let $\delta > 0$ be a positive number satisfying the conditons as in (3.4). Choose $z_0 \in Z$ such that $F^{-1}(z_0)$ is nondegenerate. Let $x_0, y_0 \in F^{-1}(z_0)$ with $x_0 \neq y_0$. Choose a sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$ of finite open coverings of X such that

(1) $\bar{\mathscr{U}}_{i+1}$ is a refinement of $\mathscr{U}_i$, and

(2) $\mathrm{mesh}(\mathscr{U}_i) < 1/2^i$ for each i.

According to (3.4), there is a sequence $\mathscr{V}_1, \mathscr{V}_2, \ldots$ of finite open coverings of X such that for some integer $N(i) \in \mathbf{Z}$,

(3) $f^{N(i)}(\bar{\mathscr{V}}_i)$ is a refinement of $\mathscr{U}_i$,

(4) $f^{N(i)}(\bar{\mathscr{V}}_i(F(x_0)))$ is a refinement of $\mathscr{U}_i(h^{N(i)}(F(x_0)))$ and $N(\mathscr{V}_i(F(x_0)))$ is a tree,

(5) if $[V_1, \ldots, V_{m(i)}]$ is a chain from x_0 to y_0 in $\mathscr{V}_i(F(x_0))$, then $[f^{N(i)}(V_1), \ldots, f^{N(i)}(V_{m(i)})]$ is crooked between U_1^i and U_2^i, where U_1^i and U_2^i are elements of $\mathscr{U}_i(h^{N(i)}(F(x_0)))$ and $d(U_1^i, U_2^i) \geqslant \delta - 2\,\mathrm{mesh}(\mathscr{U}_i)$ for each i.

Since X is compact, we may assume that $\lim_{i\to\infty} \mathrm{Cl}(U_1^i) = x_1$ and $\lim_{i\to\infty} \mathrm{Cl}(U_2^i) = y_1$. We will show that $(x_1, y_1) \in M_1$ and $d(x_1, y_1) \geqslant \delta$. Clearly, $(x_1, y_1) \in M$ $(= H)$ and $d(x_1, y_1) \geqslant \delta$ (see (5)). Let $\mathscr{U}$ be any finite open cover of X and let $\gamma > 0$. By the constructions of x_1 and y_1, there is i such that $\mathrm{Cl}(U_1^i) \subset U_{x_1}$, $\mathrm{Cl}(U_2^i) \subset U_{y_1}$ and $\bar{\mathscr{U}}_i$ is a refinement of $\mathscr{U}$ with $\mathrm{mesh}(\mathscr{U}_i) < \gamma$, where U_{x_1} (resp. U_{y_1}) is an element of $\mathscr{U}$ containing x_1 (resp. y_1). We may assume that $f^{N(i)}(\bar{\mathscr{V}}_i(F(x_0)))$ is a refinement of $\mathscr{U}(F(x_1))$, because F^{-1} is upper semicontinuous. By (4) and (5), $(f^{N(i)}(x_0), f^{N(i)}(y_0))$ and $f^{N(i)}(\mathscr{V}_i)$ satisfy the conditions of the definition of M^f. Hence $(x_1, y_1) \in M^f = M_1$.

For a countable ordinal λ, we assume that we have obtained (x_α, y_α) in M_α for all $\alpha < \lambda$ such that $d(x_\alpha, y_\alpha) \geqslant \delta$. We will define $(x_\lambda, y_\lambda) \in M_\lambda$ recursively in the following way: Consider two cases.

(i) $\lambda = \alpha + 1$. By an argument similar to the above one, we can obtain $(x_{\alpha+1}, y_{\alpha+1}) \in M_{\alpha+1}$ such that $d(x_{\alpha+1}, y_{\alpha+1}) \geqslant \delta$.

(ii) λ is a limit ordinal. Take a sequence $\alpha_1 < \alpha_2 < \ldots$ of countable ordinals such that $\lim_{i\to\infty} \alpha_i = \lambda$. Note that $d(x_{\alpha_i}, y_{\alpha_i}) \geqslant \delta$. We may assume that $\{x_{\alpha_i}\}$ (resp. $\{y_{\alpha_i}\}$) converges to a point x_λ (resp. y_λ) of X. Then $d(x_\lambda, y_\lambda) \geqslant \delta$ and $(x_\lambda, y_\lambda) \in H$. Since M_α is closed in H and $M_\alpha \supset M_\beta$ for $\alpha \leqslant \beta$, $(x_\lambda, y_\lambda) \in M_{\alpha_i}$ for all i. Hence

$$(x_\lambda, y_\lambda) \in \bigcap_{i=1}^{\infty} M_{\alpha_i} = \bigcap_{\alpha<\lambda} M_\alpha = M_\lambda.$$

Therefore $M_\lambda \neq \emptyset$ for any countable ordinal λ. Claim (II) contradicts Claim (I), which completes the proof.

Proof of (3.1). Let $f\colon X \to X$ be an expansive homeomorphism of X. Suppose, on the contrary, that X is hereditarily decomposable. By (3.3), we can choose a minimal nondegenerate subcontinuum X_1 in the set $\{Z \mid Z$ is a nondegenerate subcontinuum of X such that $f(Z) = Z\}$. Put $f_1 = f|X_1\colon X_1 \to X_1$. Note that X_1 is **F**-like. By (2.1), X_1 is a Θ_n-continuum for some $n \geqslant 0$.

For any one-dimensional continuum Y, we define an index $I(Y)$ as follows: $I(Y) \leqslant m$ if for any $\varepsilon > 0$ there is a finite open cover $\mathscr{U}$ of Y such that the nerve $N(\mathscr{U})$ is a graph, $\operatorname{mesh}(\mathscr{U}) < \varepsilon$ and the number of all simple closed curves in $N(\mathscr{U})$ is equal to or less than m. When $I(Y) \leqslant m$ and $I(Y) \leqslant m-1$ is not true, we define $I(Y) = m$. Note that if Z is any subcontinuum of Y, then $I(Z) \leqslant I(Y)$.

Since **F** is a finite collection of graphs and X_1 is **F**-like, there is a natural number N such that $I(X_1) = N < \infty$. By (2.2), there are a monotone map $F_1\colon X_1 \to G_1$ from X_1 onto a graph G_1 and a homeomorphism $h_1\colon G_1 \to G_1$ making the following diagram (C_1) commute:

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_1} \\ G_1 & \xrightarrow[h_1]{} & G_1 \end{array}$$

Since h_1 is not expansive (see [2], [7] and [8]), F_1 is not a homeomorphism.

Put $E_1 = \{y \in G_1 \mid F_1^{-1}(y)$ is nondegenerate and not tree-like$\}$. Now, we shall prove that E_1 is a finite set, in fact the cardinal number $E_1^{\#}$ of E_1 is equal to or less than N. Suppose, on the contrary, that $E_1^{\#} \geqslant N+1$. Choose $y_1, \ldots, y_{N+1}$ in E_1 such that $y_i \neq y_j$ $(i \neq j)$ and $F_1^{-1}(y_i)$ is not tree-like for each i. Choose ε such that $0 < \varepsilon < \frac{1}{2}\min\{d(F_1^{-1}(y_i), F_1^{-1}(y_j)) \mid 1 \leqslant i \neq j \leqslant N+1\}$. Let $\varepsilon = \varepsilon_1 > \varepsilon_2 > \ldots$ be a sequence of positive numbers with $\lim_{i\to\infty} \varepsilon_i = 0$. By the definition of N, we have finite open coverings $\mathscr{U}_i$ of X_1 such that $N(\mathscr{U}_i)$ is a graph, $\operatorname{mesh}(N(\mathscr{U}_i)) < \varepsilon_i$ for $i = 1, 2, \ldots,$ and the number of simple closed curves in $N(\mathscr{U}_i)$ is $\leqslant N$ for each i. Hence we can choose k_0 and a subsequence $\{i_j\}$ of $\{1, 2, \ldots\}$ such that $N(\mathscr{U}_{i_j}(F_1^{-1}(y_{k_0}))) = N(\{U \in \mathscr{U}_{i_j} \mid U \cap F_1^{-1}(y_{k_0}) \neq \emptyset\})$ is a tree, which implies that $F_1^{-1}(y_{k_0})$ is tree-like. This is a contradiction. Therefore $E_1^{\#} \leqslant N$.

Since f_1 is a homeomorphism and by (C_1), $h_1(E_1) = E_1$. By the definition of X_1, h_1 has no fixed points in E_1. If $E_1 \neq \emptyset$, we can choose y_0 in E_1 such that $h_1^{n_1}(y_0) = y_0$ for some $n_1 \geqslant 2$, and $h_1^i(y_0) \neq h_1^j(y_0)$ for $1 \leqslant i < j \leqslant n_1$. Put $X_2' = F_1^{-1}(y_0)$ and $f_2' = f_1^{n_1}|X_2'\colon X_2' \to X_2'$. Note that $\{F_1^{-1}(y_0), F_1^{-1}(h_1(y_0)), \ldots, F_1^{-1}(h_1^{n_1-1}(y_0))\}$ is a family of mutually disjoint continua which are not tree-like. Hence $I(X_2') < I(X_1)$. Take a minimal nondegenerate subcontinuum X_2 in the set $\{Z \mid Z$ is a nondegenerate subcontinuum of X_2' such that $f_2'(Z) = Z\}$ and put $f_2 = f_2'|X_2$. Note that $I(X_2) \leqslant I(X_2') < I(X_1) = N$. By (2.2), there are a monotone map $F_2\colon X_2 \to G_2$ from X_2 onto a graph G_2 and a homeomorphism $h_2\colon G_2 \to G_2$ making the following diagram (C_2) commute:

$$\begin{array}{ccc} X_2 & \xrightarrow{f_2} & X_2 \\ {\scriptstyle F_2}\downarrow & & \downarrow{\scriptstyle F_2} \\ G_2 & \xrightarrow[h_2]{} & G_2 \end{array}$$

Note that f_2 is also an expansive homeomorphism (see (3.2)). Put $E_2 = \{y \in G_2 \mid F_2^{-1}(y)$ is nondegenerate and not tree-like$\}$. If $E_2 \neq \emptyset$, we continue this process. Then we see that $\infty > N = I(X_1) > I(X_2) > I(X_3) > \ldots$ Note that $I(Y) \geqslant 0$ for any continuum Y, and $I(Y) = 0$ if and only if Y is tree-like. Clearly, we can reach the situation $E_k = \emptyset$, i.e., there are a nondegenerate subcontinuum X_k of X_{k-1}, an expansive homeomorphism $f_k\colon X_k \to X_k$ of X_k and a monotone map $F_k\colon X_k \to G_k$ from X_k onto a graph G_k such that $F_k^{-1}(y)$ is tree-like for any $y \in G_k$, and a homeomorphism h_k of G_k making the following diagram (C_k) commute:

$$\begin{array}{ccc} X_k & \xrightarrow{f_k} & X_k \\ {\scriptstyle F_k}\downarrow & & \downarrow{\scriptstyle F_k} \\ G_k & \xrightarrow[h_k]{} & G_k \end{array}$$

Since h_k is not expansive, F_k is not a homeomorphism. Note that X_k is hereditarily decomposable.

By (3.5), this is a contradiction, which completes the proof.

Let $f\colon X \to X$ be a map of a compact metric space X with a metric d and let

$$(X, f) = \{(x_i)_{i=0}^\infty \mid x_i \in X,\ f(x_{i+1}) = x_i,\ i \geqslant 0\}.$$

Define a metric $\tilde{d}$ for (X, f) by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \sum_{i=0}^{\infty} d(x_i, y_i)/2^i, \quad \text{for } \tilde{x} = (x_i)_{i=0}^\infty,\ \tilde{y} = (y_i)_{i=0}^\infty \in (X, f).$$

Then the space (X, f) is called the *inverse limit* of the map f. Note that (X, f) is a compact metric space. Also, define a map $\tilde{f}\colon (X, f) \to (X, f)$ by

$$\tilde{f}((x_i)_{i=0}^\infty) = (f(x_i))_{i=0}^\infty \quad (= (f(x_0), x_0, x_1, \ldots)).$$

Then $\tilde{f}$ is a homeomorphism and it is called the *shift homeomorphism* of f. Note that almost all examples of one-dimensional continua admitting expansive homeomorphisms are obtained as inverse limits of maps $f\colon G \to G$ of graphs whose shift homeomorphisms are expansive (see [11], [13], [14], [22], [23] and [26]).

(3.6) COROLLARY. *Let* **F** *be a finite collection of graphs. If a continuum X is homeomorphic to an inverse limit of an inverse sequence $\{G_n, f_{n,n+1}\}$ such that each G_n is an element of* **F**, *and X admits an expansive homeomorphism, then X contains an indecomposable subcontinuum. In particular, if $f\colon G \to G$ is a map of a graph G such that (G, f) is nondegenerate and the shift homeomorphism $\tilde{f}$ is expansive, then (G, f) contains an indecomposable subcontinuum.*

(3.7) Remark. In the statement of (3.1) and (3.6), we can not conclude that X itself is an indecomposable continuum.

The following problems remain open.

PROBLEM 1. Is it true that if a one-dimensional continuum X admits an expansive homeomorphism, then X contains an indecomposable subcontinuum?

PROBLEM 2. Is there a hereditarily indecomposable continuum admitting an expansive homeomorphism? In particular, does the pseudo-arc admit an expansive homeomorphism?

PROBLEM 3. Is there a nonseparating plane continuum admitting an expansive homeomorphism? Moreover, is there a tree-like continuum admitting an expansive homeomorphism?

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