sparser than $P$. By Lemma 6, for all but finitely many natural numbers $n$ the intersection $P \cap \{Q_n, Q_n+1, \ldots, Q_{n+1}-1\}$ is non-empty. Thus $P$ does not belong to $I_Q$, since in this case $I$ has to contain all but finitely many natural numbers, which is impossible as $I$ is an ideal. ■

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INSTYTUT MATEMATYKI
UNIWERSYTET ŚLĄSKI
ul. Bankowa 14
40-007 Katowice
Poland



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# The space of Lipschitz maps from a compactum to an absolute neighborhood LIP extensor

by

**Katsuro Sakai** (Tsukuba)


**Abstract.** Let $X$ be a non-discrete metric compactum and $Y$ a separable locally compact absolute neighborhood LIP extensor. The spaces of continuous maps and Lipschitz maps from $X$ to $Y$ are denoted by $C(X, Y)$ and $\mathrm{LIP}(X, Y)$, respectively. Let $l_2$ be the Hilbert space and

$$l_2^Q = \{(x_i) \in l_2 \mid \sup |i \cdot x_i| < \infty\}.$$

It is proved that $(C(X, Y), \mathrm{LIP}(X, Y))$ is an $(l_2, l_2^Q)$-manifold pair if each point of $Y$ has a neighborhood $V$ admitting a map $\gamma$: $V \to \mathrm{LIP}(I, Y)$ such that each $\gamma(y)$ is an embedding with $\gamma(y)(0) = y$ and the Lipschitz constant of each $\gamma(y)$ does not exceed some $k > 0$, e.g., Euclidean polyhedra without isolated points and Lipschitz $n$-manifolds ($n > 0$) have such a property.


**Introduction.** Let $l_2$ be Hilbert space and $l_2^Q$ the subspace of $l_2$ which is the linear span of the Hilbert cube $\prod_{i \in N} [-i^{-1}, i^{-1}] \subset l_2$, that is,

$$l_2^Q = \{(x_i) \in l_2 \mid \sup |i \cdot x_i| < \infty\}.$$

An $l_2$*-manifold* or an $l_2^Q$*-manifold* is a separable metrizable space locally homeomorphic to $l_2$ or $l_2^Q$, respectively. An $(l_2, l_2^Q)$*-manifold pair* is a pair $(M, N)$ of an $l_2$-manifold $M$ and an $l_2^Q$-manifold $N$ which admits an open cover $\mathcal{U}$ of $M$ and open embeddings $\varphi_U$: $U \to l_2$, $U \in \mathcal{U}$, such that $\varphi_U(N \cap U) = l_2^Q \cap \varphi_U(U)$.

Let $X = (X, d_X)$ and $Y = (Y, d_Y)$ be separable metric spaces. In case there is no confusion, $d$ stands for both metrics $d_X$ and $d_Y$. We assume that

*$X$ is non-discrete compact and $Y$ has no isolated point.*

The spaces of (continuous) maps and Lipschitz maps from $X$ to $Y$ are denoted by $C(X, Y)$ and $\mathrm{LIP}(X, Y)$, respectively. The topology of these spaces is induced by

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the sup-metric

$$d(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}.$$

It is known that $C(X, Y)$ is an $l_2$-manifold if $Y$ is a completely metrizable ANR [Ge$_1$, To, Sa$_2$]. Naturally arises the following:

PROBLEM. Under what conditions is $\mathrm{LIP}(X, Y)$ an $l_2^Q$-manifold? Or is $(C(X, Y), \mathrm{LIP}(X, Y))$ an $(l_2, l_2^Q)$-manifold pair?

In [SW$_1$], it has been shown that $(C(X, Y), \mathrm{LIP}(X, Y))$ is an $(l_2, l_2^Q)$-manifold pair if $Y$ is a locally compact, locally convex set in a normed linear space, where $Y$ is said to be *locally convex* if each point of $Y$ has a convex neighborhood. Locally Lipschitz maps are shortly called LIP maps. If the domain is compact, a LIP map is Lipschitz. In the category of metric spaces and LIP maps, we can introduce notions corresponding to AR, AE, ANR and ANE, namely ALR (*absolute* LIP *retract*), ALE (*absolute* LIP *extensor*), ANLR (*absolute neighborhood* LIP *retract*) and ANLE (*absolute neighborhood* LIP *extensor*) (¹). Then $Y$ is an ALR or an ANLR if and only if $Y$ is an ALE or an ANLE, respectively [Lu, Theorem 4.7]. Euclidean polyhedra and Lipschitz manifolds are ANLR's (ANLE's) [LV, 5.12]. However, a normed linear space is generally not an ALE [Lu, 3.13]. For LIP maps and ANLE's (or ANLR's) refer to [LV] and [Lu]. The Lipschitz constant of $f \in \mathrm{LIP}(X, Y)$ is denoted by $\mathrm{lip}\, f$. For each $k > 0$, let $k\text{-}\mathrm{LIP}(X, Y) = \{f \in \mathrm{LIP}(X, Y) \mid \mathrm{lip}\, f \leqslant k\}$. In this paper, we prove the following:

MAIN THEOREM. *Let $Y$ be a locally compact* ANLE. *Then $(C(X, Y), \mathrm{LIP}(X, Y))$ is an $(l_2, l_2^Q)$-manifold pair if each point of $Y$ has a neighborhood $V$ admitting a map $\gamma\colon V \to k\text{-}\mathrm{LIP}(I, Y)$ for some $k > 0$ such that each $\gamma(y)$ is an embedding with $\gamma(y)(0) = \mathrm{id}$.*

It can be shown that $\mathrm{LIP}(X, Y)$ is a $\sigma$-compact ANR which is dense in $C(X, Y)$ if $Y$ is a locally compact ANLE, but it is unknown whether $\mathrm{LIP}(X, Y)$ is an $l_2^Q$-manifold without any other condition. If so, $(C(X, Y), \mathrm{LIP}(X, Y))$ is an $(l_2, l_2^Q)$-manifold pair. (See Remarks.) A Lipschitz $n$-manifold ($n > 0$) clearly satisfies the condition of the Main Theorem. We can also show that a Euclidean polyhedron without isolated points satisfies the condition. Thus we have the following

COROLLARY. *Let $Y$ be a Euclidean polyhedron without isolated points or a Lipschitz $n$-manifold ($n > 0$). Then $(C(X, Y), \mathrm{LIP}(X, Y))$ is an $(l_2, l_2^Q)$-manifold pair.*

It is unknown whether every Lipschitz manifold has a LIP triangulation, i.e., whether it is LIP homeomorphic to a Euclidean polyhedron (cf. [SS]). By [Su, Corollary 3], every (topological) $n$-manifold can be metrized so as to be a Lipschitz manifold if $n \neq 4, 5$ or if $n = 5$ and the boundary is empty.

**Proofs of Main Theorem and Corollary.** We denote by $I$ and $Q$ the unit interval $[0, 1]$ and the Hilbert cube $I^\omega$, i.e., the countable-infinite product of $I$, respectively. Let $M = (M, d)$ be a metric space. For $x \in M$ and $A \subset M$, we write

$$\mathrm{dist}(x, A) = \inf\{d(x, y) \mid y \in A\}.$$

(¹) In [LV], an ANLE is called an ALNE (= absolute LIP neighborhood extensor) but we prefer "ANLE".



A closed set $A \subset M$ is called a *Z-set* in $M$ if for each $\varepsilon > 0$ and each map $f\colon Q \to M$, there is a map $g\colon Q \to M \setminus A$ with $d(f, g) < \varepsilon$. A *cap set* for $M$ is a subset $N = \bigcup_{i \in N} N_i \subset M$, where $N_1 \subset N_2 \subset \ldots$ is a tower of compact $Z$-sets in $M$ such that for each compact set $A \subset M$, $i \in N$ and $\varepsilon > 0$, there is a $j \geqslant i$ $(\in N)$ and an embedding $h\colon A \to N_j$ such that $h|A \cap N_i = \mathrm{id}$ and $d(h, \mathrm{id}) < \varepsilon$. It is well known that $(M, N)$ is an $(l_2, l_2^Q)$-manifold pair if and only if $M$ is an $l_2$-manifold and $N$ is a cap set for $M$ (cf. [Ch]). The following is a modification of Theorem 4.8 in [Cu] which can be easily proved, so the proof is omitted.

LEMMA. *Let $N_1 \subset N_2 \subset \ldots$ be a tower of compacta in an $l_2$-manifold $M$ satisfying the following conditions*:

(i) *for each compact set $A \subset M$, $i \in N$ and $\varepsilon > 0$, there is a $j \geqslant i$ $(\in N)$ and a map $f\colon A \to N_j$ such that $f|A \cap N_i = \mathrm{id}$ and $d(f, \mathrm{id}) < \varepsilon$; and*

(ii) *for each $i \in N$, there is a $j > i$ $(\in N)$ and an embedding $g\colon N_i \times Q \to N_j$ such that $g(x, 0) = x$ for each $x \in N_i$.*

*Then $N = \bigcup_{i \in N} N_i$ is a cap set for $M$, that is, $(M, N)$ is an $(l_2, l_2^Q)$-manifold pair.*

Proof of Main Theorem. Let $x_\infty \in X$ be a cluster point. For each $y \in Y$, let $V(y)$ be an open neighborhood of $y$ such as in the condition and let

$$G(y) = \{f \in C(X, Y) \mid f(x_\infty) \in V(y)\}.$$

Since $\{G(y) \mid y \in Y\}$ is an open cover of $C(X, Y)$, it suffices to show that each $G(y) \cap \mathrm{LIP}(X, Y)$ is a cap set for $G(y)$.

Now let $y_0 \in Y$ be fixed. For simplicity, we write $V = V(y_0)$, $G = G(y_0)$ and $L = G \cap \mathrm{LIP}(X, Y)$. Let $\gamma\colon V \to k\text{-}\mathrm{LIP}(I, Y)$ be a map such that each $\gamma(y)$ is an embedding with $\gamma(y)(0) = y$. Let $Y = \bigcup_{j \in N} Y_j$ and $V = \bigcup_{j \in N} V_j$, where $Y_j$ and $V_j$ are compact, $Y_j \subset \mathrm{int}\, Y_{j+1}$, $y_0 \in \mathrm{int}\, V_1$, $V_j \subset \mathrm{int}\, V_{j+1}$ and $\gamma(V_j) \subset k\text{-}\mathrm{LIP}(I, Y_j)$.

Since $x_\infty \in X$ is a cluster point, we can choose mutually disjoint closed balls $B_{i,n}$ $(i, n \in N)$ in $X$ with center $x_{i,n}$ and radius $r_{i,n} > 0$ so that

$$\lim_{i \to \infty} d(x_{i,1}, x_\infty) = 0, \quad r_{i,n+1} \leqslant \tfrac{1}{3} r_{i,n} \quad \text{and}$$

$$r_{i,n},\ d(x_{i,n+1}, x_\infty) \leqslant \tfrac{1}{3} \cdot d(x_{i,n}, x_\infty).$$

Let $B_i = \bigcup_{n \in N} B_{i,n}$ and $L_j = \{f \in L \mid \mathrm{lip}\, f \leqslant j,\ f(X) \subset Y_j, f(\bigcup_{i \geqslant j} B_i) = f(x_\infty) \in V_j\}$. Then each $L_j$ is compact and $\bigcup_{j \in N} L_j \subset L$. By [Ch, Theorem 6.6], it suffices to show that $\bigcup_{j \in N} L_j$ is a cap set for $G$. We verify the conditions (i) and (ii) of Lemma 1.1.

(i) Let $F$ be a compact set in $G$, $k \in N$ and $\varepsilon > 0$. Let $\mathrm{ev}\colon F \times X \to Y$ be the evaluation map, i.e., $\mathrm{ev}(f, x) = f(x)$. Since $F \times X$ is compact, by [Lu, Theorem 4.11], we have $\delta > 0$ such that any LIP map $\psi$ from a closed set in $F \times X$ to $Y$ which is $\delta$-close to ev extends to a LIP map from $F \times X$ which is $\varepsilon$-homotopic to ev. (Note that $\delta$ in [Lu, 4.11] is obtained in the proof as the restriction of $\pi$ which is independent of $A$.) Choose $j' \geqslant j$ so that

$$\mathrm{dist}\big(f(x_\infty), f(\bigcup_{i \geqslant j'} B_i)\big) < \delta \quad \text{for all } f \in F$$

and let

$$\Gamma = (F \cap L_j) \times X \cup F \times (\bigcup_{i \geqslant j'} B_i \cup \{x_\infty\}).$$

Then $\Gamma$ is closed in $F\times X$. We define a map $\psi\colon \Gamma\to Y$ by

$$\psi|(F\cap L_j)\times X = \mathrm{ev} \quad \text{and}$$
$$\psi(f, x) = \psi(f, x_\infty) = f(x_\infty) \quad \text{if } (f, x)\in F\times \bigcup_{i\geqslant j'} B_i.$$

Then as is easily observed, $\psi$ is Lipschitz. Since $\psi$ is $\delta$-close to ev, it extends to a LIP map $\tilde{\psi}\colon F\times X\to Y$ which is $\varepsilon$-homotopic to ev. Note that $\tilde{\psi}$ is Lipschitz since $F\times X$ is compact. Then we can choose $j''\in N$ so that

$$j''\geqslant \max\{j', \operatorname{lip}\tilde{\psi}\}, \quad \tilde{\psi}(F\times X)\subset Y_{j''} \quad \text{and}$$
$$f(x_\infty)\in V_{j''} \quad \text{for all } f\in F.$$

Let $\varphi\colon F\to L_{j''}$ be the map defined by $\varphi(f)(x) = \tilde{\psi}(f, x)$. Then $\varphi|F\cap L_j = \mathrm{id}$ and $d(\varphi, \mathrm{id}) < \varepsilon$. Thus (i) is satisfied.

(ii) For each $j\in N$, we define $\varphi\colon L_j\times Q\to C(X, Y)$ as follows:

$$\varphi(f, z)|X\backslash B_j = f|X\backslash B_j \quad \text{and}$$
$$\varphi(f, z)(x) = \gamma f(x_\infty)\big(z_n\cdot(r_{j,n}-d(x, x_{j,n}))\big) \quad \text{for } x\in B_{j,n}.$$

Then $\varphi$ is an embedding. In fact, if $f\neq f'\in L_j$ then $f|X\backslash B_j\neq f'|X\backslash B_j$, which implies that $\varphi(f, z)\neq\varphi(f', z')$ for any $z, z'\in Q$ by the definition. If $z\neq z'\in Q$ then $z_n\neq z'_n$ for some $n\in N$, hence

$$\begin{aligned}\varphi(f, z)(x_{j,n}) &= \gamma f(x_\infty)(z_n\cdot r_{j,n})\\ &\neq \gamma f(x_\infty)(z'_n\cdot r_{j,n}) = \varphi(f, z')(x_{j,n}).\end{aligned}$$

We will show that $\varphi(L_j\times Q)\subset L_{j'}$ for some $j'\geqslant j$. For each $(f, z)\in L_j\times Q$, $\varphi(f, z)(X)\subset Y_j$ and

$$\varphi(f, z)\Big(\bigcup_{i\geqslant j+1} B_i\Big) = \varphi(f, z)(x_\infty) = f(x_\infty)\in V_j.$$

Then it suffices to find $j'\geqslant j$ so that $\varphi(f, z)$ is $j'$-Lipschitz for all $(f, z)\in L_j\times Q$. In case $x, x'\in X\backslash B_j$,

$$d\big(\varphi(f, z)(x), \varphi(f, z)(x')\big) = d\big(f(x), f(x')\big)\leqslant j\cdot d(x, x').$$

In case $x, x'\in B_{j,n}$ for some $n\in N$,

$$\begin{aligned}d\big(\varphi(f, z)(x), \varphi(f, z)(x')\big) &\leqslant k\cdot|z_n\cdot d(x, x_{j,n})-z_n\cdot d(x', x_{j,n})|\\ &\leqslant k\cdot d(x, x').\end{aligned}$$

In case $x\in B_{j,n}$ and $x'\in B_{j,n'}$ for $n<n'$,

$$\begin{aligned}d(x, x') &\geqslant d(x, x_\infty)-d(x', x_\infty) > \tfrac{2}{3}\cdot d(x_{j,n}, x_\infty)-\tfrac{4}{3}\cdot d(x_{j,n'}, x_\infty)\\ &\geqslant \tfrac{2}{3}\cdot d(x_{j,n}, x_\infty)-\tfrac{4}{9}\cdot d(x_{j,n}, x_\infty) = \tfrac{2}{9}\cdot d(x_{j,n}, x_\infty).\end{aligned}$$

Therefore we have



$$\begin{aligned}d\big(\varphi(f, z)(x), \varphi(f, z)(x')\big) &\leqslant k\cdot(r_{j,n}+r_{j,n'})\\ &\leqslant k\cdot\tfrac{4}{3}\cdot r_{j,n}\leqslant k\cdot\tfrac{4}{9}\cdot d(x_{j,n}, x_\infty)\leqslant 2k\cdot d(x, x').\end{aligned}$$

In case $x\in B_{j,n}$ and $x'\in X\backslash B_j$, note $f(x) = f(x_\infty)$ and $d(x', x_{j,n})\geqslant r_{j,n}$; then

$$\begin{aligned}d\big(\varphi(f, z)(x), \varphi(f, z)(x')\big) &\leqslant d\big(\varphi(f, z)(x), f(x_\infty)\big)+d\big(f(x), f(x')\big)\\ &\leqslant k\cdot\big(r_{j,n}-d(x, x_{j,n})\big)+j\cdot d(x, x')\\ &\leqslant k\cdot\big(d(x', x_{j,n})-d(x, x_{j,n})\big)+j\cdot d(x, x')\\ &\leqslant (k+j)\cdot d(x, x').\end{aligned}$$

Consequently, $\varphi(f, z)$ is $(2k+j)$-Lipschitz. ∎

Proof of Corollary for a Euclidean polyhedron. Let $Y$ be a Euclidean polyhedron with $K$ a simplicial triangulation. For each vertex $v\in K^0$, let $S_1(v) = \mathrm{Sd}^2(\mathrm{St}(v, K))$ be the second barycentric subdivision of the star at $v$ in $K$, let $S_1(v)$ be the simplicial neighborhood of $S_1(v)$ in the second barycentric subdivision $\mathrm{Sd}^2K$ of $K$, that is,

$$S_2(v) = \{\tau\in\mathrm{Sd}^2K|\ \exists\tau'\in\mathrm{Sd}^2K \ \text{ s.t. } \ \tau'\cap|S_1(v)|\neq\emptyset \ \text{ and } \ \tau<\tau'\},$$

and let

$$U(v) = \operatorname{int}\big(|S_2(v)|\backslash|\mathrm{St}(v, \mathrm{Sd}^2K)|\big).$$

Then $\{U(v)|\ v\in K^0\}$ is an open cover of $Y$. We will construct a map $\gamma\colon U(v)\to k\text{-LIP}(I, Y)$ so that each $\gamma(y)$ is an embedding with $\gamma(y)(0) = y$.

Let $v_0\in K^0$ be fixed. For simplicity, we write $S_1 = S_1(v_0)$, $N_1 = |S_1|$, $S_2 = S_2(v_0)$, $N_2 = |S_2|$ and $U = U(v_0)$. We define a Lipschitz deformation $\psi'\colon N_1\times I\to N_1$ as follows:

$$\psi'(x, t) = (1-t)\cdot x+t\cdot v_0.$$

As is well known, there exists a simplicial retraction $r\colon S_2\to S_1$. Give a linear order on $S_2^0$ so that $v''<v'$ if $v'\in S_1^0$ and $v''\in S_2^0\backslash S_1^0$. For each $v\in S_2^0$, let $v^j = (v, j)$, $j = 0, 1$. Let $S_2\times J$ be the product simplicial complex of $S_2$ and the triangulation $J = \{0, 1, I\}$ of $I$, that is,

$$\begin{aligned}S_2\times J = S_2\times\{0, 1\}\cup\{v_0^0\ldots v_i^0v_{i+1}^1\ldots v_n^1|\ &v_0\ldots v_n\in S_2,\\ &v_0<\ldots<v_i\leqslant v_{i+1}<\ldots<v_n\}.\end{aligned}$$

Then $S_2\times J$ is a triangulation of $N_2\times I$. We define a simplicial deformation $\psi''\colon S_2\times J\to S_2$ by

$$\psi''(v^0) = v \quad \text{and} \quad \psi''(v^1) = r(v) \quad \text{for each } v\in S_2^0.$$

Then $\psi''\colon N_2\times I\to N_2$ is a Lipschitz deformation with $\psi''_1 = r$. Let $S_0 = \{\tau\in S_2|\ \tau\cap N_1 = \emptyset\}$, $N_0 = |S_0|$ and let $\alpha\colon S_2\to J$ be the simplicial map defined by $\alpha(S_0^0) = 0$ and $\alpha(S_1^0) = 1$. Then $\alpha\colon N_2\to I$ is a Lipschitz map with $\alpha(N_0) = 0$ and $\alpha(N_1) = 1$. Each $x\in N_2\backslash(N_0\cup N_1)$ can be written as follows:

$$x = (1-s)\cdot x'+s\cdot x'', \quad x'\in N_0,\ x''\in\mathrm{bd}N_1,\ s\in(0, 1).$$

Then $\alpha(x) = s$ and

$$\begin{aligned}\psi''(x, \alpha(x)) &= \psi''((1-s)\cdot x' + s\cdot x'', s)\\ &= \psi''((1-s)\cdot(x', 0) + s\cdot(x'', 1)) = (1-s)\cdot x' + s\cdot x'' = x.\end{aligned}$$

If $t > \alpha(x) = s$ then

$$\begin{aligned}\psi''(x, t) &= \psi''((1-s)\cdot x' + s\cdot x'', t)\\ &= \left(1-\frac{t-s}{1-s}\right)\cdot\psi''((1-s)\cdot(x', 0) + s\cdot(x'', 1))\\ &\quad + \frac{t-s}{1-s}\cdot\psi''((1-s)\cdot(x', 1) + s\cdot(x'', 1))\\ &= \left(1-\frac{t-s}{1-s}\right)\cdot((1-s)\cdot x' + s\cdot x'')\\ &\quad + \frac{t-s}{1-s}\cdot r((1-s)\cdot x' + s\cdot x'')\\ &= \left(1-\frac{t-s}{1-s}\right)\cdot x + \frac{t-s}{1-s}\cdot r(x).\end{aligned}$$

By [LV, 2.21 & 2.23], we can define a Lipschitz deformation $\psi\colon N_2 \times [-1, 1] \to N_2$ as follows:

$$\psi(x, t) = \begin{cases}\psi'(r(x), t) & \text{for } t \geqslant 0,\\ \psi''(x, t+1) & \text{for } t \leqslant 0.\end{cases}$$

Observe that for all $x \in U$, $\psi(x, \alpha(x)-1) = x$ and $\psi(x, t) \neq \psi(x, t')$ if $t \neq t' \geqslant \alpha(x)-1$. Let $\beta\colon U \to [0, 1)$ be the Lipschitz map defined by $\beta(x) = (1-\alpha(x))/2$. By [LV, 2.40], the desired Lipschitz deformation $\varphi\colon U \times I \to N_2$ can be defined by

$$\varphi(x, t) = \begin{cases}\psi\left(x, \dfrac{t-\beta(x)}{1-\beta(x)}\right) & \text{for } t \geqslant \beta(x),\\ \psi(x, 2\cdot(t-\beta(x))) & \text{for } t \leqslant \beta(x).\end{cases}$$

Then $\varphi$ induces the desired map $\gamma\colon U \to k\text{-LIP}(I, Y)$, where $k = \operatorname{lip}\varphi$. ∎

**Remarks.** The following is a modification of Theorem 4.11 in [Lu]:

LIP APPROXIMATION THEOREM. *Let $f\colon M \to Y$ be a map from an arbitrary metric space $M$ to an ANLE $Y$ such that $f|A$ is LIP for a closed set $A$ in $M$. Then there is a homotopy $h\colon M \times I \to Y$ such that $h_0 = f$, $h_t|A = f$ for each $t \in I$ and $h|M \times (0, 1]$ is LIP, where the metric for $M \times I$ is defined by*

$$d((x, t), (y, s)) = d_M(x, y) + |t-s|.$$

Proof. By Theorem 4.11 in [Lu], we have a LIP map $h'\colon M \times (0, 1] \to Y$ such that

$$h'|A \times (0, 1] = f|A \times \mathrm{id} \quad\text{and}$$

$$d(h'(x, t), f(x)) < t \quad \text{for each } (x, t) \in M \times (0, 1].$$

Then $h'$ can be extended to the desired homotopy $h$ by $h_0 = f$. ∎

As an application of the above, we have the following:

PROPOSITION. *Let $Y$ be an ANLE. Then for any metric compactum $X$ and $k > 0$, there exists a homotopy $\varphi\colon C(X, Y) \times I \to C(X, Y)$ such that $\varphi_0 = \mathrm{id}$, $\varphi_t|k\text{-LIP}(X, Y) = \mathrm{id}$ for each $t \in I$ and $\varphi(C(X, Y) \times (0, 1]) \subset \mathrm{LIP}(X, Y)$.*

Proof. We denote by ev: $C(X, Y) \times X \to Y$ the evaluation, that is, $\mathrm{ev}(f, x) = f(x)$. Then $\mathrm{ev}|k\text{-LIP}(X, Y) \times X$ is Lipschitz. By the above proposition, we have a homotopy $\psi\colon C(X, Y) \times X \times I \to Y$ such that $\psi_0 = \mathrm{ev}$, $\psi_t|k\text{-LIP}(X, Y) \times X = \mathrm{ev}$ and $\psi|C(X, Y) \times X \times (0, 1]$ is LIP. The desired homotopy $\varphi$ is induced by $\psi$, that is, each $\varphi(f, t) \in C(X, Y)$ is defined by $\varphi(f, t)(x) = \psi(f, x, t)$. ∎

Thus if $Y$ is a locally compact ANLE, then $\mathrm{LIP}(X, Y)$ is a $\sigma$-compact ANR by [Hu, Ch. IV, 6.3] since $C(X, Y)$ is an ANR. Moreover, if $\mathrm{LIP}(X, Y)$ is an $l_2^Q$-manifold then $(C(X, Y), \mathrm{LIP}(X, Y))$ is an $(l_2, l_2^Q)$-manifold pair (cf. [$\mathrm{Sa}_1$, Proposition 2.1]).

Let $l_2^f$ be the subspace of $l_2$ which is the linear span of the usual orthonormal basis, that is,

$$l_2^f = \{(x_i) \in l_2 |\ x_i = 0 \text{ except for finitely many } i\}.$$

Similarly to an $(l_2, l_2^Q)$-manifold pair, we define an *$(l_2, l_2^f)$-manifold pair* as a pair of spaces which is locally homeomorphic to $(l_2, l_2^f)$. In case $X$ and $Y$ are polyhedra, let $\mathrm{PL}(X, Y)$ be the space of PL maps from $X$ to $Y$. It is known that $(C(X, Y), \mathrm{PL}(X, Y))$ is an $(l_2, l_2^f)$-manifold pair [$\mathrm{Ge}_2$]. Analogously a triple of spaces which is locally homeomorphic to $(l_2, l_2^Q, l_2^f)$ is called an *$(l_2, l_2^Q, l_2^f)$-manifold triple*. For $(l_2, l_2^Q, l_2^f)$-manifold triples, refer to [$\mathrm{SW}_2$]. It is natural to conjecture the following:

CONJECTURE. *In case $X$ and $Y$ are Euclidean polyhedra,*

$$(C(X, Y), \mathrm{LIP}(X, Y), \mathrm{PL}(X, Y))$$

*is an $(l_2, l_2^Q, l_2^f)$-manifold triple.*

In case $Y$ is an open set in Euclidean space, this conjecture has been proved in [$\mathrm{Sa}_3$].

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA
Tsukuba-city, 305 Japan

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# Caractérisation topologique de l'espace des fonctions dérivables

par

**Robert Cauty** (Paris)

**Abstract.** Let $\mathscr{C}$ be the space of continuous functions from $[0, 1]$ into the reals, with the topology of uniform convergence. We give a topological characterization of the subspace $\mathscr{D}$ of $\mathscr{C}$ consisting of everywhere differentiable functions. We show that $\mathscr{D}$ is homeomorphic to the subspace of the countable product $\mathscr{C}^\infty$ consisting of convergent sequences, as well as to the subspace of the hyperspace $2^I$ consisting of countable sets.

**1. Introduction et notations.** Tous les espaces considérés dans cet article sont supposés métrisables et séparables. Soit $\mathscr{C}$ l'ensemble des fonctions continues de $I = [0, 1]$ dans $\boldsymbol{R}$, muni de la norme de la convergence uniforme, et soit $\mathscr{D}$ (resp. $\mathscr{D}^*$) le sous-espace de $\mathscr{C}$ formé des fonctions ayant une dérivée finie en tout point de $I$ (resp. en au moins un point de $I$). Nous nous proposons ici de caractériser topologiquement les espaces $\mathscr{D}$ et $\mathscr{D}^*$ et de donner quelques applications de ces caractérisations.

Pour formuler les caractérisations de $\mathscr{D}$ et $\mathscr{D}^*$ (théorèmes 1.1 et 1.2), il nous faut rappeler quelques définitions. Si $f$ et $g$ sont deux fonctions de $Y$ dans $X$ et si $\mathscr{U}$ est un recouvrement ouvert de $X$, nous dirons que $f$ est *$\mathscr{U}$-proche* de $g$ si, pour tout $y$ dans $Y$, il y a un élément de $\mathscr{U}$ contenant à la fois $f(y)$ et $g(y)$. Un sous-ensemble $F$ d'un rétracte absolu de voisinage $X$ est appelé un *Z-ensemble* dans $X$ s'il est fermé et si, pour tout recouvrement ouvert $\mathscr{U}$ de $X$, il existe une fonction continue $f$ de $X$ dans $X$, $\mathscr{U}$-proche de l'identité et telle que $f(X) \subset X \setminus F$; si, de plus, il est toujours possible de choisir la fonction $f$ de façon que $\overline{f(X)} \cap F = \emptyset$, alors $F$ est appelé un *Z-ensemble au sens fort* dans $X$. Une fonction $f\colon Y \to X$ est appelée un *Z-plongement* si c'est un plongement et si $f(Y)$ est un $Z$-ensemble dans $X$.

Soit $\mathscr{K}$ une classe d'espaces. Un rétracte absolu de voisinage $X$ est dit *$\mathscr{K}$-universel* si, pour tout espace $K$ appartenant à $\mathscr{K}$, toute fonction continue $f\colon K \to X$ et tout recouvrement ouvert $\mathscr{U}$ de $X$, il y a un $Z$-plongement $g\colon K \to X$ qui est $\mathscr{U}$-proche de $f$; $X$ est dit *fortement $\mathscr{K}$-universel* si, pour tout espace $K$ appartenant à $\mathscr{K}$, tout fermé $L$ de $K$, toute fonction continue $f\colon K \to X$ dont la restriction à $L$ est un $Z$-plongement, et tout recouvrement ouvert $\mathscr{U}$ de $X$, il existe un $Z$-plongement $g\colon K \to X$ qui est $\mathscr{U}$-proche de $f$ et vérifie $g|L = f|L$.

Nous noterons $\mathscr{L}_1$ (resp. $\mathscr{L}_2$) la classe des espaces analytiques (resp. coanalytiques).