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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

Another application of the Effros theorem to the pseudo-arc

by

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Abstract. Let P be a pseudo-arc. Among other things, it is shown that for each integer $n>0$ and each real number $\varepsilon>0$, there exists a $\delta>0$ such that for any unions U, U' of sequences of n continua lying in mutually distinct composants of P, every δ-homeomorphism $h\colon U\to U'$ can be extended to an ε-homeomorphism $h^*\colon P\to P$. This generalizes an earlier result of the first author.

1. Introduction. In [L], Lehner has shown the following theorem.

LEHNER'S THEOREM. *Let P be a pseudo-arc and let $K_1,\ldots,K_n$, $L_1,\ldots,L_n$ be a sequence of proper subcontinua of P which satisfies the condition*:

(0) *P is irreducible between K_i and K_j, L_i and L_j for $i\neq j$, and K_i, L_i are homeomorphic for $i=1,\ldots,n$.*

Then each homeomorphism $h\colon K_1\cup\ldots\cup K_n\to L_1\cup\ldots\cup L_n$ has a homeomorphic extension $h^*\colon P\to P$.

Previously, the first author has considered in [K] the following question, closely related to Lehner's result:

Let n be a positive integer. Given $\varepsilon>0$, does there exist a $\delta>0$ satisfying the conditions (1) and (2) below?

Let $K_1,\ldots,K_n$, $L_1,\ldots,L_n$ be a sequence of proper subcontinua of P satisfying the condition (0).

(1) If $\operatorname{dist}(K_i, L_i)<\delta$ and K_i, L_i are homeomorphic for $i=1,\ldots,n$, then there is an ε-homeomorphism $h\colon P\to P$ such that $h(K_i)=L_i$.

(2) If $h\colon K_1\cup\ldots\cup K_n\to L_1\cup\ldots\cup L_n$ is a δ-homeomorphism, then it admits an ε-homeomorphic extension $h^*\colon P\to P$.

Applying the classical method (introduced by Bing in [B]) of constructing autohomeomorphisms of the pseudo-arc with the help of chain covers, Kawamura has answered this question in the affirmative for $n\leqslant 2$. In the present paper we generalize this result to arbitrary positive integer n. Our method is completely different from that

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of [K]. Avoiding any advanced technique of chain covers or inverse limits, we only use the Effros theorem and some mapping properties of the pseudo-arc due to Lehner, Cornette and Lewis.

Throughout the paper, a compact connected metric space is called a *continuum*. A continuum is called *chainable* if it can be represented as an inverse limit of arcs. A continuum X is said to be *hereditarily indecomposable* if no subcontinuum Y of X is a union of two proper subcontinua of Y. A hereditarily indecomposable, chainable continuum is topologically unique, and it is called a *pseudo-arc*. It is known that the pseudo-arc is homogeneous.

In what follows P denotes a pseudo-arc with a metric d, and $C(P)$ its hyperspace consisting of all nonempty subcontinua equipped with the Hausdorff distance denoted by dist. The term "mapping" means always a continuous mapping. Let A be a subset of P. Then $H(A)$ denotes the space of all autohomeomorphisms of A equipped with the sup metric $\tilde{d}$. Let ε be a positive number. Two maps $f, g\colon A \to P$ are said to be *ε-near* if $\tilde{d}(f, g) < \varepsilon$. A homeomorphism $f \in H(A)$ which is ε-near to id_A is called an *ε-homeomorphism*.

The following version of the Effros theorem [E] will be used.

EFFROS' THEOREM. *Let a topological group G and a space X be separable and completely metrizable, and let $f\colon G \times X \to X$ be a transitive action (i.e. for each $x, y \in X$ there is a $g \in G$ such that $f(g, x) = y$). Fix a point $a \in X$. Then the map $T_a\colon G \to X$ defined by $T_a(g) = f(g, a)$ is an open map onto X.*

Now we formulate the main result of this paper.

THEOREM 1. *For every real number $\varepsilon > 0$ and every integer $n > 0$, there is a $\delta > 0$ such that if $K_1, \ldots, K_n, L_1, \ldots, L_n$ are proper subcontinua of P fulfilling condition* (0), *then* (1) *and* (2) *hold.*

It can easily be observed that part (2) of Theorem 1 follows from part (1) of the present theorem and the following assertion:

(2′) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $K_1, \ldots, K_n$ are subcontinua of P lying in distinct composants of P, then each δ-homeomorphism $h\colon K_1 \cup \ldots \cup K_n \to K_1 \cup \ldots \cup K_n$ can be extended to an ε-homeomorphism of P onto P.

(This is a special case of part (2) of Theorem 1, when $L_i = K_i$.)

Thus in order to prove Theorem 1, we shall first prove condition (1) and then (2′).

2. Proof of Theorem 1. Our argument starts with the following lemma (see also Question 1 in Section 3 concerning a possible generalization).

LEMMA 1. *For any finite sequence $Q_1, \ldots, Q_n$ of nondegenerate continua lying in mutually distinct composants of P, and any monotone surjections $f_i\colon Q_i \to P$, $i = 1, \ldots, n$, there is a common extension $f\colon P \to P$ of all f_i.*

Proof. Consider first the case where the f_i are homeomorphisms. By Lehner's Theorem it suffices to show that

(a) there exist continua $Q_1, \ldots, Q_n$ lying in mutually distinct composants of P, and there are homeomorphisms $f_i\colon Q_i \to P$, $i = 1, \ldots, n$, which have a common extension to P.

We will prove the lemma by induction on n. Since each subcontinuum of P is a retract of P by the result of Cornette [C], the case $n = 1$ is clear. Assume that the lemma holds for $n-1$. There is an obvious homeomorphism $h \in H(P)$ such that $h^2 = \mathrm{id}_P$ and h has a unique fixed point. There is also a sequence of nondegenerate continua $Q_1, \ldots, Q_{n-1}$ in P such that

P is irreducible between Q_i, Q_j for $i \neq j$, and between $h(Q_i), Q_j$ for any $i, j = 1, \ldots, n-1$.

We will show that there exist homeomorphisms $f_1, \ldots, f_n$ on the continua $Q_1, \ldots, Q_{n-1}, Q_n = h(Q_{n-1})$ satisfying (a). To see this, we consider the family $\{\{x, h(x)\}\colon x \in P\}$. This family is a continuous decomposition of P. Denote by g the quotient map, which is open. By [R], $g(P)$ is again a pseudo-arc. The continua $g(Q_1), \ldots, g(Q_{n-1}) = g(Q_n)$ lie in mutually distinct composants of $g(P)$. By the inductive assumption, there exist homeomorphisms $h_i\colon g(Q_i) \to g(P)$, $i = 1, \ldots, n-1$, and their common extension $h_0\colon g(P) \to g(P)$. Putting $f_1 = h_1 g | Q_1, \ldots, f_{n-1} = h_{n-1} g | Q_{n-1}, f_n = h_{n-1} g | Q_n$, we obtain homeomorphisms $f_i\colon P \to g(P)$ which have a common extension $h_0 g$. Thus (a) is satisfied, implying the lemma for n.

If some f_i's are nontrivially monotone, we consider the quotient space P' of P, induced by the decomposition of P into the sets $f_i^{-1}(x)$ and single points. The quotient map is monotone, and hence P' is a pseudo-arc (see [M1, (8.16), p. 74]). Applying the above argument to P' instead of P, we obtain the conclusion.

Let $Q_1, \ldots, Q_n$ be fixed continua lying in mutually distinct composants of P. Define $P_1, \ldots, P_n$ as follows:

$$P_i = \begin{cases} P & \text{if } Q_i \text{ is degenerate,} \\ C(P) & \text{otherwise.} \end{cases}$$

Let $\mathscr{M}$ be the space of all surjective maps $f\colon P \to P$ equipped with the sup metric. Define a map $T\colon \mathscr{M} \to P_1 \times \ldots \times P_n$ by

$$T(f) = (f(Q_1), \ldots, f(Q_n)).$$

Put

$$Z_n = Z_n(Q_1, \ldots, Q_n) = \{(x_1, \ldots, x_n) \in P_1 \times \ldots \times P_n\colon \text{if } Q_i \text{ is nondegenerate, then so is } x_i, \text{ and } x_i, x_j \text{ lie in distinct composants of } P \text{ for each } i \neq j\}.$$

Note that Z_n is a G_δ-set in $P_1 \times \ldots \times P_n$.

LEMMA 2. *For any $\varepsilon > 0$ and any $z_0 = (z_1^0, \ldots, z_n^0) \in Z_n$, there exists a $\delta > 0$ such that for any $(z_1, \ldots, z_n) \in Z_n$ with $\operatorname{dist}(z_i, z_i^0) < \delta$ $(i = 1, \ldots, n)$, there exists an ε-homeomorphism $h \in H(P)$ which carries z_i^0 onto z_i.*

Proof. Define an action $f\colon H(P) \times Z_n \to Z_n$ by $f(h, (z_1, \ldots, z_n)) = (h(z_1), \ldots, h(z_n))$. Then f is transitive by Lehner's Theorem. Apply Effros' Theorem to $G = H(P)$, $X = Z_n$, and $a = (z_1^0, \ldots, z_n^0)$.

Lemma 3. *For any $x = (x_1, \ldots, x_n) \in P_1 \times \ldots \times P_n$, there is a map $f \in \mathscr{M}$ such that $T(f) = x$ and T is interior at f (i.e. $x \in \operatorname{int} T(U)$ for every open set U in $\mathscr{M}$ containing f). Moreover, if x_i is nondegenerate, then $f | Q_i\colon Q_i \to x_i$ can be a homeomorphism.*

Proof. Let R_i be a continuum such that

$$R_i = \begin{cases} Q_i & \text{if } x_i = P, \\ \text{a proper subcontinuum of } P \text{ properly containing } Q_i & \text{otherwise.} \end{cases}$$

Let $f_i\colon R_i \to P$ be such that

$$f_i = \begin{cases} \text{a monotone open surjection} & \text{if } x_i \text{ is degenerate and } Q_i \text{ is not,} \\ \text{a homeomorphism} & \text{otherwise,} \end{cases}$$

and $f_i^{-1}(x_i) = Q_i$. Further, let $f\colon P \to P$ be a common extension of all f_i's, whose existence is guaranteed by Lemma 1. Given a neighbourhood N of id_P in $H(P)$, in view of Lemma 2 and the definition of f_i, we observe that $x \in \operatorname{int} T(fN)$, where $fN = \{fg\colon g \in N\}$. Hence the lemma follows.

Lemma 4. *For any $\varepsilon > 0$, there is a $\delta > 0$ such that if $x = (x_1, \ldots, x_n) \in Z_n$, $y = (y_1, \ldots, y_n) \in Z_n$ and $\operatorname{dist}(x_i, y_i) < \delta$ for $i = 1, \ldots, n$, then there are mappings $f, g \in \mathscr{M}$ with $T(f) = x$, $T(g) = y$, and $\tilde{d}(f, g) < \varepsilon$.*

Lemma 4 follows from Lemma 3 and the compactness of $P_1 \times \ldots \times P_n$.

Lemma 5. *For any $\varepsilon > 0$, there is a $\delta > 0$ such that if $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in Z_n$ and $\operatorname{dist}(x_i, y_i) < \delta$ for $i = 1, \ldots, n$, then there is an ε-homeomorphism $h\colon P \to P$ with $h(x_i) = y_i$.*

Proof. For a given $\varepsilon > 0$, let δ be as in Lemma 4. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in Z_n satisfy the hypothesis of the lemma, then there are maps $f, g \in \mathscr{M}$ as in Lemma 4. By [Lw3] there are homeomorphisms $f_m, g_m \in H(P)$ such that $\lim f_m = f$ and $\lim g_m = g$. Applying Lemma 2, we obtain homeomorphisms $q_m, r_m \in H(P)$ such that $q_m f_m(Q_i) = x_i$, $r_m g_m(Q_i) = y_i$ and $\lim q_m = \lim r_m = \mathrm{id}_P$. We have $\tilde{d}(q_m f_m, r_m g_m) < \varepsilon$ for sufficiently large m, and thus the required h is $r_m g_m f_m^{-1} q_m^{-1}$.

For a fixed n there are exactly 2^n distinct sets Z_n. They are constructed by taking all combinations of Q_i's, each Q_i being either degenerate or not. For any $\varepsilon > 0$ and any set Z_n, there is a δ guaranteed by Lemma 5. The minimum of these δ's satisfies conclusion (1) of Theorem 1, and thus this conclusion is proved.

Next we proceed to find, for a given $\varepsilon > 0$, a $\delta > 0$ which satisfies the conclusion (2′).

Again, consider a fixed set Z_n.

Lemma 6. *For any $(z_1, \ldots, z_n) \in Z_n$ and any $\varepsilon > 0$, there is a $\delta > 0$ such that every δ-homeomorphism $h\colon z_1 \cup \ldots \cup z_n \to z_1 \cup \ldots \cup z_n$ with $h(z_i) = z_i$ possesses an ε-homeomorphic extension $h^*\colon P \to P$.*

Proof. Put

$$H_1 = \{h \in H(P)\colon h(z_i) = z_i \text{ for } i = 1, \ldots, n\},$$
$$H_2 = \{h \in H(z_1 \cup \ldots \cup z_n)\colon h(z_i) = z_i \text{ for } i = 1, \ldots, n\}.$$

The action $a_0\colon H_1 \times H_2 \to H_2$ is defined by

$$a_0(h, f) = h_z f, \quad \text{where } h_z = h | z_1 \cup \ldots \cup z_n \quad \text{for } h \in H_1.$$

Clearly, a_0 is transitive by Lehner's Theorem. Since both H_1 and H_2 are completely metrizable, the lemma follows from Effros' Theorem.

To show the next lemma, we need a certain preparation.

Let X be a space, and $\mathscr{D}$ a continuous decomposition of X into compact sets. The following property of $\mathscr{D}$ will be considered:

(*) (a) each autohomeomorphism of $X/\mathscr{D}$ can be lifted to an autohomeomorphism of X, and
(b) for each $D \in \mathscr{D}$ and each autohomeomorphism f of D, f can be extended to an autohomeomorphism f^* of X such that $f^*(D') = D'$ for every $D' \in \mathscr{D}$.

In [Lw5] Lewis has shown that for any one-dimensional continuum X there is a one-dimensional continuum $\hat{X}$ with a continuous decomposition $\mathscr{D}$ into pseudo-arcs satisfying (*), such that $\hat{X}/\mathscr{D}$ is homeomorphic to X. Another result of Lewis [Lw4] says that in the case when $X = P$, the space $\hat{P}$, which must be hereditarily indecomposable, is again a pseudo-arc. Hence P admits a continuous decomposition into pseudo-arcs satisfying (*). In the next lemma we prove that the elements of such a decomposition can be arbitrarily small.

Lemma 7. *There are continuous decompositions $\mathscr{F}_m$, $m = 1, 2, \ldots$, of P into pseudo-arcs, satisfying (*), such that $\operatorname{mesh} \mathscr{F}_m = \sup\{\operatorname{diam} F\colon F \in \mathscr{F}_m\} \to 0$ as $m \to \infty$.*

Proof. Let $P = \varprojlim(A_i, f_i^j)$, where the A_i are arcs. Lewis' construction of $\hat{P}$ can be sketched as follows. Arcs of pseudo-arcs $\hat{A}_i$ (i.e. chainable continua with continuous decomposition into pseudo-arcs, which have the arcs A_i as quotient spaces) are considered instead of the arcs A_i with the quotient maps $g_i\colon \hat{A}_i \to A_i$. The maps $f_i^j\colon A_j \to A_i$ are lifted to some maps $\hat{f}_i^j\colon \hat{A}_j \to \hat{A}_i$. The space $\hat{P}$ is defined as $\varprojlim(\hat{A}_i, \hat{f}_i^j)$.

We modify this construction to obtain pseudo-arcs $\hat{P}_m$ defined as $\varprojlim(B_i, g_i^j)$, where

$$B_i = \begin{cases} A_i & \text{for } i < m, \\ \hat{A}_i & \text{for } i \geq m, \end{cases} \qquad g_i^j = \begin{cases} f_i^j & \text{for } j < m, \\ \hat{f}_i^j & \text{for } i, j \geq m, \\ f_i^m g_m \hat{f}_m^j & \text{for } i < m \leq j. \end{cases}$$

We assume that all considered spaces are contained in the same Hilbert cube. Being homeomorphic to P, each pseudo-arc $\hat{P}_m$ has a continuous decomposition $\mathscr{D}_m$ into pseudo-arcs, satisfying (*), and $\lim \operatorname{mesh} \mathscr{D}_m = 0$. On the other hand, $\lim \hat{P}_m = P$ and thus, by [Lw2, Theorem 1], there are ε_m-homeomorphisms $h_m\colon \hat{P}_m \to P$ such that $\lim \varepsilon_m = 0$. Then the decompositions $\mathscr{F}_m = \{h_m(D)\colon D \in \mathscr{D}_m\}$ are those desired.

Lemma 8. *For every $\varepsilon > 0$, there is a $\delta > 0$ such that for any $P' \in C(P)$, every δ-homeomorphism $h \in H(P')$ has an ε-homeomorphic extension $h^* \in H(P)$.*

Proof. Given any $\varepsilon > 0$, take $2\delta_1$ as in Lemma 5 for $\varepsilon/4$ and $n = 1$. By Lemma 7, there exists a continuous decomposition $\mathscr{F}$ of P into pseudo-arcs whose mesh is less than δ_1. Let

$$C_1 = \{A \in C(P)\colon \operatorname{diam} A \leq \delta_1\} \tag{1}$$

and assume that $P' \in C_1$. Then there exists an $\varepsilon/4$-homeomorphism $f \in H(P)$ which carries P' onto a member Q of $\mathscr{F}$. Assume that $h \in H(P')$ is a δ_1-homeomorphism. Then $fhf^{-1}|Q \in H(Q)$ has a δ_1-homeomorphic extension G by (*) and the choice of $\mathscr{F}$. An ε-homeomorphic extension of h is defined by $f^{-1}Gf$.

Next we fix a nondegenerate proper subcontinuum Q_1 of P. By Lemma 3 ($n = 1$), there exists a map $g: P \to g(P) = P$ such that

(2) $$g|Q_1: Q_1 \to P \quad \text{is a homeomorphism},$$

and

(3) $$P \in \operatorname{int} T(N) \quad \text{for each neighbourhood } N \text{ of } g \text{ in } \mathscr{M}.$$

Take $\xi > 0$ such that if $d(x, y) < \xi$, then $d(g(x), g(y)) < \varepsilon/2$ for any $x, y \in P$. Let $\eta > 0$ be a number whose existence is guaranteed by Lemma 6 for $\xi/4$ and Q_1 ($n = 1$). By (2), there exists a $\delta_2 > 0$ such that $\delta_2 < \varepsilon/16$, and the following conditions are satisfied:

(4) for any homeomorphism $k: Q_1 \to P$ which is $4\delta_2$-near to $g|Q_1$, k^{-1} and $(g|Q_1)^{-1}$ are $\eta/3$-near,

(5) for any pair $x, y \in P$ with $d(x, y) < 4\delta_2$,
$$d\big((g|Q_1)^{-1}(x), (g|Q_1)^{-1}(y)\big) < \eta/3.$$

By (3) applied to the $N = 2\delta_2$-neighbourhood of g, there exists a $\varrho > 0$ such that the ϱ-neighbourhood of P in $C(P)$ is contained in $\operatorname{int} T(N)$. Define C_2 by

(6) $$C_2 = \{A \in C(P): \operatorname{dist}(A, P) \leqslant \varrho/2\},$$

and assume that $P' \in C_2$. Take a δ_2-homeomorphism $h \in H(P')$. We will construct an ε-homeomorphic extension of h. By the choice of ϱ, [Lw3], and Lemma 5, we may assume that there exists a $k \in H(P)$ which is $4\delta_2$-near to g with $k(Q_1) = P'$. Then $\tilde{d}(k^{-1}hk|Q_1, \mathrm{id}_{Q_1}) < \eta$. Take a ξ-homeomorphic extension G of $k^{-1}hk|Q_1$. Then the required extension of h is kGk^{-1}. Let

$$C_3 = \{A \in C(P): \operatorname{diam} A \geqslant \delta_1 \text{ and } \operatorname{dist}(A, P) \geqslant \varrho/2\}.$$

Since C_3 is compact, by Lemmas 5 and 6 we can find a $\delta_3 > 0$ such that each δ_3-homeomorphism on each $P' \in C_3$ has an ε-homeomorphic extension to P.

Taking $\delta = \min(\delta_1, \delta_2, \delta_3)$ we obtain the number required in the conclusion of Lemma 8.

Lemma 9. *For every $\varepsilon > 0$, there is a $\delta > 0$ such that for any $z = (z_1, \ldots, z_n) \in Z_n$ and any δ-homeomorphism $h: z_1 \cup \ldots \cup z_n \to z_1 \cup \ldots \cup z_n$ with $h(z_i) = z_i$, there is an ε-homeomorphic extension $h^* \in H(P)$ of h.*

Proof. Consider any given $\varepsilon > 0$. Fix nondegenerate proper subcontinua $L_1, \ldots, L_n$ lying in mutually distinct composants of P, and some homeomorphisms $g_i: L_i \to P$, $i = 1, \ldots, n$. By Lemma 1, $g_1, \ldots, g_n$ have a common extension $g: P \to P$. We take positive numbers $\delta_1, \delta_2, \delta_3$ and δ which satisfy the following conditions:

(1) For any pair of points x, y in P with $d(x, y) < \delta_1$, $d(g(x), g(y)) < \varepsilon/4$.

(2) Each δ_2-homeomorphism on $L_1 \cup \ldots \cup L_n$ which carries L_i onto L_i has a δ_1-homeomorphic extension to P (Lemma 6).

(3) Each δ_3-homeomorphism on any proper subcontinuum K_i of L_i has a δ_2-homeomorphic extension to L_i (Lemma 8).

(4) For any pair of points x, y in P with $d(x, y) < \delta$, $d(g_i^{-1}(x), g_i^{-1}(y)) < \delta_3$, $i = 1, \ldots, n$.

Such a δ is the required one. To see this, we take a $z = (z_1, \ldots, z_n) \in Z_n$ and a δ-homeomorphism $h \in H(z_1 \cup \ldots \cup z_n)$ which satisfies $h(z_i) = z_i$. We will construct an ε-homeomorphic extension of h. Define a homeomorphism f on $\bigcup_{i=1}^{n} g_i^{-1}(z_i)$ by $f|g_i^{-1}(z_i) = (g_i|g_i^{-1}(z_i))^{-1} h (g_i|g_i^{-1}(z_i))$. Then $\tilde{d}(f|g_i^{-1}(z_i), \mathrm{id}_{g_i^{-1}(z_i)}) < \delta_3$ by (4). Hence we have a δ_2-homeomorphic extension $f_i \in H(L_i)$ of $f|g_i^{-1}(z_i)$ for $i = 1, \ldots, n$ by (3), and a δ_1-homeomorphic extension $F \in H(P)$ of $f_1, \ldots, f_n$ by (2).

Take $\delta_5, \delta_6, \delta_7$ satisfying the following conditions:

(5) Each $2\delta_5$-homeomorphism on $z_1 \cup \ldots \cup z_n$ has an $\varepsilon/2$-homeomorphic extension to P (Lemma 6).

(6) Each pair of points $x, y \in P$ with $d(x, y) < \delta_6$ satisfies the inequalities $d(gF(x), gF(y)) < \delta_5$ and $\delta_6 < \delta_5$.

(7) For any homeomorphism $k: g_i^{-1}(z_i) \to z_i$ which is δ_7-near to $g|g_i^{-1}(z_i)$, k^{-1} is δ_6-near to $(g|g_i^{-1}(z_i))^{-1}$, and $\delta_7 < \delta_6, \varepsilon/8$.

By [Lw3] and Lemma 5, there exists a homeomorphism $p \in H(P)$ which is δ_7-near to g, with $p(g_i^{-1}(z_i)) = z_i$. Then, since $gF(g|g_i^{-1}(z_i))^{-1} = h$, we have

$$\tilde{d}\big(h\big(pF(p|g_i^{-1}(z_i))^{-1}\big)^{-1}, \mathrm{id}_{z_i}\big) = \tilde{d}\big(h, pF(p|g_i^{-1}(z_i))^{-1}\big)$$
$$\leqslant \tilde{d}\big(pF(p|g_i^{-1}(z_i))^{-1}, gF(p|g_i^{-1}(z_i))^{-1}\big) + \tilde{d}\big(gF(p|g_i^{-1}(z_i))^{-1}, gF(g|g_i^{-1}(z_i))^{-1}\big) < 2\delta_5$$
$$\text{for } i = 1, \ldots, n.$$

Hence there exists an $\varepsilon/2$-homeomorphic extension $R \in H(P)$ of $h\big(pF(p|g_i^{-1}(z_i))^{-1}\big)^{-1}$. The extension which we want is $h^* = RpFp^{-1}$. This completes the proof.

For any fixed n, any $\varepsilon > 0$ and any Z_n there is a δ guaranteed by Lemma 9. The minimum of these δ's satisfies the conclusion (2'). The proof of Theorem 1 is complete.

3. Some other results. In this section we will prove two results related to the lemmas to Theorem 1. The first one is also related to a result of Cornette. He has shown that every subcontinuum of a pseudo-arc is a retract of the whole space. We prove

Theorem 2. *Let X be a continuum having only pseudo-arcs for proper nondegenerate subcontinua. Then X is a pseudo-arc if and only if it satisfies the following condition:*

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that each proper subcontinuum Q of X satisfying $\operatorname{dist}(Q, X) < \delta$ admits an ε-retraction $r: X \to Q$ onto Q.

Proof. This theorem can also be proved by a chain argument similar to that used by Cornette. However, using an idea due to Cook [Co], we will apply here Theorem 1 and the following theorem [Mo, p. 3].

Moore's Theorem. *Let M be an uncountable metric space. Suppose that a relation $\geqslant$ on*

M satisfies the following conditions:

(1) *For each $x, y \in M$, $x \geqslant y$ or $y \geqslant x$,*

(2) *if $x \geqslant y$ and $y \geqslant x$, then $x = y$.*

Further, assume that every uncountable subset of M has a limit point. Then there exists a $p \in M$ which is a limit point of both the set $\{x \in M: x > p\}$ and the set $\{x \in M: p > x\}$.

To prove Theorem 2, suppose that X is a pseudo-arc, and take a point $p \in X$. Let $\mathscr{P}$ be the collection of all subcontinua of X which contain p. For each $A \in \mathscr{P}$, take a retraction $r_A: X \to A$ onto A. The set $R = \{r_A: A \in \mathscr{P}\}$ with the sup metric is separable, so every uncountable subset of R has a limit point. Define a relation $\geqslant$ by

$$r_A \geqslant r_B \quad \text{if and only if} \quad A \supset B.$$

Being a linear order, $\geqslant$ satisfies the hypothesis of Moore's Theorem. Hence there is a continuum $A \in \mathscr{P}$ such that r_A is a limit point of both $\{r_B: B \subsetneqq A\}$ and $\{r_B: A \subsetneqq B\}$. Therefore there exists a sequence of subcontinua $A_i \subsetneqq A$ such that $\lim r_{A_i} = r_A$. Note that $\operatorname{Lim} A_i = A$. Since A is homeomorphic to X, we infer that there is a sequence P_i of proper subcontinua of X converging to X and a sequence r_i of retractions of X onto P_i such that $\lim r_i = \mathrm{id}_X$. Using this fact and the conclusion (1) of Theorem 1, we see that the conclusion of Theorem 2 is satisfied.

The converse implication is easy. It follows from the fact that a continuum which admits ε-retractions onto chainable continua for each $\varepsilon > 0$ is chainable.

Recall that a continuum is said to be a *triod* if it has a subcontinuum with the complement containing at least three components. A continuum containing no triod is said to be *atriodic*. A map $f: X \to Y$ is said to be *atomic* if $f^{-1}f(K) = K$ for each continuum $K \subset X$ such that $f(K)$ is nondegenerate.

The next result, connected with Lemma 7 and the remark just before it, concerns continuous decompositions of a pseudo-arc. Lewis has shown [Lw4] that a pseudo-arc is the unique hereditarily indecomposable continuum which admits a continuous decomposition into pseudo-arcs such that its decomposition space is a pseudo-arc. Here we will show that a pseudo-arc is the unique atriodic continuum which has this property. We will essentially use [Lw4].

THEOREM 3. *Let $f: M \to P$ be an open map from an atriodic continuum M onto P. If each fibre of f is a pseudo-arc, then M is a pseudo-arc.*

We need the following two lemmas for the proof. The first has been shown by Maćkowiak and Tymchatyn in [MT, (13.7), p. 32].

LEMMA 10. *If f is a monotone open mapping from an atriodic continuum, then f is atomic.*

The proof of the next lemma is easy, so we omit it.

LEMMA 11. *Let $f: X \to Y$ be an atomic surjection from a continuum X such that, for any $y \in Y$, the spaces Y and $f^{-1}(y)$ are hereditarily indecomposable continua. Then X is hereditarily indecomposable.*

Proof of Theorem 3. The map f is atomic by Lemma 10. From Lemma 11 it follows that M is hereditarily indecomposable. Applying [LW4], we see that M is a pseudo-arc. This completes the proof.

Finally, we will pose two questions related to lemmas to the main theorem. The first one is related to Lemma 1. Maćkowiak [M2] has extended the result of Cornette [C] by showing that the pseudo-arc is an absolute extensor among hereditarily indecomposable continua. In view of Lemma 1, the authors wonder whether the following generalization of the results [C] and [M2] is possible.

QUESTION 1. Let X be a hereditarily indecomposable continuum (e.g. a pseudo-arc), $Q_1, \ldots, Q_n$ be mutually disjoint subcontinua of X and $f_i: Q_i \to P$ be mappings. Does there exist a common extension $f: X \to P$ of all mappings f_i?

If all f_i's are onto, the proof similar to [M2] gives an affirmative answer. This generalizes Lemma 1, but the proof of Lemma 1 is direct and "more geometric".

The next question is related to Lemma 7.

QUESTION 2. Let $\mathscr{D}$ and $\mathscr{D}'$ be arbitrary continuous, nontrivial decompositions of P into pseudo-arcs. Do they satisfy (*)? Does there exist a homeomorphism $h \in H(P)$ such that $\mathscr{D}' = \{h(D): D \in \mathscr{D}\}$?

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