Decomposition of special Jacobi sets

by

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Abstract. In [4], Jacobi sets R are introduced and studied using the Weyl group of the usual real system of roots $\hat{R}$. In this paper, we study a special class of Jacobi sets using their combinatorial structure without recourse to the Weyl group of $\hat{R}$ and show that they can be decomposed as a sum of a classical root system and a nil root system. Finally, we give an example of a Lie algebra whose root system is a special Jacobi set.

0. Introduction. In Winter [7, 10], symmetrysets are introduced and studied, and in Winter [9], Lie rootsystems are also introduced. These variations of rootsystems occur naturally in the study of classical and symmetric Lie algebras respectively. For a classical Lie algebra L of characteristic p, with Cartan subalgebra H, we have a rootsystem called $R(L, H)$. This system is not a rootsystem in Euclidean space due to p torsion. In Winter [7, 10] it has been shown that certain structures of $R(L, H)$ alone lead to the identification of $R(L, H)$ with a rootsystem in Euclidean space, and this enabled direct classification of classical Lie algebras of characteristic p along the same lines as in the theory of complex semisimple Lie algebras [6, 7].

In Hailat [4], Jacobi sets R are introduced and studied using the Weyl group of the usual real system of roots $\hat{R}$. Jacobi sets are symmetrysets with a Jacobi condition. In this paper we continue our study of Jacobi sets using their combinatorial structure. This paper is one in a series of papers [2, 3, 4, 5, 7, 10], whose objectives are to classify rootsystems. The rootsystems under study are beginning to fill out a pattern which thus far is only partially exposed, the exposed part being the rootsystems of classical Lie algebras (systems of black roots), the rootsystems of symmetric Lie algebras (systems of black and white roots) and systems where black and grey roots appear in rootsystems of Lie algebras over algebraic number fields and other not algebraically closed fields (for more about the colors of roots see Hailat [2]).

In § 1 we introduce all the definitions and the preliminaries. In § 2 we decompose a root x into its two parts: *regular* s and *nil* n, that is, $x = s+n$. Consequently, in § 3 we decompose R as $R = S \oplus R_0$ where S is a *regular* symmetryset and R_0 is *nil*.

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1. Definitions and preliminaries. Let V be a vector space over a field of characteristic $p > 0$ and let R be a finite subset of V. We regard R as a *groupoid* with $a+b \in R$ only for certain $a, b \in R$. For $a \in V$ the relation $\{(b, b+a) \mid b+a \in R\}$ generates an equivalence relation on R. The corresponding equivalence class of $b \in R$ is the *string* $R_b(a) = \{b-ra, \ldots, b+qa\}$. We call $R_b(a)$ the *a-orbit* of b with length $q+r$. The orbit $R_b(a)$ is *bounded* if $R_b(a) \neq b+\mathbf{Z}a$. An *automorphism* of the set $R \subseteq V$ is a bijection $r\colon R \to R$ such that $a+b \in R$ if and only if $r(a)+r(b) \in R$, in which case $r(a+b) = r(a)+r(b)$, for all $a, b \in R$. We call an automorphism r_a which stabilizes all a-orbits $R_b(a)$ $(b \in R)$ a *symmetry* of R at a. If $R_b(a)$ is bounded we define the *Cartan integer* $a^*(b) = r-q$, and the *reflection* r_a by $r_a(c) = c - a^*(c)a$ if $R_c(a)$ is bounded. The element $a \in R$ *is unbounded* if there exists $b \in R$ such that $R_b(a) = b+\mathbf{Z}a$. We say, more specifically, that a is *unbounded at* b if $R_b(a) = b+\mathbf{Z}a$ $(b \in R)$.

A *symmetryset* R is a finite subset of V such that $\operatorname{Aut}R$ contains a symmetry r_a of R at a for all $a \in R$, $a \neq 0$. A symmetryset R is *unbounded* if every element in R is unbounded. Note that every symmetryset contains 0 and $-a$ for all $a \in R$. Also note that in a symmetryset we have $r_a(a) = -a$ and, therefore, $a^*(a) = 2$ for every bounded element $a \in R$. An element $a \in V$ is called a *root* if $a \in R \subseteq V$.

The *closure mapping* of R, $\wedge\colon R \to \hat{R}$, is defined by $\wedge(a) = \hat{a}$, where $\hat{R} \subseteq R^{**} = \operatorname{Hom}(R^*, \mathbf{Z})$, $\hat{a}(f) = f(a)$ and $f \in R^* = \operatorname{Hom}(R, \mathbf{Z})$. We say R is *nil* if $R = \operatorname{Ker} \wedge = R_0$.

1.1. Definition. A symmetryset R is said to be a *Jacobi* set if the following condition is satisfied: if $a, b, c, a+b, (a+b)+c \in R$ such that $a \neq -b$ and one of the following conditions holds:

(J_1) $(a+b)\hat{} = \hat{0}$ and $\hat{c} = \hat{0}$,

(J_2) $(a+b)\hat{} \neq \hat{0}$ and $\hat{c} \neq \hat{0}$,

then $a+c \in R$ or $b+c \in R$.

Note that (J_1) reflects the condition for Witt Lie algebras whose rootsets R are subgroups of the prime field $\mathbf{Z}_p$ such that $\hat{a} = \hat{0}$ for all $a \in R$. Also, (J_2) reflects the condition for complex semi-simple Lie algebras that $[[L_a, L_b], L_c] \neq 0$ implies that $[L_a, L_c] \neq 0$ or $[L_b, L_c] \neq 0$.

We fix following notations throughout the paper. $R_0 = \{a \in R \mid \hat{a} = \hat{0}\}$, $R_\infty = \{a \in R \mid a$ is unbounded element of $R\}$, $R_1 = \{a \in R \mid \hat{a} \neq \hat{0}\}$. Note that $R_\infty \subseteq R_0$ since every unbounded element is in R_0 by Proposition 1.3.

1.2. Definition. Let $\pi = \{a_1, \ldots, a_n\}$ be a subset of R_1 which satisfies the following conditions:

(C_1) $a_i - a_j \notin R$ for all $i \neq j$;

(C_2) there exists no infinite sequence $c_1, c_2, \ldots, c_s \ldots$ of elements of R such that $c_j - c_{j-1} \in \pi$ for all $j = 1, 2, \ldots$

(C_3) If $a = (\ldots((\alpha + a_{i_1}) + a_{i_2} + \ldots + a_{i_{k-1}}) + a_{i_k} \in R$ and $\alpha \in R$ be such that $\hat{\alpha} = \hat{0}$ then $\hat{a} \neq \hat{0}$ where $a_{i_1}, \ldots, a_{i_k} \in \pi$.

The subset $\pi \subset R_1$ is called *regular* if π is a maximal subset of R satisfying the conditions (C_1), (C_2) and (C_3).

Following Winter [7, 10], a symmetryset R has a unique maximally refined decomposition where the subsets $D_1, \ldots, D_n$ are symmetrysets and $R = D_1 \cup \ldots \cup D_n$ with $D_i \cap D_j = \{0\}$ for $i \neq j$ such that $a_1 + \ldots + a_n = a \in R$ if and only if $a_1, \ldots, a_n, a \in D_i$ for some $i = 1, \ldots, n$. The symmetrysets $D_1, \ldots, D_n$ are the irreducible components of R and R is *irreducible* if $n = 1$. Thus for any symmetryset R, R is the *inner direct sum* of its irreducible components, that is, $R = D_1 \oplus \ldots \oplus D_n$.

The following results are needed in the paper:

1.3. Proposition (Hailat [2]). *Let R be a symmetryset. If $\hat{a} \neq \hat{0}$, then a is bounded and $a^* \in \operatorname{Hom}(R, \mathbf{Z})$.*

1.4. Theorem (Hailat [5]). *Let a be an unbounded element in a Jacobi set R, and let b, $b+ta \in R$ for some $t \neq 0$. Then $b+i(ta) \in R$ for all $i \in \mathbf{Z}$.*

1.5. Theorem (Hailat [5]). *Let $R \subseteq \mathbf{Z}_p^n$ be an unbounded Jacobi set. Then $R = \mathbf{Z}_p^{n_1} \oplus \ldots \oplus \mathbf{Z}_p^{n_k}$ (inner direct sum).*

1.6. Theorem (Hailat [5]). *Let R be a symmetryset (Jacobi set). Then the set $R_0 = \operatorname{Ker} \wedge = \{a \in R \mid \hat{a} = \hat{0}\}$ is a nil symmetryset (Jacobi set).*

1.7. Theorem (Winter [14]). *Let R be a symmetryset such that $R_b(a)$ $(a, b \in R$, $a \neq 0)$, $R_{\hat{b}}(\hat{a})$ $(\hat{a}, \hat{b} \in \hat{R}$, $\hat{a} \neq \hat{0})$ are bounded and $r_{\hat{a}}(\hat{b}) = (r_a(b))\hat{}$ for all $a, b \in R$, $a \neq 0$, $\hat{a} \neq \hat{0}$. Then*

(1) *$\hat{a}^*(\hat{b}) = a(b)$ for all $a, b \in R$, $a \neq 0$, $\hat{a} \neq \hat{0}$;*

(2) *for any $a, b \in R$, $\hat{a} \neq \hat{0}$, there exists $c \in R$ such that the closure mapping maps $R_c(a)$ bijectively onto $R_{\hat{b}}(\hat{a})$;*

(3) *the closure mapping $R \to \hat{R}$ is an isomorphism (of groupoids) if and only if it is bijective.*

1.8. Theorem (Hailat [3]). *Let R be a Jacobi set. Then all regular subsymmetrysets of R are isomorphic to $\hat{R}$ and, therefore, are rootsystems in the sense of Bourbaki [1] with 0 added.*

Note that the symmetryset $\hat{R}$ is a system of roots in the sense of Bourbaki [1] with 0 added by Theorem 2.3 of Winter [14].

We say a symmetryset S is *regular* if there exists a regular subset $\pi \subset R_1$ such that $S = \mathbf{Z}\pi \cap R$ and in this case we write $S = S(\pi)$. Note that $S \subseteq R_1 \cup \{0\}$ since $R = R_0 \cup R_1$ and $S \cap R_0 = \{0\}$ by Proposition 2.1 below.

2. The decomposition: $x = s+n$. Let R be a Jacobi set and let $S = S(\pi)$ be a regular subsymmetry set of R. We show, in this section, that any element in R can be written as a sum of two elements, the first of which is in S and the second in R_0. But, to do this, we need the following proposition which is Proposition 2.5 of [3].

2.1. Proposition. *Let R be a Jacobi set and let S be a regular subsymmetry set of R. Then $S \cap R_0 = \{0\}$.*

We now have our decomposition theorem.

2.2. Theorem. *Let R be a Jacobi set and let S be a regular subsymmetry set of R. If $x \in R$ then $x = s+n$ for some $s \in S$ and for some $n \in R_0$.*

Proof. If $x \in S$ then set $s = x$, $n = 0$; also, if $x \in R_0$ then set $s = 0$, $n = x$. Since $R = R_0 \cup R_1$ the only remaining case for x is to be in $R_1 - S$. Let $x \in R_1 - S$. Since $\hat{S} = \hat{R}$ by Theorem 1.8, there exists $a \in S$ such that $\hat{x} = \hat{a}$. Thus $x^*(a) = \hat{x}^*(\hat{a}) = \hat{x}^*(\hat{x}) = 2$ by Theorem 1.7. This implies that $r_x(a) = a - 2x$, so that $a - x \in R$. Set $s = a$, $n = x - a$. We have $n \in R_0$ since $\hat{n} = (x-a)\hat{} = \hat{0}$.

Note that the above decomposition is unique since $S \cap R_0 = \{0\}$. We call s the *regular* part of x and n the *nil* part of x.

3. Decomposition of special Jacobi sets. In this section we decompose a special Jacobi set R (a Jacobi set with the condition $R_0 = R_\infty$) as a sum of a regular symmetryset S and a nil symmetry set R_0. But first we construct some special Jacobi sets as an example motivating further studies.

Let G be a finite subgroup of a vector space V over a field F of characteristic p and let S be a classical rootsystem over F. We may regard $S \subseteq W$ where W is a vector space. Now consider $G \oplus S = \{g \oplus a \mid g \in G,\ a \in S\} \subset V \oplus W$. Introduce the homomorphism $r_{g\oplus a}(h \oplus b) = (h \oplus b) - (g \oplus a)^\circ((h \oplus b))(g \oplus a)$ by specifying $(a \oplus b)^\circ \in \mathrm{Hom}(V \oplus W, \boldsymbol{Z}_p)$ as follows:

$$(g \oplus 0)^\circ(h \oplus b) = g^\circ(h),$$

$$(g \oplus a)^\circ(h \oplus b) = a^\circ(b), \qquad (b \neq 0).$$

Note that $(G \oplus S)_{g \oplus a}(h \oplus b) = \{(h \oplus b) - r(g \oplus a), \ldots, (h \oplus b) + q(g \oplus a)\}$ where $S_b(a) = \{b - ra, \ldots, b + qa\}$. Then $b + qa = r_a(b - ra) = (b - ra) - (a^\circ(d - rb))b$, so that $(h \oplus b) + q(g \oplus a) = r_{g \oplus a}((h \oplus b) - r(g \oplus a))$. It follows that $G \oplus S$ is a symmetryset. Let $R = G \oplus S$. Since G is a group $R_h(g) = h + \boldsymbol{Z}_p g$ implies that g is an unbounded root for all $g \in G$. Therefore, $\hat{g} = \hat{0}$ for every $g \in G$ by Proposition 1.3. Also, since S is a classical rootsystem then $\hat{a} \neq \hat{0}$ for all $a \in S - \{0\}$ (every nonzero element in S is bounded). Now let $g_1 \oplus a$, $g_2 \oplus b$, $g_3 \oplus c$, $(g_1 \oplus a) + (g_2 \oplus b)$, $((g_1 \oplus a) + (g_2 \oplus b)) + (g_3 \oplus c)$ be elements of R such that $g_1 \oplus a \neq (g_2 \oplus b)$. Then $a, b, c, a + b, (a + b) + c \in S$ from the definition of $G \oplus S$. If $((g_1 \oplus a) + (g_2 \oplus b))\hat{} = \hat{0}$ and $(g_3 \oplus c)\hat{} = \hat{0}$ then $(a + b)\hat{} = \hat{0}$ and $\hat{c} = \hat{0}$ so that $\hat{a} = -\hat{b}$, $c = 0$. It follows that:

$$(g_1 \oplus a) + (g_3 \oplus c) = (g_1 + g_3) \oplus (a + 0) = (g_1 + g_3) \oplus a \in R.$$

Also, if $((g_1 \oplus a) + (g_2 \oplus b))\hat{} \neq \hat{0}$ and $(g_3 \oplus c)\hat{} \neq \hat{0}$ then $(a + b)\hat{} \neq \hat{0}$ and $\hat{c} \neq \hat{0}$. It follows that $a + c \in S$ or $b + c \in S$ since S is a classical rootsystem. This last conclusion reflects the condition for the complex semisimple Lie algebra $L = L_S$ that $[[L_a, L_b], L_c] \neq 0$ implies that $[L_a, L_c] \neq 0$ or $[L_b, L_c] \neq 0$. Therefore, $(g_1 \oplus a) + (g_3 \oplus c) = (g_1 + g_3) \oplus (a + c) \in R$ or $(g_2 \oplus b) + (g_3 \oplus c) = (g_2 + g_3) \oplus (b + c) \in R$. Note that $R_0 = G \oplus 0 = R_\infty$. The above discussion implies the following proposition.

3.1. Proposition. *Let G be a finite subgroup of a vector space V over a field F of characteristic p and let S be a classical rootsystem over F. Then $R = G \oplus S$ is a special Jacobi set.*

Now Proposition 2.2 of Winter [8] and Proposition 3.1 imply that the rootsystem $G \oplus S$ of the generalized classical Albert–Zassenhaus (GCAZ) $L_{G \oplus S}$ is a special Jacobi set. Also, the rootsystem of the GCAZ $L_{G \oplus S}$ is of the form given by Theorem 3.10 below where $G \cong R_0$ and $S \cong \hat{R}$.

The following theorem, which is Theorem 2.3 of [4], is needed in this section.

3.2. Theorem. *Let R be a special Jacobi set and let $a \in R_1$, $\alpha, \gamma \in R_0$ be such that $\alpha \neq 0$ and $a + \alpha \in R$. Then $\alpha \pm \gamma \in R$ if and only if $a + \gamma \in R$.*

A number of lemmas will precede the proof of the main theorem. The same set up is involved in all of them. To avoid repetition the notation will now be fixed for this section.

Let R be a special Jacobi set and let S be a regular subsymmetryset of R. Also, let $x, y, x + y, x + \alpha \in R$ such that $\alpha \in R_0$ and $x \neq 0$.

3.3. Lemma. *If $x, y \in S - \{0\}$ then $y + \alpha \in R$.*

Proof. Suppose that $x, y \in S - \{0\}$ and $y + \alpha \notin R$. Then

$$R_y(x + \alpha) = \{y - r(x + \alpha), \ldots, y, \ldots, y + q(x + a)\}, \qquad r, q > 0.$$

Note that this orbit is bounded by Proposition 1.3 since $(x + \alpha)\hat{} = \hat{x} \neq 0$. It follows that

$$R_{\hat{y}}(x + \alpha) = \{\hat{y} - r\hat{x}, \ldots, \hat{y}, \ldots, \hat{y} + q\hat{x}\} \in \hat{R}.$$

But $S \cong \hat{R}$ by Theorem 1.8, therefore $y - rx, \ldots, y, \ldots, y + qx \in R$. This implies that

$$R_y(x + \alpha) = \{(y - rx) - r\alpha, \ldots, y, \ldots, (y + qx) + q\alpha\}.$$

Note that $(x + \alpha)^*(x) = (x + \alpha)\hat{}^*(\hat{x}) = \hat{x}^*(\hat{x}) = 2$ by Theorem 1.7, so that $r_{x+\alpha}(x) = x - 2(x + \alpha) = -x - 2\alpha$. Since $y + \alpha \notin R$ we have $y + i\alpha \notin R$ for all integers i by Theorem 1.4. Therefore, $(x + y) - (x + \alpha) = y - \alpha \notin R$. It follows that

$$R_{(x+y)}(x + \alpha) = \{(x + y), \ldots, (x + y) + k(x + \alpha)\} \qquad \text{and}$$

$$\hat{R}_{(x+y)}(x + \alpha) = \{\hat{x} + \hat{y}, \ldots, (k + 1)\hat{x} + \hat{y}\} \subset \hat{S} \qquad \text{for some } k.$$

This implies that $y + x, \ldots, y + (k + 1)x \in S$, since $S \cong \hat{R} = \hat{S}$, and $x, y \in S - \{0\}$. Therefore

$$R_{(y+x)}(x + \alpha) = \{(y + x), \ldots, (y + (k + 1)x) + k\alpha\}.$$

Now we summarize the above results as follows:

$$r_{x+\alpha}(x) = -x - 2\alpha, \qquad r_{x+\alpha}(y) = y - (r - q)(x + \alpha) = ((y - (r - q)x)) - (r - q)\alpha,$$

$$r_{x+\alpha}(y + x) = (y + (k + 1)x) + k\alpha.$$

But $r_{x+\alpha}(y + x) = r_{x+\alpha}(y) + r_{x+\alpha}(x)$. Hence

$$(y + (k + 1)x) + k\alpha = ((y - (r - q)x) - (r - q)\alpha) + (-x - 2\alpha).$$

It follows that $r - q = -(k + 2)$ and $r_{x+\alpha}(y) = y + (k + 2)x + (k + 2)\alpha \in R$. But $r_{\hat{x}}(\hat{y}) = r_{(x+\alpha)}\hat{}(\hat{y}) = \hat{r}_{x+\alpha}(y) = \hat{y} + (k + 2)\hat{x} \in \hat{R} \cong S$. Hence $y + (k + 2)x \in S$ and $r_{x+\alpha}(y) = (y + (k + 2)x) + (k + 2)\alpha \in R$. Choose $i \in \boldsymbol{Z}_p$ such that $i(k + 2) = k + 1$. Then $(y + (k + 2)x) + (k + 1)\alpha \in R$ by Theorem 1.4, a contradiction to the assumption that the last root to the right in $R_{y+x}(x + \alpha)$ is $(y + (k + 1)x) + k\alpha$. Therefore $y + \alpha \in R$.

3.4. Lemma. *If* $x \in R_1 - \{0\}$, $y \in R_0$ *or* $x \in R_0 - \{0\}$, $y \in R_1$ *then* $y + \alpha \in R$.

Proof. In either case we have $y+\alpha \in R$, by Theorem 3.2 since $x+y \in R$ and $x+\alpha \in R$.

3.5. Lemma. *If* $x, y \in R_0$ *then* $y+\alpha \in R$.

Proof. Since $R_0 = R_\infty = \boldsymbol{Z}_p^{n_1} \cup \ldots \cup \boldsymbol{Z}_p^{n_k}$ and $x+\alpha$, $y+x \in R$ it follows that $x, \alpha \in \boldsymbol{Z}_p^{n_i}$ for some i and $x, y \in \boldsymbol{Z}_p^{n_j}$ for some j. This implies that $i = j$ and, therefore, $y, \alpha \in \boldsymbol{Z}_p^{n_i}$. Thus $y+\alpha \in \boldsymbol{Z}_p^{n_i} \subseteq R$.

3.6. Lemma. *If* $x \in S - \{0\}$, $y \in R_1 - S$ *then* $y+\alpha \in R$.

Proof. Let $x \in S - \{0\}$ and $y \in R_1 - S$. Then $y = s+n \in S+R_0$ by Theorem 2.2. But $x+(s+n) = x+y \in R$. Hence $x+s \in R$ or $x+n \in R$ since (J_2) holds. Suppose first that $x+s \in R$. Then $s+\alpha \in R$ by Lemma 3.3 since $x+\alpha \in R$. But $(\alpha-(s+\alpha))+y = n \in R$, therefore $\alpha+y \in R$ or $y-(s+\alpha) \in R$. If $y+\alpha \in R$ then we are done. But if $y-(s+\alpha) \in R$ then $n-\alpha = (y-s)-\alpha = y-(s+\alpha) \in R$. This implies that $\pm n$ and $\pm\alpha$ are in the same component of R_0. But $y-n = s \in R$, so $y+\alpha \in R$ by Theorem 3.2 since $n+\alpha \in R$. Suppose, second, that $x+n \in R$. Then $\alpha+n \in R$ by Theorem 3.2 since $x+\alpha \in R$. It follows that $y+\alpha \in R$ by Theorem 3.2 since $y-n = s \in R$.

3.7. Lemma. *If* $x \in R_1 - S$, $y \in S$ *then* $y+\alpha \in R$.

Proof. Let $x \in R_1 - S$, $y \in S$. Then $x = s+n \in S+R_0$. This implies that $x-n = s \in R$, so that $\alpha+n \in R$ by Theorem 3.2 since $x+\alpha \in R$. But $(s+n)+y = x+y \in R$. Hence $s+y \in R$ or $n+y \in R$ since (J_2) holds. If $y+n \in R$ then $y+\alpha \in R$ by Theorem 3.2 since $\alpha+n \in R$. If $s+y \in R$ then $y+\alpha \in R$ by Lemma 3.3 since $s, y \in S$, $s+\alpha \in R$. (Note that $s+\alpha \in R$ by Theorem 3.2 since $s-n = x \in R$ and $\pm n+\alpha \in R$.)

3.8. Lemma. *If* $x, y \in R_1 - S$ *then* $y+\alpha \in R$.

Proof. Let $x, y \in R_1 - S$. Then $y = s'+n' \in S+R_0$ and $x+(s'+n') = x+y \in R$. This implies that $x+s' \in R$ or $x+n' \in R$ since (J_2) holds. If $x+n' \in R$ then $n'+\alpha \in R$ by Theorem 3.2 and, therefore, $y+\alpha \in R$ by the same theorem since $y-n' = s' \in R$. Now if $x+s' \in R$ then $s'+\alpha \in R$ by Lemma 3.7. But $s'+n' = y \in R$, so that $\pm n'+\alpha \in R$. This implies that $y+\alpha+R$ since $y-n' = s' \in R$.

3.9. Lemma. *Let R be a special Jacobi set and let $x, x+\alpha, y \in R$ such that $\alpha \in R_0$ and $x \neq 0$. If $x \pm y \in R$ then $y+\alpha \in R$.*

Proof. If $x+y \in R$ then $y+\alpha \in R$ by the previous lemmas. If $x-y \in R$ then $x+(-y) \in R$, so that $-y+\alpha \in R$. This implies that $y-\alpha = -(-y+\alpha) \in R$ so that $y+i(-\alpha) \in R$ for all $i \in \boldsymbol{Z}_p$ by Theorem 1.4. Choose $i = p-1$. Then $y+\alpha \in R$.

3.10. Theorem. *Let R be an irreducible special Jacobi set and let S be any regular subsymmetry set of R. Then $R = S \oplus R_0$ (direct sum).*

Proof. Let R be an irreducible special Jacobi set and let S be any regular subsymmetry set of R. Then $R \subseteq S \oplus R_0$ by Theorem 2.2. For the other inclusion, let $s \in S$ and $n \in R_0$. Since R is irreducible then there exists a chain of roots $c_0 = s, c_1, c_2, \ldots, c_k = n$ in R such that $c_i + c_{i+1} \in R$ for all $i = 0, 1, \ldots, k-1$. Now, we proceed by induction on k. If $k = 1$ then $s+n = c_0+c_1 \in R$. Suppose it is true for $k-1$.

That is, $s+c_{k-1} = c_0+c_{k-1} \in R$. Then $s+n = s+c_k \in R$ since $c_{k-1}+c_k \in R$ by Lemma 3.11. Therefore, $S \oplus R_0 \subseteq R$. It follows that $R = S \oplus R_0$.

Now Theorem 1.8 and Theorem 3.10 imply the following main result.

3.11. Theorem. *Any irreducible special Jacobi set R is a direct sum of a rootsystem $\hat{R}$ in the sense of Bourbaki [1] with 0 added and a nil symmetryset R_0.*

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