Boolean semigroup rings and exponentials of compact zero-dimensional spaces

by

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Abstract. We investigate the algebraic relationship between the ring $\mathrm{Clop}(X)$ of clopen subsets of some compact, zero-dimensional space X and the ring $\mathrm{Clop}(\exp X)$ of clopen subsets of its exponential. It turns out that $\mathrm{Clop}(\exp X)$ is isomorphic to the semigroup ring over the multiplicative semigroup of $\mathrm{Clop}(X)$ with coefficients in the field F_2 with two elements. In the second part of the paper we investigate the class of all Boolean rings that can be written in the form $F_2[H]$ for some semilattice H. The main result is that no complete Boolean ring and no infinite direct product of Boolean rings can belong to that class.

Let X be a topological space. The exponential of X, denoted by $\exp X$, is a new topological space whose points are the non-empty closed subsets of X. To avoid misunderstandings we sometimes write $\hat{F}$ if we consider the closed set F as a point of $\exp X$. The topology of $\exp X$ is given by a base. It consists of all sets

$$V(U_1, \ldots, U_n) = \{\hat{F} \in \exp X\colon\ F \subseteq U_1 \cup \ldots \cup U_n \text{ and } F \cap U_i \neq \emptyset \text{ for all } i\}$$

where $n < \omega$ and $U_1, \ldots, U_n$ are open subsets of X.

The construction of exponential spaces dates back to the early days of topology and is connected with the names of Hausdorff (who did the metric case, cf. [Ha, VIII, 6]) and Vietoris who gave the above definition in [V]. There is a well developed theory of exponentials of metric continua. It is quite comprehensively represented in [N]. Important results for other classes of spaces are due to Marjanović [M] and the Moscow school (cf. [S] and what is quoted there). All facts that are necessary to understand this paper can be found in the exercises (with sufficient hints) of [E]. Our point of departure is the following

Fact. *If X is compact and zero-dimensional, then so is* $\exp X$.

Compactness is [E, 3.12.26]. To prove zero-dimensionality we consider all sets $V(a_1, \ldots, a_n)$ where $n < \omega$ and $a_1, \ldots, a_n$ are clopen in X. It is an easy exercise o show that these sets are clopen in $\exp X$ and form a base of the Vietoris topology use compactness).

The main purpose of this paper is to show how the formation of exp translates into the language of Boolean algebra. To make this precise we have to introduce some further notions. To each topological space X we attach a ring, denoted by $\mathrm{Clop}(X)$. Its elements are the clopen subsets of X. Addition is set-theoretic symmetric difference and multiplication is set-theoretic intersection. $\mathrm{Clop}(X)$ is a Boolean ring, i.e. each element is idempotent. $\emptyset$ is the zero- and X is the unit-element.

Stone duality asserts that any compact zero-dimensional space X is completely determined by the ring $\mathrm{Clop}(X)$. Moreover, each abstract Boolean ring (with unit) is isomorphic to one of the form $\mathrm{Clop}(X)$. The reader is supposed to have a working knowledge of Boolean algebra, especially Stone duality. Everything needed for Sections 1–4 can be found in practically all textbooks on the subject, my favourite being still [H].

In this paper we want to study the algebraic relationship between the rings $\mathrm{Clop}(X)$ and $\mathrm{Clop}(\exp X)$. We shall also consider abstract Boolean rings $R = \langle R; +, \cdot, 0, 1\rangle$. In contrast to what has recently become fashionable our $+$ denotes addition not union. For this we keep the good old $\vee$. So, by definition, $a \vee b = a+b+a\cdot b$. Here are two more definitions: $a-b = a+a\cdot b$ and $a \leqslant b$ stands for $a\cdot b = a$.

When we are dealing with rings of sets, the symbols $\cap, \cup, \setminus$ are used alternatively to $\cdot$, $\vee$, $-$. Let us finally agree that BR will be shorthand for "unitary Boolean ring", and BS for "Boolean space", i.e. compact and zero-dimensional topological space. We tacitly assume throughout that BR's are non-trivial, i.e. $0 \neq 1$, and BS's non-empty.

1. The structure of $\mathrm{Clop}(\exp X)$**.** Every BR can be regarded as a vector space over F_2, the field with two elements. This gives a natural notion of linear independence. It will be convenient to have an independence test in terms of $\vee$ and $\leqslant$ rather than $+$.

LEMMA 1. *Suppose that R is a* BR *and consider a subset $H \subseteq R$ which is closed under multiplication. Then the following are equivalent:*

(1) *$H\setminus\{0\}$ is linearly independent.*

(2) *If $h, h_1, \ldots, h_n \in H$ are such that $h \leqslant h_1 \vee \ldots \vee h_n$, then $h \leqslant h_i$ for some $i = 1, \ldots, n$.*

Proof. For the direction (1)⇒(2) we use the formula

$$h_1 \vee \ldots \vee h_n = \sum_S \prod_{i\in S} h_i$$

where S runs through all non-empty subsets of $\{1, \ldots, n\}$. For $n = 2$ the formula becomes $h_1 \vee h_2 = h_1+h_2+h_1\cdot h_2$, the definition of $\vee$. A straightforward induction establishes the general case. The condition $h \leqslant h_1 \vee \ldots \vee h_n$ can now be expressed as an equation:

$$h = h\cdot(h_1 \vee \ldots \vee h_n) = \sum_S \big(h\cdot \prod_{i\in S} h_i\big).$$

We drop all zeros and pairs of equal terms that emerge on the right-hand side. If $h \neq 0$, then some terms have to remain and they belong, as does h, to the linearly independent set $H\setminus\{0\}$. It follows that $h = h\cdot\prod_{i\in S} h_i$ for some $S \subseteq \{1, \ldots, n\}$. Hence, $h \leqslant h_i$ for each $i \in S \neq \emptyset$. If $h = 0$, then $h \leqslant h_1$.

The proof of (2)⇒(1) also needs a formula. For arbitrary elements $a_1, \ldots, a_n$ of any BR and all $h = 1, \ldots, n$ it holds that

$$a_1 \leqslant (a_1+\ldots+a_n) \vee \bigvee_{j\neq i} a_j\,.$$

There is no problem to check this for the ring F_2. But then it is true in every BR.

Suppose now that condition (2) is satisfied and consider pairwise distinct elements $h_1, \ldots, h_n$ of $H\setminus\{0\}$. We have to show $h_1+\ldots+h_n \neq 0$. Assuming the contrary, the above formula yields $h_i \leqslant \bigvee_{j\neq i} h_j$ for all i. Applying (2) we find some $j \neq i$ such that $h_i \leqslant h_j$. This being true for all i, we can define a function $f\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $f(i) \neq i$ and $h_i \leqslant h_{f(i)}$ for all $i = 1, \ldots, n$.

In the infinite sequence $1, f(1), f(f(1)) = f^2(1), f^3(1), \ldots$ there have to be repetitions, say $f^p(1) = f^{p+q}(1)$ with $q \geqslant 1$. But then

$$h_{f^p(1)} \leqslant h_{f^{p+1}(1)} \leqslant \ldots \leqslant h_{f^{p+q}(1)}$$

implies $h_{f^p(1)} = h_{f^{p+1}(1)}$, where $f^p(1) \neq f^{p+1}(1)$.

This is the desired contradiction. For, the h_i were assumed to be distinct.

THEOREM 1. *Let X be a* BS *and consider the mapping*

$$V\colon a \mapsto V(a) = \{\hat{F} \in \exp X\colon F \subseteq a\}$$

of $\mathrm{Clop}(X)$ *into* $\mathrm{Clop}(\exp X)$*. Let H denote its image. Then the following conditions are satisfied:*

(1) *V is multiplicative and injective.*

(2) *H generates* $\mathrm{Clop}(\exp X)$.

(3) *$H\setminus\{0\}$ is linearly independent.*

Proof. (1) Multiplicativity means $V(a\cap b) = V(a)\cap V(b)$ and is obvious. If $a \neq b$, then this is witnessed by some point, say x. The closed set $\{x\}$ witnesses $V(a) \neq V(b)$.

(2) We use the folklore fact that (for BS's) a subset of Clop generates the whole ring iff it separates the points of the space. If $F \neq G$ are closed subsets of X, then there is some point x which belongs to F, say, but not to G. Choose an element a in $\mathrm{Clop}(X)$ such that $G \subseteq a$ and $x \notin a$. Then, clearly, $\hat{G} \in V(a)$, but $\hat{F} \notin V(a)$. So, $V(a) \in H$ separates $\hat{F}$ and $\hat{G}$.

(3) We use the lemma. Suppose $\emptyset \neq V(a) \subseteq V(a_1) \cup \ldots \cup V(a_n)$, but $V(a) \not\subseteq V(a_i)$ for all i. Then, clearly, $a \not\subseteq a_i$ for all i. Pick $x_i \in a\setminus a_i$ and put $F = \{x_1, \ldots, x_n\}$. Then $\hat{F}$ belongs to $V(a)$, but to none of the $V(a_i)$, contradiction.

2. Boolean semigroup rings. In this section we have a closer look at the situation described in Theorem 1 and give it a name. Suppose $R = \langle R; +, \cdot, 0, 1\rangle$ is a BR. If we forget the addition (but keep 0), we obtain the semigroup $R^* = \langle R; \cdot, 0, 1\rangle$.

Let H be a subsemigroup of R^* which generates R and contains 0 and 1. If $H\setminus\{0\}$ is linearly independent, then each non-zero element of R has a unique (up to the order of the terms) representation as a sum (= linear combination over F_2) $h_1+\ldots+h_n$ of pairwise distinct elements of $H\setminus\{0\}$.

The same situation can be manufactured from outside. The following kind of construction being very popular in algebra (just think of group rings) the reader will allow us to be a bit sketchy in its description. He should, however, pay attention to the somewhat special role of 0 (and not worry about the sign $\oplus$. The reason we prefer it to the usual + will become clear later).

Let a commutative and idempotent semigroup (otherwise known as semilattice) with 0 and 1 be given and call it H. Take the set of all formal sums $h_1\oplus\ldots\oplus h_n$ of pairwise distinct elements of $H\setminus\{0\}$. Identify all such sums that differ only in the order of their terms. Add 0 and define multiplication and addition ($\oplus$) in the natural way (with $h\oplus h = 0$). What comes out is a BR that will be denoted by $F_2[H]$ and called the Boolean semigroup ring over H. For the rest of this paper the abbreviation SG will always mean "commutative and idempotent semigroup with 0 and 1". As with BR's, all SG's under consideration will be tacitly assumed non-trivial, i.e. $0\neq 1$.

Remarks. (1) There is no need to restrict the coefficients of the formal linear combinations to F_2. Starting from an arbitrary BR A and a SG H we obtain $A[H]$ by considering all formal linear combinations $a_1h_1\oplus\ldots\oplus a_nh_n$ with $a_i\in A$ and $h_i\in H\setminus\{0\}$. It turns out, however, that $A[H]$ is isomorphic with the free (= tensor) product of A and $F_2[H]$. Therefore, we can dispense with the more general construction.

(2) Non-isomorphic SG's may well lead to isomorphic rings. If, for example, $|H| = |K| < \omega$, then $F_2[H]\cong F_2[K]$.

In what follows we have to know how homomorphisms behave with respect to the formation of semigroup rings. Statement (1) of the following lemma could be used to define $F_2[H]$ in the spirit of category theory.

LEMMA 2. (1) *Let H be a* SG *and R a* BR. *Each semigroup homomorphism $\alpha\colon H\to R^*$ has a unique extension to a ring homomorphism $\bar\alpha\colon F_2[H]\to R$. ($H$ is considered as a subsemigroup of $F_2[H]^*$.)*

(2) *If $\alpha\colon H\to K$ is a homomorphism of* SG's, *then there is a unique homomorphism $F_2[\alpha]\colon F_2[H]\to F_2[K]$ extending α.*

(3) *$F_2[\ ]$ is a covariant functor from the category of* SG's *into the category of* BR's.

Proof. (1) We leave it to the reader to check that the formula
$$\bar\alpha(h_1\oplus\ldots\oplus h_n) = \alpha(h_1)+\ldots+\alpha(h_n)$$
defines the required homomorphism.

(2) Consider α as a mapping into $F_2[K]^*$ and apply (1).

(3) The identity $F_2[\alpha\circ\beta] = F_2[\alpha]\bullet F_2[\beta]$ is an easy consequence of the uniqueness in (2).

Using the new notation we can express Theorem 1 in a more concise way.

COROLLARY 1. *If X is a* BS, *then* $\mathrm{Clop}(\exp X)\cong F_2[\mathrm{Clop}(X)^*]$.

Any BR that can be written in the form $F_2[H]$ will be called a semigroup ring (SGR, for short). The question naturally arises which BR's are SGR's. Using Lemma 1 it is easy to see that any totally ordered subset of a BR is linearly independent. It follows that all BR's with ordered bases, in particular all countable BR's, are SGR's.

Applied to the free SG on $\varkappa$ generators, $F_2[\ ]$ yields the free BR on $\varkappa$ generators. More generally, one can show that the class of SGR's is closed under free products.

On the other hand, there is no obvious example of a non-semigroup ring. We shall produce a class of them in the next section.

3. No SGR can be complete. It is well known that any Hausdorff space can be identified with a closed subspace of its exponential.

The mapping $e\colon X\to\exp X$ defined by $e(x) = \{x\}$ is a canonical embedding. Let us find its Stone dual. Generally, the dual of a continuous mapping $f\colon X\to Y$ is the homomorphism $\tilde f\colon \mathrm{Clop}(Y)\to\mathrm{Clop}(X)$ defined by the formula $\tilde f(a) = f^{-1}(a)$. Applied to our situation we find for each $a\in\mathrm{Clop}(X)$
$$\tilde e(V(a)) = e^{-1}(V(a)) = \{x\in X\colon \{x\}\in V(a)\} = a\,.$$
From Theorem 1 we know that every element W of $\mathrm{Clop}(\exp X)$ can be written as $W = V(a_1)\Delta\ldots\Delta V(a_n)$ with $a_i\in\mathrm{Clop}(X)$ and Δ denoting set-theoretic symmetric difference. $\tilde e$ being a homomorphism we must have $\tilde e(W) = a_1\Delta\ldots\Delta a_n$.

These considerations lead to the following abstract definition. Let R be an arbitrary BR. The canonical homomorphism $\varphi\colon F_2[R^*]\to R$ is the mapping defined by the formula
$$\varphi(a_1\oplus\ldots\oplus a_n) = a_1+\ldots+a_n\,.$$
Note that φ is the unique extension of the identical mapping $R^*\to R^*$ discussed in Lemma 2 (2). For $R=\mathrm{Clop}(X)$, φ is the same as $\tilde e$ up to the identification of $F_2[R^*]$ with $\mathrm{Clop}(\exp X)$.

Remark. Suppose that $\alpha\colon R\to S$ is a ring homomorphism. It gives rise to the following diagram in which φ and ψ denote the respective canonical homomorphisms just defined.

$$\begin{tikzcd}
F_2[R^*] \arrow[r, "\varphi"] \arrow[d, "F_2[\alpha]"'] & R \arrow[d, "\alpha"] \\
F_2[S^*] \arrow[r, "\psi"] & S
\end{tikzcd}$$

A straightforward verification shows that this diagram is commutative.

Next we discuss under what conditions the canonical homomorphism admits a right inverse. First a special case

LEMMA 3. *If* $R = F_2[H]$ *is a* SGR, *then the canonical homomorphism* $\varphi\colon F_2[R^*] \to R$ *has a right inverse.*

Proof. Denote the identical embedding $H \to R^*$ by ν. By Lemma 2 (2), $F_2[\nu]$ is an embedding of $F_2[H] = R$ into $F_2[R^*]$. Writing $+$ for the addition in R and $\oplus$ for the one in $F_2[R^*]$ the definition of $F_2[\nu]$ takes the form

$$F_2[\nu](h_1+\ldots+h_n) = h_1 \oplus \ldots \oplus h_n$$

whereas for the canonical homomorphism $\varphi\colon F_2[R^*] \to R$ it holds that

$$\varphi(h_1 \oplus \ldots \oplus h_n) = h_1+\ldots+h_n\,.$$

It follows that $\varphi \circ F_2[\nu]$ is the identity on R.

Recall that a BR R is called a *retract* of a BR S iff there are homomorphisms $R \xrightarrow{\alpha} S \xrightarrow{\beta} R$ such that $\beta \circ \alpha = \mathrm{id}_R$.

THEOREM 2. *For any* BR R *the following are equivalent*:

(1) *The canonical homomorphism* $\varphi\colon F_2[R^*] \to R$ *has a right inverse.*

(2) R *is a retract of* $F_2[R^*]$.

(3) R *is a retract of some* SGR.

Proof. The implications (1)$\Rightarrow$(2)$\Rightarrow$(3) are obvious.

To prove (3)$\Rightarrow$(1) we assume that R is a retract of some SGR S, say $R \xrightarrow{\alpha} S \xrightarrow{\beta} R$ with $\beta \circ \alpha = \mathrm{id}_R$.

Consider the following diagram in which ψ denotes the canonical homomorphism for S.

$$\begin{tikzcd}
F_2[R^*] \arrow[r, "\varphi"] \arrow[d, shift right, "{F_2[\alpha]}"'] & R \arrow[d, shift right, "\alpha"'] \\
F_2[S^*] \arrow[r, "\psi"] \arrow[u, shift right, "{F_2[\beta]}"'] & S \arrow[u, shift right, "\beta"']
\end{tikzcd}$$

By Lemma 3 there is a right inverse for ψ, i.e. some homomorphism $\gamma\colon S \to F_2[S^*]$ such that $\psi \circ \gamma = \mathrm{id}_S$. We are going to show that $\varphi \circ (F_2[\beta] \circ \gamma \circ \alpha) = \mathrm{id}_R$. Choose an arbitrary $a \in R$. Then $b = \gamma(\alpha(a))$ belongs to $F_2[S^*]$. By the remark preceeding Lemma 3 we have

$$\beta(\psi(b)) = \varphi(F_2[\beta](b))\,,$$

in other words

$$\varphi \circ F_2[\beta] \circ \gamma \circ \alpha(a) = \beta \circ \psi \circ \gamma \circ \alpha(a)\,.$$

From $\psi \circ \gamma = \mathrm{id}_S$ and $\beta \circ \alpha = \mathrm{id}_R$ it follows that the right-hand side reduces to a, as was to be shown.

Remark. There is no obvious reason why a retract of a SGR should be a SGR again. On the other hand, I have no counterexample even for the simplest case, a principal ideal of a SGR. It would be interesting to know if all retracts of free BR's, otherwise known as projective Boolean algebras, are SGR's.

COROLLARY 2. *If* X *is a* BS *such that* $\mathrm{Clop}(X)$ *is a* SGR, *then* X *is a retract (in the topological sense) of* $\exp X$.

Proof. Let $e\colon X \to \exp X$ be the canonical embedding introduced at the beginning of this section. The theorem yields a right inverse for $\tilde{e}$. Its dual is a left inverse for e.

The question of X being a retract of $\exp X$ has been discussed before. In [S, th. 4. 8] Shchepin gives the example $\exp_3 D^{\omega_2}$ of a BS which is not a retract of its exponential. (Here D denotes the discrete two-point space. The meaning of $\exp_3$ will be explained at the beginning of Section 4.)

For this example Clop is a subring of the free ring on ω_2 generators. It follows that subrings of SGR's need not be SGR's again. As any BR is a homomorphic image of a free one, the class of SGR's cannot be closed under homomorphic images either.

In order to get more examples of non-semigroup rings we prove the following

PROPOSITION 1. *If an infinite* BS *is a retract of its exponential, then it contains a convergent sequence of pairwise distinct points.*

Proof. Call the space in question X. We begin by mentioning an easy case that does not depend on X being a retract of $\exp X$. There may be an infinite clopen subset of X that contains only a finite number of accumulation points. Then it is easy to find a convergent sequence of distinct isolated points. We leave the details to the reader and turn to the harder case in which there is no such set.

Let f denote the assumed retraction, i.e. $f\colon \exp X \to X$ is continuous and $f(\{x\}) = x$ for all $x \in X$.

The envisaged sequence will be constructed by induction. Suppose we already have clopen subsets $a_0 \supseteq a_1 \supseteq \ldots \supseteq a_n$ of X such that all a_i are infinite and all $f(\hat{a}_i)$ are pairwise distinct. (Remember that by writing $\hat{a}$ we stress that a is considered as a point of $\exp X$.) We have to construct a_{n+1}. The set $a_n \setminus \{f(\hat{a}_0), \ldots, f(\hat{a}_n)\}$ is infinite. As we are in the hard case, it contains accumulation points of X. Therefore, we find an infinite clopen subset b of a_n that contains no $f(\hat{a}_i)$ for $i = 0, \ldots, n$. The set $W = f^{-1}(b) \cap V(b) \subseteq \exp X$ is infinite, because it contains all $\{x\}$ with $x \in b$.

Being clopen in $\exp X$, W is a finite union of basic clopen sets $V(c_1, \ldots, c_k)$. At least one of the c's occuring in this union has to be infinite. For, otherwise, each $V(c_1, \ldots, c_k)$ would be finite and so would be their union W. We choose some $V(c_1, \ldots, c_k) \subseteq W$ such that $a_{n+1} = c_1 \cup \ldots \cup c_k$ becomes infinite.

From $\hat{a}_{n+1} \in V(c_1, \ldots, c_k) \subseteq W \subseteq V(b)$ we have $a_{n+1} \subseteq b \subseteq a_n$ and $f(\hat{a}_{n+1}) \in f(W) \subseteq b$. The latter implies that $f(\hat{a}_{n+1})$ is distinct from all $f(\hat{a}_i)$ with $i = 0, \ldots, n$. This ends the inductive construction.

Next we show that the sequence $\hat{a}_n$ is convergent in $\exp X$. By compactness $\emptyset \neq F = \bigcap_{n<\omega} a_n$. Suppose that $V(U_1, \ldots, U_k)$ is a basic open neighbourhood of F. This means that $F \cap U_i \neq \emptyset$ for all i and $F \subseteq U_1 \cup \ldots \cup U_k$. Again by compactness, $a_n \subseteq U_1 \cup \ldots \cup U_k$ for all sufficiently large n, say $n > n_0$. Moreover, $a_n \cap U_i \supseteq F \cap U_i \neq \emptyset$ for all n and i. We conclude that $\hat{a}_n \in V(U_1, \ldots, U_k)$ for all $n > n_0$.

This proves $\lim \hat{a}_n = \hat{F}$. The continuity of f finally implies that $f(\hat{a}_n)$ converges to $f(\hat{F})$.

Our argument actually proved a bit more than promised, namely

COROLLARY 3. *If the* BS X *is a retract of* $\exp X$, *then each infinite clopen subset of* X *contains non-trivial convergent sequences.*

Proof. By construction, $f(\hat{a}_{n+1}) \in a_n \subseteq a_0$. And we can start with any infinite clopen a_0.

Combining the proposition with Corollary 2 we obtain

COROLLARY 4. *Suppose that the infinite* BS X *does not contain non-trivial convergent sequences. Then* $\mathrm{Clop}(X)$ *is not a semigroup ring. In particular, no homomorphic image of a complete Boolean algebra is a* SGR.

Using the full strength of Theorem 2 we find

COROLLARY 5. *No infinite direct product of* BR's *is a* SGR.

Proof. Let I be an infinite index set and $(R_i)_{i\in I}$ a family of BR's (by tacit assumption non-trivial). The ring $\{0, 1\}$ is a retract of each R_i. Consequently, $S = \{0, 1\}^I$ is a retract of $\prod_{i\in I} R_i$. If the latter were a SGR, then S were a retract of $F_2[S^*]$, by Theorem 2, and the Stone space of S would contain non-trivial convergent sequences, by Proposition 1. On the other hand, S is isomorphic to the ring of all subsets of I and, therefore, complete. Hence its Stone space does not contain non-trivial converging sequences.

4. An embedding and its applications. Again we start with well-known topological considerations. Let a BS X and a natural number $1 \leqslant n \leqslant \omega$ be given. The formula $(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}$ defines a continuous mapping e_n: $X^n \to \exp X$. Some authors call the image of X^n under e_n the nth hypersymmetric power of X and denote it by $\exp_n X$ (e.g. [S]). The following two facts are easily established (cf. [E, 2.7.20]).

(1) $\exp_n X \subseteq \exp_{n+1} X$

and

(2) $\bigcup_{n<\omega} \exp_n X$ is a dense subset of $\exp X$.

The dual homomorphisms $\tilde{e}_n$: $\mathrm{Clop}(\exp X) \to \mathrm{Clop}(X^n)$ are determined by their values on the generators and these are easy to calculate:

$$\tilde{e}(V(a)) = \{(x_1, \ldots, x_n): \{x_1, \ldots, x_n\} \subseteq a\} = \underbrace{a \times \ldots \times a}_{(n \text{ times})}.$$

The two facts from above dualize as follows.

($\tilde{1}$) $\ker(\tilde{e}_n) \supseteq \ker(\tilde{e}_{n+1})$

and

($\tilde{2}$) $\bigcup_{n<\omega} \ker(\tilde{e}_n) = \{\emptyset\}$.

Remembering that $\mathrm{Clop}(X^n)$ is isomorphic to the n-fold free product of $\mathrm{Clop}(X)$ with itself we are led to the following abstract definition.

Suppose that R is a BR and put $R^{(n)} = R * \ldots * R$ (n-fold free product). For $a \in R$ let $a^{(n)}$ denote $a * \ldots * a \in R^{(n)}$.

The mapping $a \mapsto a^{(n)}$ is, obviously, multiplicative. By Lemma 2 (1) it has a unique extension to a ring homomorphism $F_2[R^*] \to R^{(n)}$ which we denote by φ_n and call the nth canonical homomorphism. φ_1 is the canonical homomorphism considered in the previous section. Note that φ_n is no longer surjective for $n > 1$. Up to the identifications of $\mathrm{Clop}(\exp X)$ with $F_2[R^*]$ and $\mathrm{Clop}(X^n)$ with $R^{(n)}$, φ_n is the same as $\tilde{e}_n$. Together with the fundamental fact that $R = \mathrm{Clop}(X)$ is the general case, ($\tilde{1}$) and ($\tilde{2}$) yield

$$\ker(\varphi_n) \supseteq \ker(\varphi_{n+1}) \quad \text{and} \quad \bigcap_{n<\omega} \ker(\varphi_n) = \{0\}.$$

The last assertion implies that the formula $a \mapsto (\varphi_n(a))_{n<\omega}$ defines an embedding of $F_2[R^*]$ into $\prod_{n<\omega} R^{(n)}$.

We now use this embedding to explain a somewhat surprising phenomenon that was hitherto proved by an ad hoc construction due to T. Cramer. Recall that a BR is called superatomic iff it does not embed the free BR on ω generators. The topological dual notion is that of a scattered space.

PROPOSITION 2 ([C]). *For any cardinal number* $\varkappa$ *there is a countable product of superatomic* BR's *that embeds the free* BR *on* $\varkappa$ *generators.*

Proof. The argument becomes clearer using topological language. Let $\varkappa$ which we assume infinite be given and choose a compact scattered space X with $\varkappa$ pairwise disjoint clopen subsets $\{a_\alpha: \alpha < \varkappa\}$. The one-point compactification of the discrete space of power $\varkappa$ would be the simplest example of this kind.

Put $W_\alpha = \{F \in \exp X: F \cap a_\alpha \neq \emptyset\} = \exp X \setminus V(a_\alpha) \in \mathrm{Clop}(\exp X)$.

CLAIM. *The elements* W_α *are independent in* $\mathrm{Clop}(\exp X)$.

Indeed, let $\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m < \varkappa$ be pairwise distinct. We have to show that

$$[W_{\beta_1} \cap \ldots \cap W_{\beta_n}] \setminus [W_{\gamma_1} \cup \ldots \cup W_{\gamma_m}]$$

is non-empty. As $\varkappa$ is infinite, there is no harm to assume $n > 0$. Take $x_1 \in a_{\beta_1}, \ldots, x_n \in a_{\beta_n}$ and put $F = \{x_1, \ldots, x_n\}$. Then $F \in W_{\beta_i}$ for all $i = 1, \ldots, n$ and, by disjointness, $F \notin W_{\gamma_j}$ for $j = 1, \ldots, m$.

It follows that the free ring on $\varkappa$ generators embeds into $\mathrm{Clop}(\exp X)$ and, therefore, into $\prod_{n<\varkappa} \mathrm{Clop}(X^n)$. It remains to refer to the folklore fact that finite products of scattered spaces remain scattered.

Our second application of the embedding $F_2[R^*] \to \prod_{n<\omega} R^{(n)}$ uses it the other way round. Recall that the depth of a BR R is the cardinal $\sup\{|A|: A \subseteq R$ is well ordered by $\leqslant\}$.

PROPOSITION 3. $F_2[R^*]$ *and* R *have same depth.*

Proof. Using that R^* is a subsemigroup of $F_2[R^*]^*$ and the abovementioned embedding we obtain

$$\operatorname{depth}(R) \leqslant \operatorname{depth}(F_2[R^*]) \leqslant \operatorname{depth}\Big(\prod_{n<\omega} R^{(n)}\Big).$$

It remains to apply two general facts concerning the behaviour of depth (cf. [MM]):

$$\operatorname{depth}\Big(\prod_{n<\omega} S_n\Big) = \sup\{\operatorname{depth}(S_n): n < \omega\}$$

and

$$\operatorname{depth}(S * T) = \max\{\operatorname{depth}(S), \operatorname{depth}(T)\}.$$

Remark. The length of R is defined as the cardinal

$$\sup\{|A|: A \subseteq R \text{ is totally ordered by } \leqslant\}.$$

If we try to evaluate the length of $F_2[R^*]$ by the same method that worked for depth, we run into difficulties. For, length$(\prod_{n<\omega} S_n)$ is in general, not equal to $\sup\{\text{length}(S_n): n<\omega\}$ (nor to any other reasonable function, cf. [MM]).

It is, however, true that R and $F_2[R^*]$ have the same length. The proof is somewhat lengthy and will be given elsewhere.

5. Non-unitary rings and locally compact spaces. In Section 2 we defined $F_2[H]$ for semigroups with unit-element. Looking back the reader will convince himself that this assumption is not necessary for the construction to yield a Boolean ring (in general without unit). Moreover, all purely algebraic results that were proved for unitary Boolean rings and semigroups (in particular Lemmas 1 and 2, and Theorem 2) hold true in the non-unitary setting. The proofs remain the same; there is no mention of 1 in them. Stone duality connects non-unitary Boolean rings with non-compact, locally compact zero-dimensional spaces. (Strangely enough, this part of the theory is not reflected in any of the popular textbooks. Therefore, I refer the reader to the classical paper [St] by Stone.)

Let X be such a space. The corresponding ring is the subring of $\text{Clop}(X)$ that consists of all compact and open subsets of X. Let us denote that non-unitary Boolean ring by $\text{Coop}(X)$. In order to obtain a result analogous to Theorem 1 we have to consider not the space $\exp X$, but its subspace $\text{comp}\, X$ consisting of all non-empty compact subsets of X. It can be shown that $\text{comp}\, X$ is again locally compact ([E, 3.12.26]). To prove that it is zero-dimensional too, one establishes that the family of all $V(a_1, \ldots, a_n)$, with $n<\omega$ and $a_1, \ldots, a_n$ compact and clopen in X, forms a clopen base of $\text{comp}\, X$. With the appropriate change in notation and the occasional addition of the word "compact" the proof of Theorem 1 can now be repeated. In analogy to Corollary 1 the result can be formulated as follows.

COROLLARY 6. *If X is a locally compact, zero-dimensional space, then* $\text{Coop}(\text{comp}\, X) \cong F_2[\text{Coop}(X)^*]$.

Added in proof (February 1990). As I have learned only now Boolean semigroup rings have been studied before by Elliott Evans in *The Boolean ring universal over a meet semilattice*, J. Austral. Math. Soc. 23 (1977), 402–415.

Other relevant information on SGR's, though from a completely different point of view, is contained in the lecture note *Pontryagin duality of compact 0-dimensional semilattices and its applications* by K. H. Hofmann, M. Mislove and A. Stralka. When writing the present paper I was not aware of that source either.

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