[11] M. Magidor, *Combinatorial characterization of supercompact cardinals*, Proc. Amer. Math. Soc. 42 (1974), 279–285.
[12] R. Solovay, W. Reinhardt, and A. Kanamori, *Strong axioms of infinity and elementary embeddings*, Ann. Math. Logic 13 (1978), 73–116.
[13] W. Zwicker, $P_\kappa(\lambda)$ *Combinatorics* I: *Stationary coding sets rationalize the club filter*, Contemp. Math. 31 (1984), 243–259.

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Functions provably total in $I^-\Sigma_1$

by

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Abstract. We estimate the rapidity of the growth of recursive functions which are provably total in a finite fragment of Σ_1 parameter-free induction subject to the size of the fragment.

The aim of this paper is to bound the rapidity of the growth of recursive (Σ_1 definable) functions which are provably total in $I^-\Sigma_1$ (induction for parameter free Σ_1 formulas). We show that if in the proof of the totality of a recursive function f from N to N Σ_1 induction is applied n times then the function can be bounded by the $n+1$'s function in the Wainer hierarchy (see [W]).

The result is proved by means of a proof-theoretic analysis of proofs of sentences of the form $(\forall t)_{\geqslant 0}\varphi(t)$ in $I^-\Sigma_1$ (an analogous analysis for $\exists_1$ formulas and $I^-\exists_1$ can be found in [A]). We consider here Σ_1 formulas φ without parameters.

Here PA^- denotes the theory of discretely ordered rings. If φ is a formula then $\mathrm{Ind}\,\varphi$ denotes the following sentence:

$$\mathrm{PA}^- \,\&\, [\varphi(0) \,\&\, \big((\forall t)_{\geqslant 0}\varphi(t) \Rightarrow \varphi(t+1)\big) \Rightarrow (\forall t)_{\geqslant 0}\varphi(t)]\,.$$

To simplify the notation we will assume that for every formula of the form $\varphi(y, \vec{x})$ the sentence $\varphi(-1, 0, \ldots, 0)$ is true. Formally, this can be assumed since we can replace φ be the formula φ^* defined as $\big(y \geqslant 0 \,\&\, \varphi(y, x)\big) \vee y < 0$. Then φ is equivalent to φ^* for all non-negative y's we are interested in. Without causing confusion we identify φ and φ^*.

DEFINITION 1. Let the formulas $\varphi_1, \ldots, \varphi_n$ be of the form

$$\varphi_i(t) = (\exists \vec{s})\varphi_i'(t, \vec{s}) \quad \text{where } \varphi_i' \in \Delta_0,\ i = 1, \ldots, n.$$

We assume that the quantifiers in the formulas φ_i' are bounded by the free variable or by one of the variables of $\vec{s}$. Let $M \models \mathrm{PA}^-$, $v_1, \ldots, v_m \in M$. Assume that we have a fixed enumeration of polynomials. Let $K \in N$. A set $H \subseteq M$ is called a *K-closure of* $\{v_1, \ldots, v_m\}$ *with respect to* $\{\varphi_1, \ldots, \varphi_n\}$ if there exist sets $H_0, \ldots, H_K$ such that

1. $H = H_0 \cup H_1 \cup \ldots \cup H_K$ and $\{v_1, \ldots, v_m\} = H_0$.
2. If $x \in H_j$ for a certain $j < K$ then for every $i \in \{1, \ldots, n\}$ there is an $\vec{y} \in H_{j+1}$ such that $M \models \varphi_i'(x, \vec{y})$.

3. If $x \in H_j$, $j < K$, $y < x$ then $y \in H_{j+1}$.

4. If $\vec{x} \in H_j$, $j < K$, p is a polynomial with the number smaller than K then $p(\vec{x}) \in H_{j+1}$.

5. H is a minimal set with the properties (1)–(4).

LEMMA. *Let $\varphi_1, \ldots, \varphi_n$, φ be as above. Then the following conditions are equivalent*:

1. $\mathrm{Ind}\,\varphi_1 \,\&\, \ldots \,\&\, \mathrm{Ind}\,\varphi_n \vdash (\forall t)_{\geq 0}\,\varphi(t)$.

2. *There is a $K \in N$ such that for any sets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ and $\{j_1, \ldots, j_l\} = \{1, \ldots, n\} - \{i_1, \ldots, i_m\}$ the following is true in every model of* PA^-:

$(\forall t)_{\geq 0} (\forall t_{j_1} \ldots t_{j_l})_{\geq -1} (\forall \vec{s}_{j_1} \ldots \vec{s}_{j_l}) \big((\varphi'_{j_1}(t_{j_1}, \vec{s}_{j_1}) \,\&\, \ldots \,\&\, \varphi'_{j_l}(t_{j_l}, \vec{s}_{j_l})\big) \Rightarrow$ *in every K-closure of the set $\{t, t_{j_1}, \ldots, t_{j_l}, \vec{s}_{j_1}, \ldots, \vec{s}_{j_l}\}$ with respect to $\{\varphi_{i_1}, \ldots, \varphi_{i_m}\}$ there exists an $\vec{s}$ such that*

$$\varphi_{j_1}(t_{j_1}+1, \vec{s}) \vee \ldots \vee \varphi_{j_l}(t_{j_l}+1, \vec{s}) \vee \varphi(t, \vec{s})\big).$$

Proof. (1)$\Rightarrow$(2). Suppose that (1) and $\neg$(2) holds. Then there is a subset $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ such that (2) does not hold for an infinite set X of K's. Let $\{j_1, \ldots, j_l\} = \{1, \ldots, n\} - \{i_1, \ldots, i_m\}$. Add to the language of arithmetic the constants: $\underline{u}, \underline{u}_{j_1}, \ldots, \underline{u}_{j_l}, \underline{\vec{s}}_{j_1}, \ldots, \underline{\vec{s}}_{j_l}$ and predicate symbols H, H_i. Then for $K \in X$ the following theories are consistent:

$T_K = \mathrm{PA}^- \cup \{\varphi'_{j_1}(\underline{u}_{j_1}, \underline{\vec{s}}_{j_1})\} \cup \ldots \cup \{\varphi'_{j_l}(\underline{u}_{j_l}, \underline{\vec{s}}_{j_l})\} \cup \{H$ is a K-closure of $\{\underline{u}, \underline{u}_{j_1}, \ldots, \underline{u}_{j_l}, \underline{\vec{s}}_{j_1}, \ldots, \underline{\vec{s}}_{j_l}\}$ w.r.t. $\{\varphi_{i_1} \ldots \varphi_{i_m}\}$ such that for every $\vec{y} \in H$, $\neg\varphi'_{j_1}(\underline{u}_{j_1}+1, \vec{y})$ & ... & $\neg\varphi'_{j_l}(\underline{u}_{j_l}+1, \vec{y})$ & $\neg\varphi(\underline{u}, \vec{y})\} \cup$
$\cup \{\underline{u} \geq 0, \underline{u}_{j_1} - \underline{u}_{j_l} \geq -1\}$.

Note that those theories can be formulated in our language. There are formulas $\theta_i(\vec{x}, H_i)$ expressing the fact that H_i is an i-closure of $\vec{x}$ w.r.t. $\{\varphi_{i_1}, \ldots, \varphi_{i_m}\}$. It follows that for $K \in X$ the following theories are consistent:

$$\begin{aligned} T_K = \mathrm{PA}^- &\cup \{\varphi'_{j_1}(\underline{u}_{j_1}, \underline{\vec{s}}_{j_1})\} \cup \ldots \cup \{\varphi'_{j_l}(\underline{u}_{j_l}, \underline{\vec{s}}_{j_l})\} \\ &\cup \{\theta_0(\underline{u}, \underline{u}_{j_1}, \ldots, \underline{u}_{j_l}, \underline{\vec{s}}_{j_1}, \ldots, \underline{\vec{s}}_{j_l}, H_0), \ldots, \theta_K(\underline{u}, \underline{u}_{j_1}, \ldots, \underline{u}_{j_l}, \underline{\vec{s}}_{j_1}, \ldots, \underline{\vec{s}}_{j_l}, H_K)\} \\ &\cup \{(\forall \vec{y})_{H_0} \neg\varphi'_{j_1}(\underline{u}_{j_1}+1, \vec{y}) \,\&\, \ldots \,\&\, \neg\varphi'_{j_l}(\underline{u}_{j_l}+1, \vec{y}) \,\&\, \neg\varphi(\underline{u}, \vec{y}), \ldots \\ &\ldots, (\forall \vec{y})_{H_K} \neg\varphi'_{j_1}(\underline{u}_{j_1}+1, \vec{y}) \,\&\, \ldots \,\&\, \neg\varphi'_{j_l}(\underline{u}_{j_l}+1, \vec{y}) \,\&\, \neg\varphi(\underline{u}, \vec{y})\} \\ &\cup \{H_0 \subseteq H_1 \subseteq \ldots \subseteq H_K\} \cup \{\underline{u} \geq 0, \underline{u}_{j_1} \geq -1, \ldots, \underline{u}_{j_l} \geq -1\}. \end{aligned}$$

Then the theory $T = \bigcup_K T_K$ is consistent, by compactness. Let M be a model of T and let $M' \subseteq M$ be the submodel of M whose universe consists of elements of M satisfying H_i in M. Then M' is an initial segment of M (it may happen that it is the whole M). Let u, u_j, $\vec{s}_j$ interpret $\underline{u}$, $\underline{u}_j$, $\underline{\vec{s}}_j$ respectively in M. Then $M' \models \mathrm{PA}^-$ since it is closed under polynomials in M.

$M' \models \varphi_j(u_j)$ for $j \in \{j_1, \ldots, j_l\}$ since $M \models \varphi'_j(u_j, \vec{s}_j)$, $\vec{s}_j \in M'$ and φ'_j is absolute w.r.t. M' being Δ_0.

$M' \models \neg\varphi_j(u_j+1)$ for $j \in \{j_1, \ldots, j_l\}$-immediate.

$M' \models \neg\varphi(u)$. Finally, $M' \models (\forall t)_{\geq 0}\,\varphi_i(t)$ for $i \in \{i_1, \ldots, i_m\}$. It follows that $M' \models \mathrm{Ind}\,\varphi_1 \,\&\, \ldots \,\&\, \mathrm{Ind}\,\varphi_n \,\&\, \neg\varphi(u)$ which contradicts (1).

(2)$\Rightarrow$(1). Let M be a model of $\mathrm{Ind}\,\varphi_1 \,\&\, \ldots \,\&\, \mathrm{Ind}\,\varphi_n$. Assume that $\{i_1, \ldots, i_m\}$ is the set of those i's from $\{1, \ldots, n\}$ for which $M \models (\forall t)_{\geq 0}\,\varphi_i(t)$. Let $\{j_1, \ldots, j_l\} = \{1, \ldots, n\} - \{i_1, \ldots, i_m\}$. Then there are $u_{j_1}, \ldots, u_{j_l}$ in M such that

$$M \models \varphi_j(u_j) \,\&\, \neg\varphi_j(u_j+1) \quad \text{for } j \in \{j_1, \ldots, j_l\}.$$

Let $u \in M$ be arbitrary $u \geq 0$. Since $M \models (\forall t)_{\geq 0}\,\varphi_i(t)$ for $i \in \{i_1, \ldots, i_m\}$, there exists a K-closure of $\{u, u_{j_1}, \ldots, u_{j_l}, \vec{s}_{j_1}, \ldots, \vec{s}_{j_l}\}$ w.r.t. $\{\varphi_{i_1}, \ldots, \varphi_{i_m}\}$ in M, where $\vec{s}_j$ are such that $M \models \varphi'_j(u_j, \vec{s}_j)$ for $j \in \{j_1, \ldots, j_l\}$ and K is taken from (2). In this closure there exists an $\vec{s}$ such that $M \models \varphi(u, \vec{s})$, by (2). Since u was arbitrary, $M \models (\forall t)_{\geq 0}\,\varphi(t)$. ■

Now we are going to formulate the main theorem. We define a hierarchy of functions based on the hierarchy defined by Grzegorczyk [G] and Wainer [W].

DEFINITION 2. Let F_i be functions from N to N defined as:

$$F_0(t) = 2^t, \quad F_{n+1}(t) = F_n^{t+1}(t)$$

THEOREM. *Let $n \in N$. Let $\varphi_1, \ldots, \varphi_n$, φ be as before. Define the function $f\colon N \to N$ as follows*:

$$f_\varphi(t) = w \Leftrightarrow (\exists \vec{s})(\varphi(t, \vec{s}) \,\&\, \vec{s} \leqslant w) \,\&\, (\forall \vec{w})_{<w} \neg\varphi(t, \vec{w}').$$

Assume that $\mathrm{Ind}\,\varphi_1 \,\&\, \ldots \,\&\, \mathrm{Ind}\,\varphi_n \vdash (\forall t)_{\geq 0}\,\varphi(t)$. *Then $f_\varphi(t) < F_n(t)$ for almost all $t \in N$.*

Proof. We need the following definition:

DEFINITION 3. Let $L, K \in N$. Let $a_1, \ldots, a_n \in N$. We say that a subset H of N is an *Lth K-closure of $\{a_1, \ldots, a_n\}$ with respect to $\varnothing$* if $H = H_1 \cup \ldots \cup H_L$ where H_1 is a K-closure of $\{a_1, \ldots, a_n\}$ w.r.t. $\varnothing$ and H_i is a K-closure of H_{i-1} w.r.t. $\varnothing$ for $i = 2, \ldots, n$.

Proof. From the lemma it follows that in consecutive K-closures of the set $\{t, -1, 0\}$ w.r.t. $\varnothing$ there appear new sequences $\langle m, \vec{s}\rangle$ satisfying one of the formulas $\varphi'_1, \ldots, \varphi'_n$, φ'. We shall say that a sequence $\langle m, \vec{s}\rangle$ satisfying φ'_i or φ' is the *mth sequence* for φ_i or φ, respectively. Assume without loss of generality that for a given m there is just one such sequence.

It cannot happen that there appear only consecutive sequences $\langle 0, \vec{s}_0\rangle, \langle 1, \vec{s}_1\rangle, \ldots$ for one of the formulas $\varphi_1, \ldots, \varphi_n$, say for φ_1, and the tth sequence for φ does not appear. Indeed, if it was so, then a K-closure of $\{t, -1, 0\}$ w.r.t. φ_1 would exist. But in this K-closure there must be the 0th sequence for one of the formulas $\varphi_2, \ldots, \varphi_n$, say for φ_2, or the tth sequence for φ (by the lemma), contradiction.

Reasoning in the same way, we infer it cannot happen that building the consecutive K-closures of $\{t, -1, 0\}$ w.r.t. $\emptyset$ only sequences for two of the formulas $\varphi_1, \ldots, \varphi_n$, say for φ_1, φ_2, are generated and the tth sequence for φ does not appear. Similarly for three, etc.

If so, then there is an $L \in N$ such that in the Lth K-closure of the set $\{t, -1, 0\}$ w.r.t. $\emptyset$ there are all sequences for $\varphi_1, \ldots, \varphi_n$ which are needed to generate the tth sequence for φ ($L \in N$ since the theorem is formulated for functions from N to N, in the case of a nonstandard model L can be nonstandard).

It either can happen that these are all the sequences for $\varphi_1, \ldots, \varphi_n$ which are in the K-closure of $\{t\}$ w.r.t. $\{\varphi_1, \ldots, \varphi_n\}$ (those certainly generate the tth sequence for φ by the lemma) or the tth sequence for φ is generated somewhere earlier in the procedure.

The number L is the biggest in the following situation: for every $i = 1, \ldots, n$ the 0th sequence for φ_i does not appear in any K-closure of $\{t, -1, 0\}$ until all the sequences for $\varphi_1, \ldots, \varphi_{i-1}$ occurring in the K-closure of $\{t, -1, 0\}$ w.r.t. $\{\varphi_1, \ldots, \varphi_{i-1}\}$ are generated. Neither does the tth sequence for φ appear until a K-closure of $\{t, -1, 0\}$ w.r.t. $\{\varphi_1, \ldots, \varphi_n\}$ is generated.

Assume that we are dealing with the above (worst) situation. Then we define $L_{ij} \in N$ to be the first number such that in the L_{ij}th K-closure of $\{t, -1, 0\}$ w.r.t. $\emptyset$ there appears the jth sequence for φ_i $(i = 1, \ldots, n)$. Then $L_{1,0} = 1$, $L_{1,j} = j+1$.

Define the functions $f_i(t)$ bounding the elements of the $L_{i,0}$th K-closure of $\{t, -1, 0\}$ w.r.t. $\emptyset$:

(1) Assume $i = 1$. We have to consider the superpositions of the polynomials whose numbers are less than K iterated K times. If $u > 1$ then there is an $m \in N$ such that whenever $\bar{x} < u$ and q is a polynomial whose number is less than K then $q(\bar{x}) < u^m$. Hence, since $L_{1,0} = 1$, we can define $f_1(t) = \underbrace{((t^m) \overset{m}{\ldots})^m}_{K \text{ times}} = t^{m^K}$. Then $f_1^{t+1}(t) < F_0^{\bar{K}}(t)$ for almost all t and a $\bar{K} \in N$, $\bar{K} \geq K$.

(2) Assume $i = 2$. Note that the function $f_1^{j+1}(t)$ bounds the elements of the $L_{1,j}$th K-closure of $\{t, -1, 0\}$ w.r.t. $\emptyset$ (since $L_{1,j} = j+1$). Let H denote the K-closure of $\{t, -1, 0\}$ w.r.t. $\{\varphi_1\}$. To find the number $L_{2,0}$ it is enough to bound the elements of H. By the definition of a K-closure, $H = H_0 \cup \ldots \cup H_K$. By the definition of a K-closure it follows that the greatest element of H_1 is the tth sequence for φ_1 (we can assume that it is greater than the values of the polynomials having numbers less than K at t). Hence if $a \in H_1$ then $a < f_1^{t+1}(t)$. Similarly, if $a \in H_2$, then $a < f_1^{f_1^{t+1}(t)+1}(t)$ (the $f_1^{t+1}(t)$th sequence for φ_1). And so on. If $a \in H_K$ then

$$a < f_1^{f_1^{\cdot^{\cdot^{f_1^{f_1^{t+1}(t)+1}(t)+1}}(t)+1}}(t) \quad K \text{ times}$$

Hence we can define $f_2(t)$ to be equal to the above expression. Applying the same reasoning to the set $\{\bar{s}, -1, 0\}$ where $\bar{s}$ is the 0th sequence for φ_2 we infer that the function $f_2(f_2(t))$ bounds the elements of the $L_{2,2}$th K-closure of $\{t, -1, 0\}$ w.r.t. $\emptyset$. Repeating the same argument j times, we infer that the function $f_1^{j+1}(t)$ bounds the elements of the $L_{2,j}$th K-closure of $\{t, -1, 0\}$ w.r.t. $\emptyset$.

(3) The general case. Assume that we have defined the function $f_i(t)$. Similarly as before, we note that the function $f_i^{j+1}(t)$ bounds the elements of the $L_{i,j}$th K-closure of $\{t, -1, 0\}$ w.r.t. $\emptyset$. Let H denote the K-closure of $\{t, -1, 0\}$ w.r.t. $\{\varphi_1, \ldots, \varphi_i\}$, $H = H_0 \cup \ldots \cup H_K$ by the definition of a K-closure. To define the function $f_{i+1}(t)$ it suffices to bound the elements of H. By the definition of a K-closure we obtain successively: if $a \in H_1$ then $a < f_i^{t+1}(t)$ (since the greatest element of H_1 is the greatest of the tth sequences for $\varphi_1, \ldots, \varphi_i$, those could be bound by $f_1^{t+1}(t) \ldots f_i^{t+1}(t)$ respectively and $f_1^{t+1}(t) < f_2^{t+1}(t) < \ldots < f_i^{t+1}(t)$); if $a \in H_2$ then $a < f_i^{f_i^{t+1}(t)+1}(t)$; if $a \in H_K$ then

$$a < f_i^{f_i^{\cdot^{\cdot^{f_i^{f_i^{t+1}(t)+1}(t)+1}}(t)+1}}(t) \quad K \text{ times}.$$

We can define $f_{i+1}(t)$ to be equal to the above expression.

Let us show by induction w.r.t. i that, for every $i = 1, \ldots, n$, $f_i^{t+1}(t) < F_{i-1}^{\bar{K}^i}(t)$ for almost all t.

If $i = 1$ then $f_1^{t+1}(t) < F_0^{\bar{K}}(t)$ as we have already noticed. Assume that $f_i^{t+1}(t) < F_{i-1}^{\bar{K}^i}(t)$ for almost all t and we shall show that $f_{i+1}^{t+1}(t) < F_i^{\bar{K}^{i+1}}(t)$ for almost all t.

We make the following estimation for $t > K$:

$$f_{i+1}(t) = \underbrace{f_i^{f_i^{\cdot^{\cdot^{f_i^{t+1}(t)+1}}(t)+1}}(t)}_{K \text{ times}} < \underbrace{f_i^{f_i^{\cdot^{\cdot^{f_i^{(F_{i-1}^{\bar{K}^i})^{t+1}(t)+1}(t)+1}}(t)+1}}(t)}_{K-1 \text{ times}}$$

$$< \underbrace{f_i^{f_i^{\cdot^{\cdot^{f_i^{(F_{i-1}^{\bar{K}^i})^{F_i^{\bar{K}^i}(t)+1}(t)+1}(t)+1}}(t)+1}}(t)}_{K-2 \text{ times}}$$

$$< \underbrace{f_i^{f_i^{\cdot^{\cdot^{f_i^{F_i^{2\bar{K}^i}(t)+1}(t)+1}}(t)+1}}(t)}_{K-2 \text{ times}} < F_i^{K \cdot \bar{K}^i}(t) = F_i^{\bar{K}^{i+1}}(t).$$

In this way we obtain $f_n(t) < F_{n-1}^{\bar{K}^n}(t)$ for almost all t. Taking a K-closure of $\{t\}$ w.r.t. $\{\varphi_1, \ldots, \varphi_n\}$ and reasoning as before we conclude that $f_\varphi(t) < F_{n-1}^{\bar{K}^n}(t)$ for almost all t; therefore, $f_\varphi(t) < F_n(t)$ for almost all t (f_φ is the function defined in the way described earlier). ∎

COROLLARY. *The theory $I^-\Sigma_1$ is not finitely axiomatizable.*

Proof. Suppose that it is finitely axiomatisable and let T be a finite set of sentences axiomatizing it. Then there is a finite set of Σ_1 formulas $\{\varphi_1, \ldots, \varphi_n\}$ such that $\mathrm{Ind}\,\varphi_1 \,\&\, \ldots \,\&\, \mathrm{Ind}\,\varphi_n \vdash T$. Since all the functions $F_i(t)$ are provably total in $I^-\Sigma_1$, T proves that F_{n+2} is total. By our theorem, $F_{n+2} < F_{n+1}$ almost everywhere, contradiction.

References

[A] Z. Adamowicz, *Algebraic approach to $\exists_1$ induction*, Seminarbericht Nr. 70, Proceedings of the Third Easter Conference on Model Theory, Groß Köris, April 8–13, 1985, pp. 5–15

[G] A. Grzegorczyk, *Some classes of recursive functions*, Rozprawy Matematyczne 4 (1953).

[W] S. Wainer, *Ordinal recursion and a refinement of the extended Grzegorczyk hierarchy*, J. Symbolic Logic 37 (1972), 281–292.

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An isomorphism theorem of Hurewicz type in the proper homotopy category

by

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Abstract. Numerous mathematicians have proved theorems of Hurewicz type in different contexts shape theory, pro-categories, coherent categories. In this paper we obtain a Hurewicz Theorem in the proper homotopy category. In particular, we prove:

THEOREM. *Let (X, A) be a proper pair such that $\pi_0(X)$, $\pi_0(A)$, $\tau_0(X)$, $\tau_0(A)$ are trivial. Suppose that for $n \geqslant 2$ (X, A) is (π) n-connected and (τ) $(n-1)$-connected. Then for each proper ray α in A, $\varrho_\tau\colon \tau_n(X, A, \alpha)/(\Omega^n_\tau(X, A, \alpha) \to J_{n+1}(X, A)$ is an isomorphism. In the case where (X, A) is (τ) n-simple, for example if $\pi_1(A, \alpha) = 0$, then $\varrho_\tau\colon \tau_n(X, A, \alpha) \to J_{n+1}(X, A)$ is an isomorphism.*

1. Introduction. A natural relationship between singular homology groups and Hurewicz homotopy groups is displayed by Hurewicz's Theorem. This theorem was established in terms of simplicial homology and absolute homotopy groups by Hurewicz [11] in 1935 for simply connected polyhedra. In 1944, Eilenberg proved that the fundamental group modulo the commutator subgroup is the first singular homology group. Blakers [2], the proposer of the concept of relative homology groups, proved in 1948 the Hurewicz Theorem in the relative case given the kernel of a homomorphism.

There are more Hurewicz type theorems in other homotopy theories. For example, in 1969 Artin and Mazur [1] proved a Hurewicz Theorem in the category pro-$\mathscr{R}_0$, where $\mathscr{R}_0$ is the pointed homotopy category of connected pointed CW-complexes, and pro-$\mathscr{R}_0$ is the category of inverse systems of objects of $\mathscr{R}_0$. Relative Hurewicz type theorems for pro-$\mathscr{R}_0^2$ and Sh^2 were proved by Mardešić and Ungar [15] and independently by Morita [16]. Raussen [17] proved a Hurewicz type theorem in pro-Ho(Top$_*$), where Ho(Top$_*$) is the homotopy category of pointed topological spaces. In 1972, Kuperberg [13] proved another Hurewicz type Theorem between the homotopy groups defined by Borsuk and the Vietoris-Čech homology groups. In 1979, Kodama and Koyama [12] proved a Hurewicz type theorem between the Quigley approaching groups and the Steenrod homology groups. In a recent paper, Koyama proved a Hurewicz Theorem in the coherent homotopy category of inverse systems of spaces CPHTop.