and hence $\sum_i N(h_i, y) \leqslant 1$ on a positive measure subset of $B \setminus Y_0$. This implies that there is a k and there is a closed set $C \subset B \setminus Y_0$ such that $\lambda_n(C) > 0$ and for every $y \in C$ we have

$$N(h_i, y) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

We put $D = h_k^{-1}(C)$. Then D is measurable, and $\lambda_n(D) > 0$ since $\lambda_n(D) = 0$ would imply $\lambda_n(C) = \lambda_n(h_k(D)) = 0$.

We prove that $D \subset A_k \setminus X_0$. Obviously, $D \subset A \setminus X_0$ since $A \setminus X_0$ is the domain of h_k. Let $x \in D$ and suppose that $x \notin A_k \setminus X_0$. Since $h_k(x) \in C \subset B \setminus Y_0 \subset \bigcup_i h_i(A_i \setminus X_0)$, we have $h_k(x) = h_i(x_i)$ with some $x_i \in A_i \setminus X_0$. If $i = k$ then $x_i = x_k \in A_k \setminus X_0$, and hence $x \neq x_i$. Thus $N(h_k, h_k(x)) \geqslant 2$ which is impossible since $h_k(x) \in C$. If $i \neq k$, then we get $N(h_i, h_k(x)) \geqslant 1$ which also contradicts $h_k(x) \in C$.

Therefore $D \subset A_k \setminus X_0$ and, consequently, $D \cap A_i = \emptyset$ for $i \neq k$. This implies that $\bigcup_i f_i(A_i \cap D) = f_k(D) = h_k(D) = C$, where C is measurable and

$$0 < \lambda_n(C) \leqslant M_k^n \lambda_n(D) \leqslant M \lambda_n(D).$$

In other words, $D \in \mathscr{K}$. However, $D \cap X_0 = \emptyset$, and hence D is disjoint from the elements of $\mathscr{K}$ which contradicts the maximality of $\mathscr{K}$. This contradiction completes the proof. ∎

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An atriodic tree-like continuum with positive span which admits a monotone mapping to a chainable continuum

by

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Abstract. In this paper an example of an atriodic tree-like continuum with positive span is constructed. It is shown that there is a monotone mapping of this continuum onto a chainable continuum such that the only nondegenerate point inverse under the mapping is an arc.

1. Introduction. The following problems appear in the University of Houston Mathematics Problem Book. The first was raised by Howard Cook, the second by Cook and J. B. Fugate.

PROBLEM 92. If M is a continuum with positive span such that each of its proper subcontinua has span zero, does every nondegenerate, monotone, continuous image of M have positive span?

PROBLEM 105. Suppose M is an atriodic 1-dimensional continuum and G is an upper semi-continuous collection of continua filling up M such that M/G and every element of G are chainable. Is M chainable?

These problems also appeared as problems 163 and 15, respectively, in [9]. Several partial positive results concerning these problems have appeared ([2] and [8] for instance).

In this paper we construct an example which answers both questions in the negative. The example is constructed as an inverse limit of simple triods with a single bonding map and has positive span. It is similar in this respect to the examples constructed in [4, 5]. The inspiration for this example was an example of an attractor of a discrete dynamical system presented by Marcy Barge at the 1986 Spring Topology Conference at the University of Southwestern Louisiana [1]. However, this example is not the example he discussed.

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All spaces considered in this paper are metric. A *continuum* is a compact connected space and a *mapping* is a continuous function.

Suppose X and Y are spaces and d is a metric for Y. If f is a mapping of X into Y, the *span of* f, denoted by σf, is the least upper bound of the set of numbers ε such that there is a connected subset Z of $X\times X$ with equal first and second projections such that $d(f(x), f(y))\geqslant\varepsilon$ for each (x, y) in Z. The *span of* X, denoted by σX is the span of the identity mapping on X.

An *inverse sequence* is a pair $\{X_i, f_i\}$ whose first term is a sequence $X_1, X_2, X_3, \ldots$ of spaces and whose second term is a sequence of mappings (called the *bonding maps* of the system) $f_i\colon X_{i+1}\to X_i$. The *inverse limit* of the inverse sequence $\{X_i, f_i\}$ is the subspace X of $\prod_{i>0} X_i$ consisting of all points $(x_1, x_2, x_3, \ldots)$ in $\prod_{i>0} X_i$ such that $f_i(x_{i+1}) = x_i$ for each i. The projection of X onto the ith factor space will be denoted by π_i.

2. The continua X and Y and the mapping μ. Let A, B, C, and O denote the complex numbers i, -1, 1, and 0 respectively. Let T be the simple triod $[O, A]\cup[O, B]\cup[O, C]$ lying in the complex plane. Let $f\colon T\to T$ be the unique piecewise linear mapping such that $f(O) = \frac{B}{2}$, $f(A) = B$, $f(B) = C$, $f\left(\frac{B}{2}\right) = \frac{C}{2}$, $f\left(\frac{B}{4}\right) = O$, $f(C) = C$, $f\left(\frac{C}{2}\right) = \frac{C}{2}$, $f\left(\frac{3C}{8}\right) = O$, $f\left(\frac{C}{4}\right) = A$, and $f\left(\frac{C}{8}\right) = O$, and such that f is linear on $[O, A]$, $[O, B/4]$, $[B/4, B/2]$, $[B/2, B]$, $[O, C/8]$, $[C/8, C/4]$, $[C/4, 3C/8]$, $[3C/8, C/2]$, and $[C/2, C]$. Figure 1 depicts this mapping by showing the domain simple triod embedded in a "thickened" simple triod, following the pattern given by f. The diagram might be thought of as a "graph" of the function f.

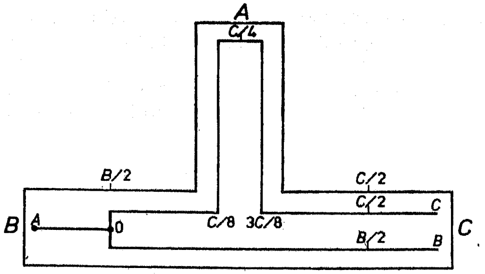

Fig. 1

For each positive integer n, let $T_n = T$ and $f_n = f$. Let X be the inverse limit of the inverse sequence $\{T_n, f_n\}$. An embedding of X in the plane is indicated in the left half of Fig. 2.

Denote the unit interval $[0, 1]$ by I. Define $g\colon I\to I$ to be the mapping such that $g(0) = 0$, $g(1/8) = 1/2$, $g(3/8) = 1/2$, $g(1/2) = 1$, $g(1) = 0$, and such that g is linear on each of $[0, 1/8]$, $[1/8, 3/8]$, $[3/8, 1/2]$, $[1/2, 1]$. The graph of g is shown in Fig. 3. For each positive integer n, let $Y_n = I$ and $g_n = g$. Let Y be the inverse limit of the inverse sequence $\{Y_n, g_n\}$. The continuum Y is homeomorphic to the Brouwer–Janiszewski–Knaster Continuum [6, p. 204], shown in the right half of Fig. 2.

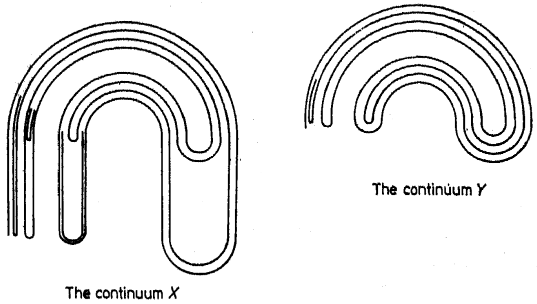

Fig. 2

Define $h\colon T\to I$ as the mapping such that $h([C, C/2]) = 0$, $h([O, A]) = 1/2$, $h([B/2, B]) = 1$, and such that h is linear on $[C/2, O]$, and $[O, B/2]$. The graph of h restricted to $[C, O]\cup[O, B]$ is also shown in Fig. 3.

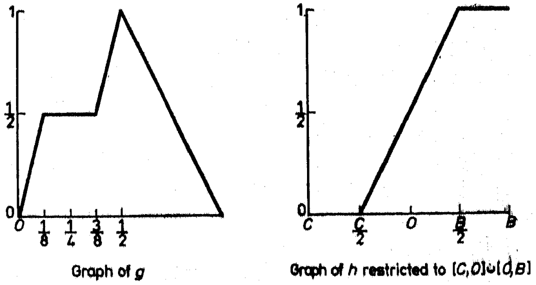

Fig. 3

LEMMA 1. *The mapping h maps T onto I, and $hf = gh$. Thus, $\mu\colon X\to Y$ defined by $\mu(x_1, x_2, \ldots) = (h(x_1), h(x_2), \ldots)$ is a mapping of X onto Y.*

Proof. Since each of f, g, and h is piecewise linear, it suffices to check that $hf(x) = gh(x)$ at each of the points x in $\{A, C, C/2, 3C/8, C/4, C/8, B, B/2, B/4, O\}$. We leave this to the reader.

THEOREM 1. *The mapping μ is monotone, and the only nondegenerate point inverse under μ is the arc*

$$\mu^{-1}(0,0,0,\ldots) = \{(t,t,t,\ldots)\colon\ t\in[C/2, C]\}.$$

Proof. That $\mu^{-1}(0,0,0,\ldots)$ is the set indicated above follows from the facts that $h^{-1}(0) = [C/2, C]$ and f restricted to $[C/2, C]$ is the identity. This set is clearly homeomorphic to an arc and is thus connected.

Suppose that $y = (y_1\ y_2\ y_3,\ldots)$ is a point of Y and that $\mu^{-1}(y)$ is nondegenerate. Let $w = (w_1, w_2, w_3,\ldots)$ and $x = (x_1, x_2, x_3,\ldots)$ be points of $\mu^{-1}(y)$ such that $w\neq x$. Then there is a positive integer n such that $w_n\neq x_n$.

Suppose that $i>n$. Then $w_{i+2}\neq x_{i+2}$. Recall, from the definition of μ, that $h(w_{i+2}) = h(x_{i+2}) = y_{i+2}$. Since h is a homeomorphism on $(B/2, O)\cup(O, C/2)$, both w_{i+2} and x_{i+2} belong to $[B, B/2]\cup[C/2, C]\cup[O, A]$. Now $f^2([O, A]) = f([B/2, B)] = f([C/2, C]) = [C/2, C]$, hence w_i and x_i are both in $[C/2, C]$, regardless of which of the intervals w_{i+2} and x_{i+2} are in. If $j<i$, $w_j = w_i$ and $x_j = x_i$ since f restricted to $[C/2, C]$ is the identity. Thus w_0 and x_0 are in $[C/2, C]$, and $w_j = w_0$ and $x_j = x_0$ for all j. Therefore w and x both belong to $\mu^{-1}(0,0,0,\ldots)$.

3. Atriodicity of X. A continuum M is a *triod* provided there is a subcontinuum K of M such that $M\setminus K$ has at least three components. A continuum is *atriodic* if it contains no triod.

THEOREM 2. *Every proper subcontinuum of X is an arc, and thus X is atriodic.*

Proof. Suppose that H is a proper subcontinuum of X. There exists a positive integer n such that if $i>n$ then O is not in $\pi_i(H)$. If not, for infinitely many integers j, O belongs to $\pi_j(H)$. Then every projection of H contains $C/2$ since $f(O) = B/2$ and $f(B/2) = C/2$ which is a fixed point for f. Therefore, for infinitely many integers j, $[O, C/2]\subset\pi_j(H)$. However, $f^3([O, C/2]) = T$, which implies that every projection of H is all of T. This contradicts the assumption that H is a proper subcontinuum.

Note that there is a positive integer k such that A is not in $\pi_i(H)$ if $i>k$. If not there are infinitely many projections of H containing A, but $f(f(A)) = C$ and C is fixed by f so all projections of H contain C. Thus infinitely many of the projections of H contain both A and C so infinitely many projections contain O, which we have just shown to be impossible.

Since $f^{-1}(B/2) = \{O\}$, for $i>n-1$, $B/2$ is not in $\pi_i(H)$. Thus for $i>n+k$, $f|\pi_i(H)$ is a homeomorphism of an arc, so H is an arc.

Thus every proper subcontinuum of X is an arc, and by [3, Theorem 3], X is atriodic.

4. The continuum X has positive span. In the definition and theorems that follow we employ the following notation, similar to that used in [4] and [5]. *The symbols $\langle t, u\rangle$ and $\langle t, u\rangle'$ will be used only to denote subcontinua of $T\times T$ such that the first projection is the subarc t of T and the second projection is the subarc u of T.* If p and q

are points of T the unique arc in T which is irreducible from p to q will be denoted by pq unless O is a point of the arc and is not an endpoint of it, in which case it will be denoted by pOq.

DEFINITION. A subcontinuum of $T\times T$ is said to have *property L* provided that (a) it is the union of twenty subcontinua

$$\left\langle O\frac{B}{2}, O\frac{C}{2}\right\rangle,\ \left\langle O\frac{C}{2}, O\frac{B}{2}\right\rangle,\ \left\langle O\frac{B}{2}, AO\frac{C}{8}\right\rangle,\ \left\langle AO\frac{C}{8}, O\frac{B}{2}\right\rangle,\ \left\langle AO\frac{B}{2}, O\frac{C}{2}\right\rangle,$$

$$\left\langle O\frac{C}{2}, AO\frac{B}{2}\right\rangle,\ \left\langle OB, O\frac{C}{2}\right\rangle,\ \left\langle O\frac{C}{2}, OB\right\rangle,\ \left\langle O\frac{B}{2}, \frac{3B}{8}B\right\rangle,\ \left\langle\frac{3B}{8}B, O\frac{B}{2}\right\rangle,$$

$$\left\langle O\frac{B}{2}, \frac{C}{4}C\right\rangle,\ \left\langle\frac{C}{4}C, O\frac{B}{2}\right\rangle,\ \left\langle O\frac{C}{2}, \frac{C}{4}C\right\rangle,\ \left\langle\frac{C}{4}C, O\frac{C}{2}\right\rangle,\ \left\langle AO\frac{C}{8}, \frac{C}{4}C\right\rangle,$$

$$\left\langle\frac{C}{4}C, AO\frac{C}{8}\right\rangle,\ \left\langle AO\frac{C}{8}, O\frac{C}{2}\right\rangle,\ \left\langle O\frac{C}{2}, AO\frac{C}{8}\right\rangle,$$

$$\left\langle AO\frac{B}{2}, OB\right\rangle\quad\text{and}\quad\left\langle OB, AO\frac{B}{2}\right\rangle;$$

(b) $\langle t, u\rangle^{-1} = \langle u, t\rangle$ for each $\langle t, u\rangle$ in the list above;

(c) there exist eight points $x_1, x_2, x_3, x_4, z_1, z_2, z_3$ and z_4 such that

(c.1) x_1 is in $\frac{B}{4}\frac{B}{2}$ and (x_1, O) is in $\left\langle O\frac{B}{2}, O\frac{C}{2}\right\rangle$ and $\left\langle O\frac{B}{2}, AO\frac{C}{8}\right\rangle$,

(c.2) x_2 is in $AO\frac{C}{32}$ and (x_2, O) is in $\left\langle AO\frac{C}{8}, O\frac{B}{2}\right\rangle$ and $\left\langle AO\frac{C}{8}, O\frac{C}{2}\right\rangle$,

(c.3) x_3 is in $\frac{3B}{8}B$ and (x_3, O) is in $\left\langle OB, O\frac{C}{2}\right\rangle$ and $\left\langle\frac{3B}{8}B, O\frac{B}{2}\right\rangle$,

(c.4) x_4 is in $\frac{C}{4}\frac{13C}{32}$ and (x_4, O) is in $\left\langle\frac{C}{4}C, O\frac{B}{2}\right\rangle$, $\left\langle\frac{C}{4}C, O\frac{C}{2}\right\rangle$ and $\left\langle\frac{C}{4}C, AO\frac{C}{8}\right\rangle$,

(c.5) z_1 is in $O\frac{B}{4}$ and $\left(z_1, \frac{C}{2}\right)$ is in $\left\langle O\frac{B}{2}, O\frac{C}{2}\right\rangle$ and $\left\langle O\frac{B}{2}, \frac{C}{4}C\right\rangle$,

(c.6) z_2 is in $AO\frac{C}{8}$ and $\left(z_2, \frac{C}{2}\right)$ is in $\left\langle AO\frac{C}{8}, \frac{C}{4}C\right\rangle$ and $\left\langle AO\frac{C}{8}, O\frac{C}{2}\right\rangle$,

(c.7) z_3 is in $\frac{3C}{8}\frac{C}{2}$ and (z_3, A) is in $\left\langle O\frac{C}{2}, AO\frac{C}{8}\right\rangle$ and $\left\langle O\frac{C}{2}, AO\frac{B}{2}\right\rangle$, and

(c.8) z_4 is in $O\frac{B}{4}$ and (z_4, A) is in $\left\langle O\frac{B}{2}, AO\frac{C}{8}\right\rangle$ and $\left\langle OB, AO\frac{B}{2}\right\rangle$; and

(d) $\left\langle O\frac{B}{2}, O\frac{C}{2}\right\rangle$ is a subset of both $\left\langle AO\frac{B}{2}, O\frac{C}{2}\right\rangle$ and $\left\langle OB, O\frac{C}{2}\right\rangle$ while $\left\langle\frac{3B}{8}B, O\frac{B}{2}\right\rangle$ is a subset of $\left\langle OB, AO\frac{B}{2}\right\rangle$.

LEMMA 2. *If Z is a subcontinuum of $T\times T$ with property L then there is a subcontinuum Z' of $T\times T$ with property L such that $f\times f(Z') = Z$.*

Proof. If $\langle t, u\rangle$ is a subcontinuum of Z and v and w are arcs in T such that $f|v$ is a homeomorphism throwing v onto t and $f|w$ is a homeomorphism throwing w onto u then $L(\langle t, u\rangle, v, w)$ denotes the continuum $(f|v)^{-1}\times(f|w)^{-1}(\langle t, u\rangle)$, called the lift of $\langle t, u\rangle$ with respect to v and w, and having the property that its first and second projections are v and w respectively.

The construction of the twenty continua whose union is Z' may be read from the following table. The first column contains the name, $\langle t, u\rangle'$, of the subcontinuum of Z' under construction: $\langle t, u\rangle'$ is the union of the subcontinua L_i listed as corresponding to it in the second column. Each L_i is the lift of $\langle v, w\rangle$ with respect to r and s, where $\langle v, w\rangle$ is the subcontinuum of Z shown in the third column, and r and s are shown in the fourth and fifth columns. The construction is identical in nature to that of [4].

If $\langle t, u\rangle'$ is one of the ten liftings constructed below let $\langle u, t\rangle'$ denote $(\langle t, u\rangle')^{-1}$ and let Z' be the union of these twenty sets.

To see that Z' is connected, note that $\left(\left(f|O\frac{B}{4}\right)^{-1}(x_1), \frac{3C}{8}\right)$ is common to L_1 and L_2, $\left(\frac{B}{4}, \left(f|\frac{C}{4}\frac{13C}{32}\right)^{-1}(x_2)\right)$ is in L_2 and L_3, and $\left(\left(f|\frac{B}{4}\frac{B}{2}\right)^{-1}(z_3), \frac{C}{4}\right)$ is in L_3 and L_4. Then note that $\left(\frac{B}{4}, \left(f|AO\frac{C}{32}\right)^{-1}(x_3)\right)$ is in L_5 and L_6 (since x_3 is in $\frac{3B}{8}B$, $\left(f|AO\frac{C}{8}\right)^{-1}(x_3)$ is in $AO\frac{C}{32}$). The only observation necessary to see that the third of the ten sets constructed below is connected is to note that L_1 is a subset of L_7 and this also shows that $\left\langle O\frac{B}{2}, O\frac{C}{2}\right\rangle'$ is a subset of $\left\langle AO\frac{B}{2}, O\frac{C}{2}\right\rangle'$. Further note that L_8 and L_4 both contain the point $\left(\frac{B}{2}, \left(f|O\frac{C}{8}\right)^{-1}(z_1)\right)$. The point $\left(\left(f|\frac{3B}{8}B\right)^{-1}(x_4), \frac{B}{4}\right)$ belongs to L_9 and L_{10}. The point $\left(\left(f|O\frac{B}{2}\right)^{-1}(z_1), \frac{C}{2}\right)$ belongs to L_1 and L_{11} while L_{11} and L_{12} both contain $\left(\frac{B}{4}, \left(f|\frac{7C}{16}C\right)^{-1}(x_4)\right)$. Both L_{13} and L_{14} contain the point $\left(\left(f|O\frac{C}{2}\right)^{-1}(x_1), \frac{3C}{8}\right)$ while $\left(\frac{C}{4}, \left(f|\frac{3C}{8}\frac{C}{2}\right)^{-1}(z_3)\right)$

Continuum	ref	Lift	wrt & wrt	
$\left\langle O\frac{B}{2}, O\frac{C}{2}\right\rangle'$	L_1	$\left\langle O\frac{B}{2}, O\frac{C}{2}\right\rangle$	$O\frac{B}{4}$	$\frac{3C}{8}\frac{C}{2}$
	L_2	$\left\langle O\frac{B}{2}, AO\frac{C}{8}\right\rangle$	$O\frac{B}{4}$	$\frac{C}{4}\frac{13C}{32}$
	L_3	$\left\langle O\frac{C}{2}, AO\frac{C}{8}\right\rangle$	$\frac{B}{4}\frac{B}{2}$	$\frac{C}{4}\frac{13C}{32}$
	L_4	$\left\langle O\frac{C}{2}, AO\frac{B}{2}\right\rangle$	$\frac{B}{4}\frac{B}{2}$	$O\frac{C}{4}$
$\left\langle O\frac{B}{2}, AO\frac{C}{8}\right\rangle'$	L_5	$\left\langle O\frac{C}{2}, OB\right\rangle$	$\frac{B}{4}\frac{B}{2}$	$AO\frac{C}{8}$
	L_6	$\left\langle O\frac{B}{2}, \frac{3B}{8}B\right\rangle$	$O\frac{B}{4}$	$AO\frac{C}{32}$
$\left\langle AO\frac{B}{2}, O\frac{C}{2}\right\rangle'$	L_7	$\left\langle OB, O\frac{C}{2}\right\rangle$	$AO\frac{B}{4}$	$\frac{3C}{8}\frac{C}{2}$
	L_2	$\left\langle O\frac{B}{2}, AO\frac{C}{8}\right\rangle$	$O\frac{B}{4}$	$\frac{C}{4}\frac{13C}{32}$
	L_3	$\left\langle O\frac{C}{2}, AO\frac{C}{8}\right\rangle$	$\frac{B}{4}\frac{B}{2}$	$\frac{C}{4}\frac{13C}{32}$
	L_4	$\left\langle O\frac{C}{2}, AO\frac{B}{2}\right\rangle$	$\frac{B}{4}\frac{B}{2}$	$O\frac{C}{4}$
$\left\langle OB, O\frac{C}{2}\right\rangle'$	L_8	$\left\langle\frac{C}{4}C, O\frac{B}{2}\right\rangle$	$\frac{3B}{8}B$	$O\frac{C}{8}$
	L_4	$\left\langle O\frac{C}{2}, AO\frac{B}{2}\right\rangle$	$\frac{B}{4}\frac{B}{2}$	$O\frac{C}{4}$
	L_3	$\left\langle O\frac{C}{2}, AO\frac{C}{8}\right\rangle$	$\frac{B}{4}\frac{B}{2}$	$\frac{C}{4}\frac{13C}{32}$
	L_2	$\left\langle O\frac{B}{2}, AO\frac{C}{8}\right\rangle$	$O\frac{B}{4}$	$\frac{C}{4}\frac{13C}{32}$
	L_1	$\left\langle O\frac{B}{2}, O\frac{C}{2}\right\rangle$	$O\frac{B}{4}$	$\frac{3C}{8}\frac{C}{2}$

Continuum	ref	Lift	wrt & wrt	
$\left\langle \frac{3B}{8}B,\ O\frac{B}{2}\right\rangle'$	L_9	$\left\langle \frac{C}{4}C,\ O\frac{B}{2}\right\rangle$	$\frac{3B}{8}B$	$O\frac{B}{4}$
	L_{10}	$\left\langle \frac{C}{4}C,\ O\frac{C}{2}\right\rangle$	$\frac{3B}{8}B$	$\frac{B}{4}\frac{B}{2}$
$\left\langle O\frac{B}{2},\ \frac{C}{4}C\right\rangle'$	L_2	$\left\langle O\frac{B}{2},\ AO\frac{C}{8}\right\rangle$	$O\frac{B}{4}$	$\frac{C}{4}\frac{13C}{32}$
	L_1	$\left\langle O\frac{B}{2},\ O\frac{C}{2}\right\rangle$	$O\frac{B}{4}$	$\frac{3C}{8}\frac{C}{2}$
	L_{11}	$\left\langle O\frac{B}{2},\ \frac{C}{4}C\right\rangle$	$O\frac{B}{4}$	$\frac{7C}{16}C$
	L_{12}	$\left\langle O\frac{C}{2},\ \frac{C}{4}C\right\rangle$	$\frac{B}{4}\frac{B}{2}$	$\frac{7C}{16}C$
$\left\langle O\frac{C}{2},\ \frac{C}{4}C\right\rangle'$	L_{13}	$\left\langle O\frac{B}{2},\ AO\frac{C}{8}\right\rangle$	$O\frac{C}{8}$	$\frac{C}{4}\frac{13C}{32}$
	L_{14}	$\left\langle AO\frac{B}{2},\ O\frac{C}{2}\right\rangle$	$O\frac{C}{4}$	$\frac{3C}{8}\frac{C}{2}$
	L_{15}	$\left\langle AO\frac{C}{8},\ O\frac{C}{2}\right\rangle$	$\frac{C}{4}\frac{13C}{32}$	$\frac{3C}{8}\frac{C}{2}$
	L_{16}	$\left\langle AO\frac{C}{8},\ \frac{C}{4}C\right\rangle$	$\frac{C}{4}\frac{13C}{32}$	$\frac{7C}{16}C$
	L_{17}	$\left\langle O\frac{C}{2},\ \frac{C}{4}C\right\rangle$	$\frac{3C}{8}\frac{C}{2}$	$\frac{7C}{16}C$
$\left\langle AO\frac{C}{8},\ \frac{C}{4}C\right\rangle'$	L_{18}	$\left\langle O\frac{B}{2},\ \frac{C}{4}C\right\rangle$	$O\frac{C}{8}$	$\frac{7C}{16}C$
	L_{19}	$\left\langle OB,\ O\frac{C}{2}\right\rangle$	$AO\frac{C}{8}$	$\frac{3C}{8}\frac{C}{2}$
	L_{13}	$\left\langle O\frac{B}{2},\ AO\frac{C}{8}\right\rangle$	$O\frac{C}{8}$	$\frac{C}{4}\frac{13C}{32}$

Continuum	ref	Lift	wrt & wrt	
$\left\langle AO\frac{C}{8},\ O\frac{C}{2}\right\rangle'$	L_{19}	$\left\langle OB,\ O\frac{C}{2}\right\rangle$	$AO\frac{C}{8}$	$\frac{3C}{8}\frac{C}{2}$
	L_{13}	$\left\langle O\frac{B}{2},\ AO\frac{C}{8}\right\rangle$	$O\frac{C}{8}$	$\frac{C}{4}\frac{13C}{32}$
	L_{20}	$\left\langle OB,\ AO\frac{B}{2}\right\rangle$	$AO\frac{C}{8}$	$O\frac{C}{4}$
$\left\langle OB,\ AO\frac{B}{2}\right\rangle'$	L_{10}	$\left\langle \frac{C}{4}C,\ O\frac{C}{2}\right\rangle$	$\frac{3B}{8}B$	$\frac{B}{4}\frac{B}{2}$
	L_9	$\left\langle \frac{C}{4}C,\ O\frac{B}{2}\right\rangle$	$\frac{3B}{8}B$	$O\frac{B}{4}$
	L_{21}	$\left\langle O\frac{C}{2},\ OB\right\rangle$	$\frac{B}{4}\frac{B}{2}$	$AO\frac{B}{4}$
	L_{22}	$\left\langle O\frac{B}{2},\ \frac{3B}{8}B\right\rangle$	$O\frac{B}{4}$	$AO\frac{B}{16}$

is in L_{14} and L_{15}, $\left(\left(f\mid\frac{C}{4}\frac{13C}{32}\right)^{-1}(z_2),\ \frac{C}{2}\right)$ is in L_{15} and L_{16}, and $\left(\frac{3C}{8},\ \left(f\mid\frac{7C}{16}C\right)^{-1}(x_4)\right)$, is in L_{16} and L_{17}. Note that L_{18} and L_{19} both contain the point $\left(\left(f\mid O\frac{C}{8}\right)^{-1}(z_1),\ \frac{C}{2}\right)$ while $\left(\left(f\mid O\frac{C}{8}\right)^{-1}(x_1),\ \frac{3C}{8}\right)$ is in L_{19} and L_{13}. Note that L_{13} and L_{20} both contain the point $\left(\left(f\mid O\frac{C}{8}\right)^{-1}(z_4),\ \frac{C}{4}\right)$. Finally, L_9 and L_{21} both contain the point $\left(\frac{B}{2},\ \left(f\mid O\frac{B}{4}\right)^{-1}(z_1)\right)$ while $\left(\frac{B}{4},\ \left(f\mid AO\frac{B}{16}\right)^{-1}(x_3)\right)$ is in L_{21} and L_{22}.

Because each of these liftings constructed projects onto its corresponding arcs there exist points $y_1, y_2, y_3, y_4, y_5, y_6$, and y_7 such that

$\left(y_1, \frac{B}{2}\right)$ is in $\left\langle O\frac{C}{2},\ O\frac{B}{2}\right\rangle$, $\left(y_2, \frac{B}{2}\right)$ is in $\left\langle \frac{3B}{8}B,\ O\frac{B}{2}\right\rangle$, $\left(y_3, \frac{B}{2}\right)$ is in $\left\langle AO\frac{C}{8},\ O\frac{B}{2}\right\rangle$, $\left(y_4, \frac{B}{2}\right)$ is in $\left\langle \frac{C}{4}C,\ O\frac{B}{2}\right\rangle$, $\left(y_5, \frac{C}{2}\right)$ is in $\left\langle O\frac{B}{2},\ O\frac{C}{2}\right\rangle$, (y_6, B) is in $\left\langle O\frac{C}{2},\ OB\right\rangle$ and (y_7, B) is in $\left\langle O\frac{B}{2},\ \frac{3B}{8}B\right\rangle$.

Let $x_1' = \left(f\,|\,\frac{B}{4}\frac{B}{2}\right)^{-1}(y_1)$, $x_2' = \left(f\,|\,AO\frac{C}{32}\right)^{-1}(y_2)$, $x_3' = \left(f\,|\,\frac{3B}{8}B\right)^{-1}(y_4)$, $x_4' = \left(f\,|\,\frac{C}{4}\frac{13C}{32}\right)^{-1}(y_3)$, $z_1' = \left(f\,|\,O\frac{B}{4}\right)^{-1}(y_5)$, $z_2' = \left(f\,|\,AO\frac{C}{8}\right)^{-1}(y_5)$, $z_3' = \left(f\,|\,\frac{3C}{8}\frac{C}{2}\right)^{-1}(y_6)$ and $z_4' = \left(f\,|\,O\frac{B}{4}\right)^{-1}(y_7)$. So far, we have shown that each of the ten liftings listed above is connected. Observe that for the collection $C_1 = \left\{\left\langle O\frac{B}{2}, O\frac{C}{2}\right\rangle', \left\langle AO\frac{B}{2}, O\frac{C}{2}\right\rangle', \left\langle OB, O\frac{C}{2}\right\rangle', \left\langle O\frac{B}{2}, \frac{C}{4}C\right\rangle'\right\}$, the first member listed is a subset of the second and third members and intersects the fourth. Thus C_1^* is connected. Let $C_2 = \left\{\left\langle \frac{3B}{8}B, O\frac{B}{2}\right\rangle', \left\langle OB, AO\frac{B}{2}\right\rangle'\right\}$ and note that C_2^* is connected. Let $C_3 = \left\{\left\langle O\frac{C}{2}, \frac{C}{4}C\right\rangle', \left\langle AO\frac{C}{8}, O\frac{C}{2}\right\rangle', \left\langle AO\frac{C}{8}, \frac{C}{4}C\right\rangle'\right\}$ and since the first member listed intersects each of the other two, C_3^* is a continuum. Observe that (x_1', O) is in $\left\langle O\frac{B}{2}, O\frac{C}{2}\right\rangle'$ and $\left\langle O\frac{B}{2}, AO\frac{C}{8}\right\rangle'$ while (z_4', A) is in $\left\langle O\frac{B}{2}, AO\frac{C}{8}\right\rangle'$ and $\left\langle OB, AO\frac{B}{2}\right\rangle'$ and (O, x_4') is in $\left\langle O\frac{B}{2}, \frac{C}{4}C\right\rangle'$ and $\left\langle O\frac{C}{2}, \frac{C}{4}C\right\rangle'$. These observations imply that $C_1^*\cup C_2^*\cup C_3^*\cup\left\langle O\frac{B}{2}, AO\frac{C}{8}\right\rangle'$ is a continuum, H. Since the point (x_2', O) is in $\left\langle AO\frac{C}{8}, O\frac{B}{2}\right\rangle'$ and $\left\langle AO\frac{C}{8}, O\frac{C}{2}\right\rangle'$, $Z' = H\cup H^{-1}$ is a continuum.

LEMMA 3. *Suppose that for each integer $n>1$ f_1^n is the $(n-1)$-fold composite of f with itself. Then if n is an integer, $\sigma(f_1^n)\geqslant 1/4$.*

Proof. Let

$$\left\langle O\frac{B}{2}, O\frac{C}{2}\right\rangle = \left(O\frac{B}{2}\times\left\{\frac{C}{2}\right\}\right)\cup\left(\left\{\frac{B}{2}\right\}\times O\frac{C}{2}\right),$$

$$\left\langle O\frac{B}{2}, AO\frac{C}{8}\right\rangle = \left(O\frac{B}{2}\times\{A\}\right)\cup\left(\left\{\frac{B}{2}\right\}\times AO\frac{C}{8}\right),$$

$$\left\langle O\frac{C}{2}, AO\frac{B}{2}\right\rangle = \left(\left\{\frac{C}{2}\right\}\times AO\frac{B}{2}\right)\cup\left(O\frac{C}{2}\times\left\{\frac{B}{2}\right\}\right),$$

$$\left\langle O\frac{C}{2}, OB\right\rangle = \left(O\frac{C}{2}\times\left\{\frac{B}{2}\right\}\right)\cup\left(\left\{\frac{C}{2}\right\}\times OB\right),$$

$$\left\langle O\frac{B}{2}, \frac{3B}{8}B\right\rangle = \left(O\frac{B}{2}\times\{B\}\right)\cup\left(\{O\}\times\frac{3B}{8}B\right),$$

$$\left\langle O\frac{B}{2}, \frac{C}{4}C\right\rangle = \left(O\frac{B}{2}\times\left\{\frac{C}{2}\right\}\right)\cup\left(\{O\}\times\frac{C}{4}C\right),$$

$$\left\langle O\frac{C}{2}, \frac{C}{4}C\right\rangle = \left(O\frac{C}{2}\times\{C\}\right)\cup\left(\{O\}\times\frac{C}{4}C\right),$$

$$\left\langle AO\frac{C}{8}, \frac{C}{4}C\right\rangle = \left(AO\frac{C}{8}\times\left\{\frac{3C}{8}\right\}\right)\cup\left(\{A\}\times\frac{C}{4}C\right),$$

$$\left\langle AO\frac{C}{8}, O\frac{C}{2}\right\rangle = \left(AO\frac{C}{8}\times\left\{\frac{C}{2}\right\}\right)\cup\left(\{A\}\times O\frac{C}{2}\right).$$

$$\left\langle AO\frac{B}{2}, OB\right\rangle = \left(AO\frac{B}{2}\times\{B\}\right)\cup(\{A\}\times OB)\cup\left(\{O\}\times\frac{3B}{8}B\right).$$

Let H denote the union of these ten continua, and let $Z_1 = H\cup H^{-1}$. Letting $x_1 = \frac{B}{2}$, $x_2 = A$, $x_3 = \frac{B}{2}$, $x_4 = \frac{3C}{8}$, $z_1 = \frac{B}{4}$, $z_2 = A$, $z_3 = \frac{C}{2}$ and $z_4 = O$, it is easy to check that Z_1 has property L. Suppose that $i>1$ is an integer and that Z_i has been constructed so that Z_i has property L and that $f_1^i\times f_1^i(Z_i) = Z_1$. By Lemma 2 there is a subcontinuum Z_{i+1} of $T\times T$ with property L so that $f\times f(Z_{i+1}) = Z_i$, and thus $f_1^{i+1}\times f_1^{i+1}(Z_{i+1}) = Z_1$. Thus, by induction, if $n>1$ there is a subcontinuum Z_n of $T\times T$ such that $f_1^n\times f_1^n(Z_n) = Z_1$. Noting that if (p, q) is in Z_1, $d(p, q)\geqslant 1/4$, we have that $\sigma(f_1^n)\geqslant 1/4$.

THEOREM 3. *The continuum X has positive span, and thus is not chainable.*

Proof. Apply Lemma 2 and Theorem 4 of [4]. That the continuum is not chainable follows from [7, p. 210].

5. Final remarks. To see that the continuum X gives negative answers to the questions in the introduction we need only note that chainable continua have zero span [7, p. 210] and apply Theorems 1, 2, and 3.

THEOREM 4. *The continuum X has positive span, each proper subcontinuum of X has zero span, and there is a monotone mapping of X onto a continuum with zero span.*

THEOREM 5. *The atriodic continuum X is not chainable, and there is an upper semi-continuous collection G of continua filling up X such that each member of G is chainable and X/G is chainable.*

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Solution to a compactification problem of Sklyarenko

by

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Dedicated to Professor Yukihiro Kodama
On his 60th birthday

Abstract. Concerning a function Skl originally introduced by Sklyarenko to study compactness deficiency def, we establish a theorem that $\mathrm{Skl}\,X = \mathrm{def}\,X$ for every separable metrizable space X. This answers a problem of Sklyarenko affirmatively.

1. Introduction. All spaces considered in this paper are assumed to be separable and metrizable. By a compactification of a space X, we mean a compact metrizable space containing X as a dense subspace. For undefined notion see [3] and [5].

The *compactness deficiency* $\mathrm{def}\,X$ of a space X is the least integer n for which X has a compactification αX with $\dim(\alpha X - X) = n$.

J. de Groot [4] proved that a space X has a compactification αX with $\dim(\alpha X - X) \leqslant 0$ if and only if X is rim-compact. Motivated by this result, to study further def he introduced the *small* (resp. *large*) *inductive compactness degree* $\mathrm{cmp}\,X$ (resp. $\mathrm{Cmp}\,X$) of a space X. In general, the inequality $\mathrm{cmp}\,X \leqslant \mathrm{Cmp}\,X \leqslant \mathrm{def}\,X$ holds [5]. The well-known conjecture of de Groot that $\mathrm{cmp}\,X = \mathrm{def}\,X$ has been negatively solved by R. Pol [9]; the space X of Pol's example has $\mathrm{cmp}\,X = 1$ and $\mathrm{Cmp}\,X = \mathrm{def}\,X = 2$. It is unknown whether there is a space X with $\mathrm{Cmp}\,X < \mathrm{def}\,X$. [1]

Another condition to study def is due to E. Sklyarenko [10], [11], which is denoted by $\mathrm{Skl}\,X \leqslant n$ as in Isbell's book [6]; a space X has $\mathrm{Skl}\,X \leqslant n$ if X has a base $\mathscr{B}$ such that $\mathrm{Bd}\,B_0 \cap \mathrm{Bd}\,B_1 \cap \ldots \cap \mathrm{Bd}\,B_n$ is compact for any $n+1$ distinct members of $\mathscr{B}$. Sklyarenko proved that $\mathrm{Skl}\,X \leqslant \mathrm{def}\,X$ [10] and asked whether $\mathrm{Skl}\,X = \mathrm{def}\,X$ for every space X [11]. Recently, J. M. Aarts, J. Bruijning and J. van Mill [2] proved that $\mathrm{Cmp}\,X \leqslant \mathrm{Skl}\,X$. In this paper we give an affirmative answer to Sklyarenko's problem above. Namely, we shall establish a theorem that $\mathrm{Skl}\,X = \mathrm{def}\,X$ for every space X. As an application it will be shown that a non-compact space X has a compactification αX with $\dim(\alpha X - X) = n$ if and only if $\mathrm{Skl}\,X \leqslant n$.

2. Preliminaries and lemmas. Let $\mathscr{S}$ be a collection of subsets of a space X. We shall write $[\mathscr{S}]^n$ for $\{\mathscr{T}\colon \mathscr{T}$ is a subcollection of $\mathscr{S}$ with $|\mathscr{T}| = n\}$, $\bigcap \mathscr{S}$

[1] Added in proof. Recently the author has constructed such a space.