A recursion-theoretic characterization of instances of $B\Sigma_n$ provable in $\Pi_{n+1}(N)$

by

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Abstract. Let $B\psi$ denote the sentence

$$(\forall a, t)((\forall x)_{\leqslant t}(\exists y)\psi(x, a, y) \Rightarrow (\exists z)(\forall x)_{\leqslant t}(\exists y)_{<z}\psi(x, a, y))$$

where ψ is Π_{n-1}, $n \geqslant 1$.

We give a recursion-theoretic characterization of those Π_{n-1} formulas ψ that $B\psi$ is provable in $\Pi_{n+1}(N)$.

As a corollary we infer that $\Pi_{n+1}(N)$ does not prove $B\Sigma_n$.

The independence of $B\Sigma_n$ from the theory of true Π_{n+1} sentences — $\Pi_{n+1}(N)$ — was first proved by Parsons in [P] by a combination of a proof-theoretic and recursion-theoretic argument. There is also a model-theoretic proof possible. Paris and Kirby in [PK] prove that $B\Sigma_n$ is independent from $I\Sigma_{n-1}$ in a model theoretic way. One can modify their argument so as to show the unprovability of $B\Sigma_n$ in $\Pi_{n+1}(N)$.

Note that $\Pi_{n+1}(N)$ is a stronger theory than $I\Sigma_{n-1}$.

We give another proof of the independence of $B\Sigma_n$ from $\Pi_{n+1}(N)$. It is similar to Parsons's proof, but in addition to the independence result, it gives a full characterization of formulas ψ such that the corresponding instance of $B\Sigma_n$ is provable in $\Pi_{n+1}(N)$. To formulate our characterization theorem we need two definitions of recursion-theoretic character.

DEFINITION 1. A relation $R(a, x) \subseteq N \times N$ is called *almost a function* if there is a number $L \in N$ such that for every a, the set $\{x\colon R(a, x)\}$ has at most L elements.

DEFINITION 2. A relation $\hat{R}(a, x)$ *uniformizes* $R(a, x)$ if for every a

$$(\exists x)\ R(a, x) \Rightarrow (\exists x)\ (R(a, x)\ \&\ \hat{R}(a, x)).$$

Our main theorem is the following:

THEOREM 1. *$\Pi_{n+1}(N) \vdash B\psi$ where ψ is Π_{n-1} iff the Π_n relation $R(\langle t, a\rangle, x)$:*

$$(\forall y)\neg\psi(x, a, y)\ \&\ x \leqslant t$$

is uniformizable in N by a Σ_n almost function.

Proof. First we prove the theorem in the "$\Leftarrow$" direction, which is easier. Let ψ always denote a Π_{n-1} formula.

LEMMA 1. *Assume that there is a Σ_{n+1} formula $\varphi(t, a)$ such that*

$$\Pi_{n+1}(N) \vdash (\forall t, a)\big((\forall x)_{\leqslant t}(\exists y)\psi(x, a, y) \Leftrightarrow \varphi(t, a)\big).$$

Then $\Pi_{n+1}(N) \vdash B\psi$.

Proof. By the assumption, $B\psi$ is equivalent in $\Pi_{n+1}(N)$ to

$$(\forall t, a)\big(\varphi(t, a) \Rightarrow (\exists z)(\forall x)_{\leqslant t}(\exists y)_{<z}\psi(x, a, y)\big).$$

The above sentence is Π_{n+1} and true, hence it is provable in $\Pi_{n+1}(N)$. Thus $\Pi_{n+1}(N) \vdash B\psi$. ■

LEMMA 2. *Assume that $R(a, x)$ is Π_n and is uniformizable by a Σ_n almost function. Then the formula $\varphi(a)$: $(\exists x)R(a, x)$ is provably Π_{n+1} in $\Pi_{n+1}(N)$, i.e. there is a Π_{n+1} formula $\Phi(a)$ such that $\Pi_{n+1}(N) \vdash (\forall a)(\varphi(a) \Leftrightarrow \Phi(a))$.*

Proof. Let $\hat{R}(a, x)$ be a Σ_n almost function uniformizing R. Let L be given by Definition 1 for $\hat{R}$. We have

$$(\forall a)\{(\exists x)R(a, x) \Leftrightarrow \bigvee_{i=1}^{L} [(\exists x_1 \ldots x_i)(\bigwedge_{j,k=1}^{i} x_j \neq x_k \,\&\, \bigwedge_{j=1}^{i} \hat{R}(a, x_j)) \,\&$$
$$\& \,(\forall x_1 \ldots x_i)((\bigwedge_{j,k=1}^{i} x_j \neq x_k \,\&\, \bigwedge_{j=1}^{i} \hat{R}(a, x_j)) \Rightarrow \bigvee_{j=1}^{i} R(a, x_j))]\}.$$

Then the "$\Rightarrow$" part of the above equivalence is a Π_{n+1} sentence and, being true, it is provable in $\Pi_{n+1}(N)$. The "$\Leftarrow$" part is provable in logic.

Moreover, the right hand side of the equivalence is a Π_{n+1} formula (in fact it is a boolean combination of Σ_n formulas).

Thus $\varphi(a)$ is equivalent in $\Pi_{n+1}(N)$ to the Π_{n+1} formula

$$\bigvee_{i=1}^{L} [(\exists x_1 \ldots x_i)(\bigwedge_{j,k=1}^{i} x_j \neq x_k \,\&\, \bigwedge_{j=1}^{i} \hat{R}(a, x_j)) \,\&\, (\forall x_1 \ldots x_i)((\bigwedge_{j,k=1}^{i} x_j \neq x_k \,\&$$
$$\& \bigwedge_{j=1}^{i} \hat{R}(a, x_j)) \Rightarrow \bigvee_{j=1}^{i} R(a, x_j))]. ■$$

Proof of the "$\Leftarrow$" part of the theorem. Assume that the relation $R(\langle t, a\rangle, x)$: $(\forall y)\neg\psi(x, a, y) \,\&\, x \leqslant t$ is uniformizable by a Σ_n almost function. Then by Lemma 2 the formula $(\exists x)(\forall y)(\neg\psi(x, a, y) \,\&\, x \leqslant t)$ is provably Π_{n+1} in $\Pi_{n+1}(N)$. Thus, there is a Σ_{n+1} formula $\varphi(t, a)$ such that $\Pi_{n+1}(N)$ proves

$$(\forall t, a)\big((\forall x)_{\leqslant t}(\exists y)\psi(x, a, y) \Leftrightarrow \varphi(t, a)\big).$$

Now, by Lemma 1, $\Pi_{n+1}(N) \vdash B\psi$.

Proof of the "$\Rightarrow$" part of the theorem. Consider the following definition.

DEFINITION 3. Let $\varphi(\vec{x})$ be a formula of the form:

$$(\forall z_0)(\exists w_0) \ldots (\forall z_k)(\exists w_k)\varphi'(x, z_0, w_0, \ldots, z_k, w_k)$$

where φ' is open.

Since we allow irrelevant quantifiers, an arbitrary formula can be written in this form.

Fix an enumeration of Z-polynomials. Let $K \in N$.

A finite set H is a K-closure of $\vec{t} = \langle t_0 \ldots t_m\rangle$ w.r.t. φ if there are sets $H_0, \ldots, H_K$ such that

(1) $H = H_0 \cup \ldots \cup H_k$, $H_i \subseteq H_{i+1}$;

(2) $H_0 = \{t_0, \ldots, t_m\}$;

(3) if $i < K$, r is a polynomial of l variables whose number is less than i and $\vec{x} \in H_i$ is of length 1 and $r(\vec{x}) \geqslant 0$ then $r(\vec{x}) \in H_{i+1}$ (by $\vec{x} \in H_i$ we mean that every term of $\vec{x}$ belongs to H_i);

(4) there are partial functions $w_j(z_0 \ldots z_j)$ from H^{j+1} to H for $j = 0, \ldots, k-1$ such that

(a) if there are $i_0, \ldots, i_j < K$ such that $i_0 < i_1 \ldots < i_j$ and $z_0 \in H_{i_0}, \ldots, z_j \in H_{i_j}$ then the sentence

$$(\forall z_{j+1})(\exists w_{j+1}) \ldots (\forall z_k)(\exists w_k)$$
$$\varphi'(\vec{t}, z_0, w_0(z_0), z_1, w_1(z_0, z_1) \ldots z_j, w_j(z_0 \ldots z_j), z_{j+1}, w_{j+1} \ldots z_k, w_k)$$

is true.

Remark 1. For every $K \in N$ there is a number $L(K) \in N$ such that a minimal K-closure of any $\vec{t}$ w.r.t. φ has at most $L(K)$ elements.

Moreover there is a formula $\theta^{\varphi}_{L(K)}(\vec{t}, \vec{h})$ of $L(K)+m$ variables stating the following: $h_0, \ldots, h_{L(K)-1}$ are consecutive, in a certain canonical ordering, elements of a minimal K-closure of $\vec{t}$ w.r.t. φ.

We show how to build $\theta^{\varphi}_{L(K)}$ on a concrete example. Assume that $\varphi(x)$ is of the form

$$(\forall z_0)(\exists w_0)(\forall z_1)\varphi'(x, z_0, w_0, z_1).$$

Let r_0, r_1 be polynomials of one variable whose numbers are 0,1 respectively. We have:

$L(0) = 1$, $\quad \theta^{\varphi}_{L(0)}(t, h_0)$: $h_0 = t$,

$L(1) = 3$, $\quad \theta^{\varphi}_{L(1)}(t, h_0, h_1, h_2)$: $h_0 = t \,\&\, h_1 = r_0(t) \,\&$
$\qquad \& \,(\forall z_1)\varphi'(t, h_0, h_2, z_1)$,

$L(2) = 10$, $\quad \theta^{\varphi}_{L(2)}(t, h_0 \ldots h_9)$: $h_0 = t \,\&\, h_1 = r_0(t) \,\&\, (\forall z_1)\varphi'(t, h_0, h_2, z_1)$
$\qquad \& \,h_3 = r_0(h_1) \,\&\, h_4 = r_0(h_2) \,\&\, h_5 = r_1(h_0) \,\&\, h_6 = r_1(h_1) \,\&$
$\qquad \& \,h_7 = r_1(h_2) \,\&\, (\forall z_1)\varphi'(t, h_1, h_8, z_1) \,\&\, (\forall z_1)\varphi'(t, h_2, h_9, z_1)$.

Note that, if φ is a Π_{n+1} formula then $\theta^{\varphi}_{L(K)}$ is Σ_n (even Π_{n-1}).

Remark 2. We have for any K

$$\mathrm{PA}^- \vdash (\forall t)(\varphi(t) \Rightarrow (\exists \vec{h})\theta^{\varphi}_{L(K)}(t, \vec{h})).$$

The next lemma is a version of Herbrand's theorem.

LEMMA 3. *Let φ be a formula. The following are equivalent*:

(1) $\mathrm{PA}^- \vdash (\forall \vec{t})\varphi(\vec{t})$;

(2) *there is a number $K \in N$ such that*

$$\mathrm{PA}^- \vdash (\forall \vec{t}, \vec{h})\neg\theta^{\neg\varphi}_{L(K)}(\vec{t}, \vec{h}).$$

Proof (due to Z. Ratajczyk). Assume (1). Add to the language the constants: $d_0, \ldots, d_m, c_0, c_1, \ldots$ and let $\vec{d} = \langle d_0 \ldots d_m\rangle$, $\vec{c}_K = \langle c_0 \ldots c_{L(K)-1}\rangle$. Suppose that (2) does not hold. Then for every K the theory

$$T_K\colon \mathrm{PA}^- + \{\theta^{\neg\varphi}_{L(0)}(\vec{d}, \vec{c}_0),\ \theta^{\neg\varphi}_{L(1)}(\vec{d}, \vec{c}_1), \ldots, \theta^{\neg\varphi}_{L(K)}(\vec{d}, \vec{c}_K)\}$$

is consistent.

Then the theory $T = \bigcup_K T_K$ is consistent, by compactness. Let M be a model of T and let $M' \subseteq M$ be the submodel of M whose universe consists of interpretations of the constants:

$$t_0, \ldots, t_m, h_0, h_1, \ldots$$

CLAIM. *$M' \Vdash \mathrm{PA}^-$ & $\neg\varphi(\vec{t})$ where $\vec{t} = \langle t_0 \ldots t_m\rangle$*

Assume that $\neg\varphi(\vec{t})$ is $(\forall z_0)(\exists w_0) \ldots (\forall z_k)(\exists w_k)\varphi'(\vec{t}, z_0, w_0 \ldots z_k w_k)$. We have $M' \Vdash \mathrm{PA}^-$ since it is closed under polynomials in M. Moreover, for any $z_0, \ldots, z_k \in M$, there are elements $w_0(z_0), w_1(z_0, z_1), \ldots, w_k(z_0 \ldots z_k)$ interpreting appropriate elements of the K-closures: $h_0 \ldots h_{L(K)-1}$. Then

$$M' \Vdash \varphi'(\vec{t}, z_0, w_0(z_0), z_1, w_1(z_0, z_1) \ldots z_k, w_k(z_0 \ldots z_k)).$$

Thus the claim is proved and hence a contradiction. The implication (2) $\Rightarrow$ (1) is easy. ■

Now we are able to prove the "$\Rightarrow$" part of the theorem.

Assume that $\Pi_{n+1}(N)$ proves $B\psi$. Especially, there is a Σ_n formula $\sigma(t, a)$ such that

$$\Pi_{n+1}(N) \vdash (\forall a, t)((\forall x)_{\leqslant t}(\exists y)\psi(x, a, y) \Leftrightarrow \sigma(t, a)).$$

Let $\varphi_1, \ldots, \varphi_m$ be a finite fragment of $\Pi_{n+1}(N)$ which proves the above equivalence. Let $\varphi(a, t)$ be the formula

$$(\varphi_1 \,\&\, \ldots \,\&\, \varphi_m) \Rightarrow ((\forall x)_{\leqslant t}(\exists y)\psi(x, a, y) \Rightarrow \sigma(t, a)).$$

Then $\mathrm{PA}^- \vdash (\forall a, t)\varphi(a, t)$.

By Lemma 3 there is a $K \in N$ such that

$$\mathrm{PA}^- \vdash (\forall t, a, \vec{h})\neg\theta^{\neg\varphi}_{L(K)}(t, a, \vec{h})$$

Here, $\neg\varphi$ is the formula

$$\varphi_1 \,\&\, \ldots \,\&\, \varphi_m \,\&\, (\forall x)_{\leqslant t}(\exists y)\psi(x, a, y) \,\&\, \neg\sigma(t, a).$$

This formula is Π_{n+1}, hence the formulas $\theta^{\neg\varphi}_{L(i)}$ are Σ_n. We can assume that $\theta^{\neg\varphi}_{L(i)}(t, a, \vec{h})$ define graphs of partial functions of t, a — otherwise take Σ_n formulas defining graphs of partial functions of t, a and uniformizing $\theta^{\neg\varphi}_{L(i)}$.

Consider the following relation $\hat{R}(\langle t, a\rangle, x)$ in $N \times N$:

$$\bigvee_{i=0}^{K-1} (\exists \vec{h})(\theta^{\neg\varphi}_{L(i)}(t, a, \vec{h}) \,\&\, (x = h_0 \vee x = h_1 \vee \ldots \vee x = h_{L(i)-1})).$$

Then $\hat{R}$ is a Σ_n almost function (to satisfy Definition 1 one can take $L = \sum_{i<K} L(i)$).

CLAIM. *$\hat{R}$ uniformizes R where $R(\langle t, a\rangle, x)$ is the relation defined in N by*

$$(\forall y)\neg\psi(x, a, y) \,\&\, x \leqslant t.$$

Proof of the claim. Assume $(\exists x)R(\langle t, a\rangle, x)$ for a pair $\langle t, a\rangle \in N$. Then $(\exists x)(x \leqslant t \,\&\, (\forall y)\neg\psi(x, a, y))$. We have $\neg\sigma(t, a)$ since

$$\sigma(t, a) \Rightarrow (\forall x)_{\leqslant t}(\exists y)\psi(x, a, y)$$

in N. Let $i < K$ be maximal such that $(\exists \vec{h})\theta^{\neg\varphi}_{L(i)}(t, a, \vec{h})$. Such an i exists since $(\exists \vec{h})\theta^{\neg\varphi}_{L(0)}(t, a, \vec{h})$ and $(\forall \vec{h})\neg\theta^{\neg\varphi}_{L(K)}(t, a, \vec{h})$. Take $\vec{h}$ such that $\theta^{\neg\varphi}_{L(i)}(t, a, \vec{h})$.

If for every $h_j \leqslant t$ there was a y such that $\psi(h_j, a, y)$ then, since $\varphi_1, \ldots, \varphi_m$ are true and $\neg\sigma(t, a)$ is true, we would be able to extend $\vec{h}$ to an $(i+1)$th closure of t, a w.r.t. $\neg\varphi$ contradicting the choice of i. Hence

$$\bigvee_{j=0}^{L(i)-1} (h_j \leqslant t \,\&\, (\forall y)\neg\psi(h_j, a, y)).$$

The claim has thus been proved and part "$\Rightarrow$" of the theorem follows. ■

COROLLARY. *$\Pi_{n+1}(N)$ does not prove $B\Sigma_n$.*

Proof. We shall show that $\Pi_{n+1}(N)$ does not prove the following form of $B\Sigma_n$:

$$(\forall t)((\forall x)_{\leqslant t}(\exists y)\psi(x, t, y) \Rightarrow (\exists z)(\forall x)_{\leqslant t}(\exists y)_{<z}\psi(x, t, y))$$

where ψ runs over Π_{n-1} formulas.

The corollary follows from the following remark:

Remark 5. There is a Π_n relation $R(t, x)$ in $N \times N$ such that the relation $R'(t, x)$ defined as $R(t, x) \,\&\, x \leqslant t$ is not uniformizable by a Σ_n almost function.

Proof of the remark Let $S(s, t, x) \subseteq N \times N \times N$ be a universal Σ_n relation. Let $R(t, x)$ be $\neg S(t, t, x)$. Then $R'(t, x)$: $R(t, x) \,\&\, x \leqslant t$ is not uniformizable by any Σ_n almost function.

Suppose the converse. Let $\hat{R}(t, x)$ be a Σ_n almost function and uniformize R'. Let L be the bound on the number of x's satisfying $\hat{R}$ with a given t. Let s be such that $s > L$ and

$$(\forall x, t)(\hat{R}(t, x) \Leftrightarrow S(s, t, x)).$$

Let us show that $(\exists x)\, R'(s, x)$. Indeed if $(\forall x) \neg R'(s, x)$ then $(\forall x)_{\leqslant s} \neg R(s, x)$ hence $(\forall x)_{\leqslant s}\, S(s, s, x)$ and hence $(\forall x)_{\leqslant s}\, \hat{R}(s, x)$. But this contradicts the choice of L.

Thus $(\exists x) R'(s, x)$. Hence, by uniformization $(\exists x)(R'(s, x) \,\&\, \hat{R}(s, x))$.

But $R'(s, x) \Leftrightarrow x \leqslant s \,\&\, \neg S(s, s, x) \Rightarrow \neg \hat{R}(s, x)$. Contradiction. Since R is Π_n it is as required. ■

Proof of the corollary. Let ψ be a Π_{n-1} formula such that

$$R(t, x) \Leftrightarrow (\forall y) \neg \psi(x, t, y)$$

where R is taken from the remark. If

$$(\forall t)\big((\forall x)_{\leqslant t} (\exists y) \psi(x, t, y) \Rightarrow (\exists z)(\forall x)_{\leqslant t} (\exists y)_{<z} \psi(x, t, y)\big)$$

was provable in $\Pi_{n+1}(N)$ then, from the proof of Theorem 1 it follows that $R(t, x)$ would be uniformizable by a Σ_n almost function and it is not the case. ■

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234^2	$\mathrm{Pa}^- \vdash (\forall t)(\varphi(t) \Rightarrow (\exists)\vec{h})\Theta^{\varphi}_{L(K)}(t, \vec{h}))$	$\mathrm{Pa}^- \vdash (\forall \vec{t})(\varphi(\vec{t}) \Rightarrow (\exists \vec{h})\Theta^{\varphi}_{L(K)}(t, \vec{h}))$
234^{17}	$M' \Vdash$	$M' \models$
234^{19}	$M' \Vdash$	$M' \models$

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