On the class of all spaces of weight not greater than ω_1 whose Cartesian product with every Lindelöf space is Lindelöf

by

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Abstract. Assuming Continuum Hypothesis, we characterize the class of all spaces whose Cartesian product with every Lindelöf space is Lindelöf and whose weight is not greater than ω_1. Using that characterization we prove that if X belongs to the class thus defined then X^ω is a Lindelöf space.

Let us denote by $\mathscr{L}$ the class of all spaces whose Cartesian product with every Lindelöf space is Lindelöf. E. Michael conjectured that if X belongs to $\mathscr{L}$ then X^ω is Lindelöf. The same question was raised by T. C. Przymusiński, who also asked for a characterization of $\mathscr{L}$ (see [T], Problem 5, page 822).

In this note, assuming Continuum Hypothesis, abbreviated (CH), we characterize all elements of $\mathscr{L}$ whose weight is not greater than ω_1. Using that characterization we prove that if X belongs to the class $\mathscr{L}$ then X^ω is Lindelöf.

We adopt the topological terminology from [E]. In the sequel $I = [0, 1]$, ω, ω_1 stand for the unit interval, the first infinite ordinal number and the first uncountable ordinal number, respectively. We shall identify a given ordinal number α with the set of all ordinals less than α. The symbol Lim will stand for the set of all countable limit ordinal numbers. For every $\alpha < \omega_1$, $\mathscr{B}_\alpha$ will denote the standard basis in I^α and p_α the projection from I^{ω_1} onto I^α. If X is a topological space then $\mathscr{K}(X)$ and $\mathscr{G}(X)$ stand for the set of all compact subsets of X and the set of all G_δ-subsets of X, respectively.

We shall say that X satisfies (*) if for every $f\colon \mathscr{K}(X) \to \mathscr{G}(X)$ such that $K \subset f(K)$ for every K of $\mathscr{K}(X)$ there is a countable set $\mathscr{S} \subset \mathscr{K}(X)$ satisfying

$$\bigcup \{f(S)\colon S \in \mathscr{S}\} = X.$$

The aim of this note is to prove

THEOREM 1 (CH). *If the weight of X is not greater than ω_1 then the following conditions are equivalent:*

(a) *X belongs to $\mathscr{L}$,*

(b) *X satisfies (*).*

As an easy corollary to Theorem 1 we obtain

THEOREM 2 (CH). *If X belongs to $\mathscr{L}$ and the weight of X is not greater than ω_1 then X^ω is Lindelöf.*

The proof of Theorem 2 is an immediate consequence of Theorem 1 and the following

LEMMA 1. *If X satisfies $(*)$ then X^ω is Lindelöf.*

Proof. Suppose not. Then there is an open cover $\mathscr{U}$ of X^ω which does not have a countable subcover. In order to finish the proof of the lemma it is enough to define a sequence $(K_n)_{n=0}^{\infty}$ of compact subsets of X such that for every $m\in\omega$, every finite sequence $(G_i)_{i=0}^{m}$ of G_δ-subsets of X satisfying $K_i\subset G_i$ and any countable subfamily $\mathscr{U}'$ of $\mathscr{U}$ we have

$$\Big(\mathop{\mathbf{P}}_{i=0}^{m} G_i\times X\times X\times\ldots\Big)\setminus\big(\bigcup\mathscr{U}'\big)\neq\emptyset .$$

Indeed, since $K=\mathop{\mathbf{P}}\limits_{n=0}^{m} K_n$ is a compact subset of X^ω, there exists $m\in\omega$, a sequence $(H_i)_{i=0}^{m}$ of open subsets of X such that $K_i\subset H_i$ and a finite subfamily $\mathscr{V}$ of $\mathscr{U}$ which covers $\mathop{\mathbf{P}}\limits_{i=0}^{m} H_i\times X\times X\times\ldots$ The last fact contradicts the property of $(K_n)_{n=0}^{\infty}$. We define the sequence $(K_n)_{n=0}^{\infty}$ by induction. Since X satisfies $(*)$, there exists $K_0\in\mathscr{K}(X)$ such that, for every countable subfamily $\mathscr{U}'$ of $\mathscr{U}$ and every $K_0\subset G_0\in\mathscr{G}(X)$, $(G_0\times X\times X\times\ldots)\setminus\bigcup\mathscr{U}'$ is not empty. If $K_0,\ldots,K_n$ are defined then there is $K_{n+1}\in\mathscr{K}(X)$ such that for every sequence $(G_i)_{i=0}^{n+1}$ of G_δ-subsets of X satisfying $K_i\subset G_i$ and every countable $\mathscr{U}'$ of $\mathscr{U}$ we have

$$\Big(\mathop{\mathbf{P}}_{i=0}^{n+1} G_i\times X\times X\times\ldots\Big)\setminus\big(\bigcup\mathscr{U}'\big)\neq\emptyset .$$

If K_{n+1} does not exist then using $(*)$ we would find a sequence $(G_i)_{i=0}^{n}$ of G_δ-subsets of X such that $\mathop{\mathbf{P}}\limits_{i=0}^{n} K_i\subset\mathop{\mathbf{P}}\limits_{i=0}^{n} G_i$ and a countable subfamily $\mathscr{U}'$ of $\mathscr{U}$ covering $\mathop{\mathbf{P}}\limits_{i=0}^{n} G_i\times X\times X\times\ldots$, contradicting the property of $(K_i)_{i=0}^{n}$.

Now let us pass to

Proof of Theorem 1. Let us assume that X satifies $(*)$. We shall prove that X belongs to $\mathscr{L}$. In the proof of this implication (CH) is not needed. Let Y be an arbitrary Lindelöf space and $\mathscr{U}$ an open cover of $Y\times X$. Since Y is a Lindelöf space, we can find for every compact set K in X a G_δ-subset $f(K)$ of X with $K\subset f(K)$ and a countable family $\mathscr{U}(K)\subset\mathscr{U}$ such that $Y\times f(K)\subset\bigcup\mathscr{U}(K)$. By $(*)$ there is a countable set $\mathscr{S}\subset\mathscr{K}(X)$ such that $\bigcup\{f(S)\colon S\in\mathscr{S}\}=X$. From the last equation it follows that $\mathscr{U}'=\bigcup\{\mathscr{U}(S)\colon S\in\mathscr{S}\}$ is a countable subvover of $\mathscr{U}$.

Proof of (a $\to$ b). Since the weight of X is not greater than ω_1, we may assume that X is a subspace of I^{ω_1}. Let us suppose that the implication does not hold. This means that X belongs to $\mathscr{L}$ and there exists $f\colon\mathscr{K}(X)\to\mathscr{G}(X)$ such that for every $K\in\mathscr{K}(X)$ and every countable subfamily $\mathscr{S}$ of $\mathscr{K}(X)$ we have $K\subset f(K)$ and the family $\{f(S)\colon S\in\mathscr{S}\}$ does not cover X. Without loss of generality we can assume that for every $K\in\mathscr{K}(X)$ there is $\alpha<\omega_1$ such that $f(K)=r_\alpha^{-1}(r_\alpha(K))$, where $r_\alpha=p_\alpha|X$. From (CH) and the form of f it follows that $|\{f(K)\colon K\in\mathscr{K}(X)\}|\leqslant\omega_1$; so let us put

$$\{f(K)\colon K\in\mathscr{K}(X)\}=\{G_\alpha\colon \alpha<\omega_1\}.$$

In the sequel we shall need the following

LEMMA 2 (CH). *If Z is a metric element of $\mathscr{L}$ then Z is σ-compact.*

It seems to me that Lemma 2 is well known. For the proof see [A], Proposition 1.

Since $X\in\mathscr{L}$ and $\mathscr{L}$ is closed with respect to continuous images and by Lemma 2 we infer that for every $\alpha<\omega_1$ $p_\alpha(X)=\bigcup\{F_{\alpha n}\colon n\in\omega\}$, where $F_{\alpha n}$ is a compact subset of I^α.

In order to finish the proof of the implication it is enough to define $A\subset I^{\omega_1}\setminus X$ of cardinality ω_1 and to endow $Y=X\cup A$ with a topology such that

(1) X is a set of isolated points,
(2) every $a\in A$ has a countable base of neighbourhoods in Y,
(3) A is a Lindelöf subspace of Y,
(4) A is not a G_δ-subset of Y,
(5) $\{(x,x)\colon x\in X\}$ is a closed subset of $X\times Y$.

Indeed, from (2), (3) and (CH) it follows that there is a family $\{H_\alpha\colon H_\alpha$ is open in Y and $\alpha<\omega_1\}$ such that for every open $A\subset H$ in Y there is $\alpha<\omega_1$ with $H_\alpha\subset H$. By (4) we infer that there is $P=\{x_\alpha\colon \alpha<\omega_1\}\subset X$ such that $x_\alpha\neq x_\beta$ if $\alpha\neq\beta$ and $x_\alpha\in\bigcap\{H_\lambda\colon\lambda\leqslant\alpha\}\cap X$.

Observe that $Y'=P\cup A$ is a Lindelöf space. To prove this it is enough to see that if $A\subset H$ and H is open in Y' then $Y'\setminus H$ is countable. Since $\{(x,x)\colon x\in P\}$ is a discrete uncountable and closed subset of $X\times Y'$, we infer that the product $X\times Y'$ is not a Lindelöf space, contradicting the assumption that X belongs to $\mathscr{L}$.

Now let us pass to the construction of A and Y.

We shall need more of notation. If $\alpha<\beta\leqslant\omega_1$ and $z\in I^\beta$ then let $z|\alpha$ be the restriction of z to α and

$$\mathscr{D}_\alpha=[D\subset I^{\omega_1}\colon \text{there are } n\in\omega,\ \beta_i\leqslant\alpha,\ n_i\in\omega, \text{ for } i\leqslant n,\ U\in\mathscr{B}_\alpha \text{ and } D=\bigcap\{p_{\beta_i}^{-1}(F_{\beta_i n_i})\cap p_\alpha^{-1}(U)\colon i\leqslant n\}.$$

Set for $\alpha\in\mathrm{Lim}$

$$C_\alpha=\{c\in I^\alpha\setminus p_\alpha(X)\colon \text{for every } \beta<\alpha\ c|\beta\in p_\beta(X)\}\setminus\{c\in I^\alpha\colon \text{there are } \beta \text{ and } \beta' \text{ less than } \alpha \text{ and } K\in\mathscr{K}(X) \text{ such that } f(K)=G_\beta=r_{\beta'}^{-1}r_{\beta'}(K) \text{ and } c|\beta'\in r_{\beta'}(G_\beta)\}.$$

If C_α is not empty and $c\in C_\alpha$ then $a(c)\in I^{\omega_1}$ is defined by

$$a(c)(\lambda)=\begin{cases} c(\lambda), & \text{if } \lambda<\alpha,\\ 0, & \text{otherwise}.\end{cases}$$

Put $A_\alpha=\{a(c)\colon c\in C_\alpha\}$ and $A=\bigcup\{A_\alpha\colon C_\alpha\neq\emptyset \text{ and } \alpha\in\mathrm{Lim}\}$.

In the sequel we shall use

LEMMA 3. $\Lambda = \{\alpha \in \mathrm{Lim}\colon C_\alpha \neq \emptyset\}$ *is unbounded.*

In the proof of Lemma 3 we shall need the notion of a big set. We say that $D \subset X$ is a big set if, for every $\alpha < \omega_1$, $D \setminus \bigcup \{G_\beta\colon \beta < \alpha\}$ is not empty. Observe that X is a big set by assumption and if $D = \bigcup \{D_n\colon n < \omega\}$ is a big set then there is an $n \in \omega$ such that D_n is a big set. The last observation will be called the countable union argument.

We shall also need

LEMMA 4. *If D is a big set then there is an ordinal $\gamma(D) < \omega_1$ such that, for every $\gamma(D) < \alpha < \omega_1$, $\alpha' < \omega_1$ and any compact subset K of*

$$p_\alpha(X),\ D \setminus (r_\alpha^{-1}(K) \cup \bigcup \{G_\beta\colon \beta < \alpha'\})$$

is not empty.

Proof of Lemma 4. Suppose not. Then there are an unbounded set $\Lambda' \subset \omega_1$, a compact subset K_α of $p_\alpha(X)$ and an ordinal $\beta(\alpha) < \omega_1$, for $\alpha \in \Lambda'$, such that

$$D \subset r_\alpha^{-1}(K_\alpha) \cup \bigcup \{G_{\beta'}\colon \beta' < \beta(\alpha)\} \tag{6}$$

or equivalently

$$D \setminus (\bigcup \{G_{\beta'}\colon \beta' < \beta(\alpha)\}) \subset r_\alpha^{-1}(K_\alpha)\,. \tag{7}$$

Since D is a big set, the family $\mathscr{F} = \{r_\alpha^{-1}(K_\alpha)\colon \alpha \in \Lambda'\}$ has the countable intersection property. Put $K = \bigcap \{\overline{F}^{I^{\omega_1}}\colon F \in \mathscr{F}\}$. Then K is a compact subset of X. To prove this it is enough to observe that for every $y \in K$ and $\alpha < \omega_1$ there is $x \in X$ such that $y|\alpha = x|\alpha$ and to apply the Lindelöf property of X. Since K is a compact subset of X, so there are $\alpha_1, \alpha_2, \alpha_3$ less than ω_1 such that $f(K) = G_{\alpha_1} = r_{\alpha_2}^{-1} r_{\alpha_2}(K)$ and for every $\alpha_3 < \alpha$

$$\begin{aligned} r_{\alpha_2}(\bigcap \{r_{\alpha'}^{-1}(K_{\alpha'})\colon \alpha' < \alpha \text{ and } \alpha' \in \Lambda'\} \subset \\ p_{\alpha_2}(\bigcap \{\overline{r_{\alpha'}^{-1}(K_{\alpha'})}^{I^{\omega_1}}\colon \alpha' < \alpha \text{ and } \alpha' \in \Lambda'\} \subset p_{\alpha_2}(K) = r_{\alpha_2}(K)\,, \end{aligned} \tag{8}$$

and consequently

$$\bigcap \{r_{\alpha_1}^{-1}(K_{\alpha'})\colon \alpha' < \alpha \text{ and } \alpha' \in \Lambda'\} \subset r_{\alpha_2}^{-1} r_{\alpha_2}(K) = G_{\alpha_1}\,. \tag{9}$$

From (7) and (9) it follows that $D \setminus \bigcup \bigcup \{G_{\beta'}\colon \beta' < \beta(\alpha') \text{ and } \alpha' < \alpha\} \subset G_{\alpha_1}$; so $D \subset \bigcup \bigcup \{G_{\beta'}\colon \beta' < \beta(\alpha') \text{ and } \alpha' < \alpha\} \cup G_{\alpha_1}$, contradicting the assumption that D is a big set.

Proof of Lemma 3. Let λ_0 be an arbitrary countable ordinal number. We shall show that there exists $\lambda_0 < \lambda$ such that C_λ is not empty. Since X is a big set, there exists $k_0 \in \omega$ such that $r_{\lambda_0}^{-1}(F_{\lambda_0 k_0})$ is a big set. Put $\mathscr{I}_0 = \{r_{\lambda_0}^{-1}(F_{\lambda_0 k_0})\}$, $\lambda_1 = \gamma(r_{\lambda_0}^{-1}(F_{\lambda_0 k_0})) + \lambda_0 + \omega$ and $\mathscr{I}_1 = \{D \subset X\colon D$ is a big set and there is $D' \in \mathscr{D}_{\lambda_1}$ such that $D = r_{\lambda_0}^{-1}(F_{\lambda_0 k_0}) \cap D'\}$. Since $\mathscr{D}_{\lambda_1}$ is a countable family, $r_{\lambda_0}^{-1}(F_{\lambda_0 k_0})$ is a big set contained in $\cup \mathscr{D}_{\lambda_1}$ and by the countable union argument we infer that there

is a set $D' \in \mathscr{D}_{\lambda_1}$ such that $D = r_{\lambda_0}^{-1}(F_{\lambda_0 k_0}) \cap D'$ is a big set and we conclude that $\mathscr{I}_1$ is not empty. If a countable family $\mathscr{I}_n$ of big sets and $\lambda_n < \omega_1$ are defined, we put

$$\lambda_{n+1} = \sup\{\gamma(D)\colon D \in \mathscr{I}_n\} + \lambda_n + \omega$$

and

$$\mathscr{I}_{n+1} = \{D \subset X\colon D \text{ is a big set and there is } D' \in D_{\lambda_{n+1}} \text{ such that } D = r_{\lambda_0}^{-1}(F_{\lambda_0 k_0}) \cap D'\}\,.$$

Arguing in the same way as in the case of $\mathscr{I}_1$ we show that $\mathscr{I}_{n+1}$ is not empty. Put $\lambda = \sup\{\lambda_n\colon n \in \omega\}$. We shall prove that C_λ is not empty. Let us put, for $n \in \omega$, $\{G_\beta\colon \beta < \lambda_n$ and there is $\beta' < \lambda_n$ and $K \in \mathscr{K}(X)$ such that $G_\beta = r_{\beta'}^{-1} r_{\beta'}(K)\} = \{Z_{nt}\colon t \in \omega\} = \mathscr{Z}_n$. Put $D_0 = r_{\lambda_0}^{-1}(F_{\lambda_0 k_0})$ and $s_0 = 0$. Let us assume that $s_n, k_n \in \omega$ and $D_n \in \mathscr{I}_{s_n}$ are defined in such a way that

$$D_n \subset r_{\lambda_{s_n}}^{-1}(F_{\lambda_{s_n} k_n}) \tag{10}$$

and

$$\overline{D_n}^{I^{\omega_1}} \cap (\bigcup \{Z_{rt}\colon r, t < n\} \cup r_\lambda^{-1}(\bigcup \{F_{\lambda t}\colon t < n\})) = \emptyset \quad \text{for } 0 < n\,. \tag{11}$$

Since $E = \bigcup \{Z_{rt}\colon r, t < n+1\} \cup r_\lambda^{-1}(\bigcup \{F_{\lambda t}\colon t < n+1\})$ is a closed subset of X determined by its projection onto λ, it means that $x \in E$ if and only if $x|\lambda \in r_\lambda(E)$; λ is a limit ordinal number, D_n is a big set, $\gamma(D_n) \leqslant \lambda_{s_{n+1}} < \lambda$; thus, by the countable union argument, there is an $s_n < s_{n+1}$ and $V \in \mathscr{B}_{\lambda_{s_{n+1}}}$ such that

$$p_{\lambda_{s_{n+1}}}^{-1}(\overline{V}^{I^{\lambda_{s_{n+1}}}}) \cap E = \emptyset \quad \text{and} \quad D'_{n+1} = D_n \cap r_{\lambda_{s_{n+1}}}^{-1}(V)$$

is a big set.

Applying the countable union argument to D'_{n+1} we can find $k_{n+1} \in \omega$ such that

$$D_{n+1} = D'_{n+1} \cap r_{\lambda_{s_{n+1}}}^{-1}(F_{\lambda_{s_{n+1}} k_{n+1}})$$

is a big set. It is easy to see that $D_{n+1} \in \mathscr{I}_{s_{n+1}}$ and the conditions (10) and (11) ar satisfied for $n+1$. Put

$$K_n = \overline{r_\lambda(D_n)}^{I^\lambda}\,.$$

By (10) we infer that

$$K_n \subset p_\lambda(p_{\lambda_{s_n}}^{-1}(F_{\lambda_{s_n} k_n})) \quad \text{for } n \in \omega\,. \tag{12}$$

Since $K_{n+1} \subset K_n$ and K_n is a compact subset of I^λ, we infer that $K = \bigcap \{K_n\colon n \in \omega\}$ is not empty. From $\lim_{n \to \omega} \lambda_n = \lambda$, (12) and (11) it follows that $K \subset C_\lambda$. Hence C_λ is not empty and we conclude that Λ is unbounded.

Now we are in position to define the topology of Y. If $y \in X$ then y is isolated in Y. Let us assume that $\alpha \in \Lambda$ and $a \in A_\alpha$. Then the base at a, denoted by $\mathscr{B}(a, Y)$, consists of all sets containing the point a of the form

$$B = \bigcap \{p_{\beta_i}^{-1}(F_{\beta_i n_i}) \cap U \cap Y\colon i \leqslant k\}\,,$$

where $k \in \omega$, $\sup\{\beta_i\colon i \leqslant k\} = \alpha' < \alpha$ and $U = p_{\alpha'}^{-1}(V)$ for $V \in \mathscr{B}_{\alpha'}$.

Since there is i_0 such that $\beta_{i_0} = \alpha'$, we infer that $B \subset p_{\alpha'}^{-1}(F_{\alpha' n_{i_0}})$ and conclude, applying $\alpha'|\gamma \notin p_\gamma(X)$ for $\gamma \leqslant \alpha'$ and $a' \in A_\gamma$, that

$$B \cap (\bigcup \{A_\lambda \colon \lambda \leqslant \alpha'\}) = \emptyset .$$

From the last equation it follows that the topology is well defined. It is easy to see that Y is a regular first countable space and that (5) holds. By Lemma 3 and (CH) the cardinality of A is equal to ω_1. If $a \in A$ and $B \in \mathscr{B}(a, Y)$, we put

$$\theta(B) = \inf\{\alpha \colon \text{there is } B' \in \mathscr{D}_\alpha \text{ and } B' = B\} .$$

Observe that

(13) $\quad$ if $a \in A$, $a \in A_\alpha$ and $B \in \mathscr{B}(a, Y)$ then $\theta(B) < \alpha$.

In order to finish the proof of Theorem 1 it is enough to show that (3) and (4) hold.

Proof of (3). Let $\mathscr{H}$ be an open (in Y) family which covers A. Without loss of generality we can assume that $\mathscr{H} \subset \bigcup \{\mathscr{B}(a, Y) \colon a \in A\}$. Then $\mathscr{H} = \bigcup \{\mathscr{H}_\alpha \colon \alpha \in \mathrm{Lim}\}$, where $\mathscr{H}_\alpha = \{H \in \mathscr{H} \colon \theta(H) < \alpha\}$. Observe that

(14) $\quad \mathscr{H}_\alpha$ is countable and $\bigcup \{\mathscr{H}_\beta \colon \beta < \alpha\} \subset \mathscr{H}_\alpha \quad$ for $\alpha \in \mathrm{Lim}$.

We claim that

(15) $\quad$ if $\alpha < \lambda$, $\lambda \in A$, $a \in A_\lambda$, $B \in \mathscr{B}(a, Y)$ and $\alpha \leqslant \theta(B)$ then $A_\alpha \cap B = \emptyset$

Indeed, $B = \bigcap \{p_{\beta_i}^{-1}(F_{\beta_i n_i}) \cap U \cap Y \colon i \leqslant k\}$, where $k \in \omega$, $\sup\{\beta_i, i \leqslant k\} = \beta' < \lambda$ and $U = p_{\beta'}^{-1}(V)$, where $V \in \mathscr{B}_{\beta'}$. There is an i_0 such that $\beta_{i_0} = \beta'$. Hence $B \subset p_{\beta'}^{-1}(F_{\beta' n_{i_0}})$. By definition, $\alpha \leqslant \theta(B) \leqslant \beta'$. Since $p_{\beta'}(X) \cap p_{\beta'}(\bigcup \{A_\gamma \colon \gamma \leqslant \beta'\})$ is empty, we have $B \cap (\bigcup \{A_\gamma \colon \gamma \leqslant \beta'\}) = \emptyset = B \cap A_\alpha$.

From (13), (14) and (15) it follows that

(16) $\quad$ for every $\alpha \in A \quad \bigcup \{A_\beta \colon \beta \leqslant \alpha\} \subset (\bigcup \mathscr{H}_\alpha) \setminus (\bigcup \bigcup (\{\mathscr{H}_\lambda \setminus \mathscr{H}_\alpha \colon \alpha < \lambda\}))$.

In order to show that $\mathscr{H}$ contains a countable subfamily which covers A, it is enough to prove

LEMMA 5. *There is δ_1 such that $X \setminus \bigcup \mathscr{H}_{\delta_1} \subset \bigcup \{G_\beta \colon \beta < \delta_1\}$.*

Indeed, assume that Lemma 5 holds. Then for every $\beta < \delta_1$ there exist $\alpha_\beta < \omega_1$ and a compact set K_β in X such that $r_{\alpha_\beta}^{-1} r_{\alpha_\beta}(K_\beta) = G_\beta$. Put

$$\delta' = \max\{\sup\{\alpha_\beta \colon \beta < \delta_1\}, \delta_1\}$$

and let $\delta' < \delta$ and $a \in A_\delta$. Since $a|\alpha_\beta \notin r_{\alpha_\beta}(G_\beta)$ for every $\beta < \delta_1$ (see the definition of C_δ and A_δ), we conclude that

(17) $$a|\delta' \notin r_{\delta'}(\bigcup \{G_\beta \colon \beta < \delta'\}) .$$

From (17), $a|\delta' \in r_{\delta'}(X)$ and Lemma 5 it follows that $a|\delta' \in p_{\delta'}((\bigcup \mathscr{H}_{\delta_1}) \cap X)$. Since $(p_{\delta'}^{-1} p_{\delta'}(\bigcup \mathscr{H}_{\delta_1})) \cap Y = \bigcup \mathscr{H}_{\delta_1}$, we infer that $a \in \bigcup \mathscr{H}_{\delta_1}$. From (16) and (14) it follows that $\mathscr{H}_{\delta'}$ covers A.

Proof of Lemma 5. Suppose that Lemma 5 does not hold. Then, for every $\delta < \omega_1$, $X \setminus (\bigcup \mathscr{H}_\delta \cup \bigcup \{G_\beta \colon \beta < \delta\})$ is not empty. In the sequel we shall need the notation of a $\mathscr{H}$-big set. We say that $D \subset X$ is a $\mathscr{H}$-big set if for every $\alpha < \omega_1$

$$D \setminus (\bigcup \{G_\beta \colon \beta < \alpha\} \cup \bigcup \mathscr{H}_\alpha) \text{ is not empty} .$$

Note that X is a $\mathscr{H}$-big set by assumption and if $D = \bigcup \{D_n \colon n \in \omega\}$ is a $\mathscr{H}$-big set then there is an $n \in \omega$ such that D_n is a $\mathscr{H}$-big set. The last observation will be called, as in the case of big sets, the countable union argument.

In the sequel we shall need

LEMMA 6. *If D is a $\mathscr{H}$-big set then there is $\gamma_{\mathscr{H}}(D) < \omega_1$ such that for every $\gamma_{\mathscr{H}}(D) < \alpha < \omega_1$, $\alpha' < \omega_1$, and any compact subset K of $p_\alpha(X)$,*

$$D \setminus (r_\alpha^{-1}(K) \cup (\bigcup \mathscr{H}_{\alpha'} \cup \bigcup \{G_\beta \colon \beta < \alpha'\})) \text{ is not empty} .$$

We omit the proof of Lemma 6 because it involves exactly the same reasoning as the proof of Lemma 4.

In order to prove Lemma 5 it is enough to show

LEMMA 7. *There exists $\lambda' \in \mathrm{Lim}$ such that $C'_{\lambda'} = C_{\lambda'} \setminus p_{\lambda'}(\bigcup \mathscr{H}_{\lambda'})$ is not empty.*

Indeed, if Lemma 7 holds and $c \in C'_{\lambda'}$ then $a(c) \notin \bigcup \mathscr{H}_{\lambda'}$, because $a(c)|\lambda' = c|\lambda'$. From (16), for $\alpha = \lambda'$ and (14) it follows that $a(c) \notin \bigcup (\mathscr{H} \setminus \mathscr{H}_{\lambda'})$, contrary to the assumption that $\mathscr{H}$ covers A.

Proof of Lemma 7. The proof of Lemma 7 is very similar to the proof of Lemma 3.

If $H = \bigcap \{p_{\beta_i}^{-1}(F_{\beta_i n_i}) \cap U \cap Y \colon i \leqslant k\} \in \mathscr{H}_\alpha$, where $\sup\{\beta_i \colon i \leqslant k\} = \alpha' < \alpha$, $k \in \omega$, $U = p_{\alpha'}^{-1}(V)$ and $V \in \mathscr{B}_{\alpha'}$ then there is a family $\mathscr{T} = \{T_n \colon n \in \omega\}$ of compact subsets in $I^{\alpha'}$ such that $V = \bigcup \{T_n \colon n \in \omega\}$. Put

$$\mathscr{E}(H) = \{\bigcap \{p_{\beta_i}^{-1}(F_{\beta_i n_i}) \cap p_{\alpha'}^{-1}(T_n) \cap Y \colon i \leqslant k\} \colon n \in \omega\}$$

and $\mathscr{E}_\alpha = \bigcup \{\mathscr{E}(H) \colon H \in \mathscr{H}_\alpha\}$. Since $\mathscr{H}_\alpha$ is countable, we conclude that $\mathscr{E}_\alpha$ is countable. Put $\lambda'_0 = \omega$. Since X is an $\mathscr{H}$-big set, there exists k_0 such that $r_\omega^{-1}(F_{\omega k_0})$ is an $\mathscr{H}$-big set. Put $\lambda'_1 = \gamma_{\mathscr{H}}(r^{-1}(F_{\omega k_0})) + \omega$. Similarly to the definition of $\mathscr{F}_n$ and λ_n in the proof of Lemma 3, we define $\mathscr{F}'_n$ and λ'_n. It is just enough to replace the words big sets by $\mathscr{H}$-big sets and use Lemma 6 instead of Lemma 4. Put $\lambda' = \sup\{\lambda'_n \colon n \in \omega\}$. The proof of the fact that $C'_{\lambda'}$ is not empty is similar to proving that C_λ is not empty. It suffices to replace λ by λ' and $\mathscr{Z}_n$ by $\mathscr{Z}'_n = \{Z'_{nm} \colon m < \omega\} = \{G_\beta \colon \beta < \lambda'_n$ and there are $K \in \mathscr{K}(X)$ and $\beta' < \lambda'_n$ such that $G_\beta = r_{\beta'}^{-1} r_{\beta'}(K)\} \cup \mathscr{E}_{\lambda'_n}$, and to replace λ_n by λ'_n.

Proof of (4). Since A is a Lindelöf space, for every open U in Y containing A there is a set W such that $A \subset W \subset U$ and W is determined by some $\alpha < \omega_1$. It is easy to see that the same holds if U is a G_δ-subsets of Y. Since for every $\alpha < \omega_1$ there are $\alpha < \beta < \omega_1$, $a \in A$ and $x \in X$ such that $a|\beta = x|\beta$, we conclude that A is not a G_δ-subset of Y.

THEOREM 3 (CH). *If the weight of X is not greater than ω_1, $X \in \mathscr{L}$ and every compact subset of X is a G_δ-set then X is σ-compact.*

Proof. By Theorem 1, X satisfies (*). Put $f(K) = K$ for every $K \in \mathscr{K}(X)$. Then it follows from (*) that X is σ-compact.

THEOREM 4 (CH). *If the weight of X is not greater than ω_1, $X \in \mathscr{L}$ and X does not contain uncountable compact subsets then X with the topology induced by G_δ-subsets, with respect to the original topology, is a Lindelöf space* (see [N], for a related result).

Proof. It is enough to observe that if $F = \{x_n: n < \omega\}$ is a compact subset of X and $x_n \in G_n$ is a G_δ-subset of X for $n < \omega$ then there is a G_δ-subset H of X such that $F \subset H \subset \bigcup \{G_n: n \in \omega\}$.

Remark 1. Theorem 3 may be improved a little bit, namely the following statement is true if the weight of X is not greater than ω_1, $X \in \mathscr{L}$ and every compact subset of X is of the G_δ-type then X is σ-compact if and only if every metric element of $\mathscr{L}$ is σ-compact. Hint: Put $C_\alpha = \{c \in I^\alpha \setminus p_\alpha(X)$: for every $\beta < \alpha$ $c|\beta \in p_\beta(X)\}$, define big-sets as non-σ-compact sets and $P = \{x_\alpha: \alpha < \omega_1\} \subset X$ such that for every α there is $a_\alpha \in A$ satisfying $a_\alpha|_\alpha = x_\alpha|_\alpha$

Remark 2. It follows from Theorem 4 that X from [A] does not belong to $\mathscr{L}$ as an uncountable space without uncountable compact subsets in which every point is of the G_δ-type.

Let me finish this note with the following

QUESTION. Assume that (CH) holds and X is such that every closed subset of X of weight not greater than ω_1 satisfies (*). Does X necessarily satisfy (*)?

Remark 3. Positive answer to this question would yield a positive answer to Michael's conjecture.

References

[A] K. Alster, *On spaces whose product with every Lindelöf space is Lindelöf*, Coll. Math., to appear.
[E] R. Engelking, *General topology*, Warszawa (1977).
[N] N. Noble, *Products with closed projection* 11, Trans. Amer. Math. Soc. 160 (1971), 169–183.
[P] T. C. Przymusiński, *Products of normal spaces*, Handbook of Set-Theoretic Topology (1984), 781–826.

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Correction to: Adding a random or a Cohen real: topological consequences and the effect on Martin's axiom

by

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This paper appeared in Fundamenta Mathematicae 103 (1979), 47–60 pp. and Shelah has recently written to me that there is a serious problem with Theorem 5.3, p. 57. This states that if $MA_{\Sigma\text{-linked}}$ holds in a model M then it still holds in $M[x]$ where x is a Cohen or random real over M; and if $MA_{\Sigma\text{-centered}}$ holds in a model M then it still holds in $M[x]$ where x is a Cohen real over M. The statement about $MA_{\Sigma\text{-linked}}$ is false: Todorčević noticed that when x is Cohen the statement conflicts with a result of Shelah's that appears in his paper on taking the inaccessible away from Solovay's proof that all sets are Lebesgue measurable (Israel Journal of Mathematics 48 (1984) 1–47 pp.). Shelah then noticed that his result can be modified to show that the statement about $MA_{\Sigma\text{-linked}}$ is false when x is random. The problems with the proof of this false theorem are, in the Cohen case, that the auxilliary partial order Q^* relies on maximal finite antichains being able to decide nearly everything, when, in fact, they seldom do; in the random case Q^* was not carefully defined and, in fact, fails to be transitive.

On the other hand, the second part of Theorem 5.3 — if $MA_{\Sigma\text{-centered}}$ holds in M then it holds in $M[x]$ where x is Cohen over M — is true. Perhaps the easiest proof was noticed several years ago by Baumgartner and Tall, and is sketched here.

Reall that $MA_{\Sigma\text{-centered}}$ is equivalent to the statement $P(\mathbf{C})$: for every centered family $\mathscr{B}$ on ω of size less than $\mathbf{C}$ there is some infinite $A \subset \omega$ with $A \subset B$ mod finite for all $B \in \mathscr{B}$.

So assume $\mathring{\mathscr{B}} = \{\mathring{B}_i: i \in I\}$ is a Cohen forcing name for a centered family on ω of size less than $\mathbf{C}$. We may assume that $\mathring{\mathscr{B}}$ is forced to be closed under finite intersections. Let Q be the set of all triples $\langle s, t, \mathring{B}_i\rangle$ where s is a finite Cohen condition, t is a finite subset of ω, and $i \in I$. The order on Q is: $\langle s, t, \mathring{B}\rangle \leqslant \langle s', t', \mathring{B}'\rangle$ iff $s \subset s'$, $t \subset t'$, and $s \Vdash$ if $n \in t - t'$ then $n \in \mathring{B}'$. Q is easily seen to be Σ-centered and if G is Q-generic for the obvious dense sets and x is Cohen over M then $A = \bigcup \{t: \exists s \in x\, \exists \mathring{B}_i$ with $\langle s, t, \mathring{B}_i\rangle \in G\}$ is the required set.

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