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The Lévy continuity theorem for nuclear groups

by

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Abstract. Let G be an abelian topological group. The Lévy continuity theorem says that if G is an LCA group, then it has the following property (PL): a sequence of Radon probability measures on G is weakly convergent to a Radon probability measure μ if and only if the corresponding sequence of Fourier transforms is pointwise convergent to the Fourier transform of μ. Boulicaut [Bo] proved that every nuclear locally convex space G has the property (PL). In this paper we prove that the property (PL) is inherited by nuclear groups, a variety of abelian topological groups containing LCA groups and nuclear locally convex spaces, introduced in [B1].

1. Introduction. Let G be an LCA group and Γ the dual group. The Bochner theorem may be formulated in the following way:

(α) *Every continuous positive-definite function on G is the inverse Fourier transform of a (unique) finite positive Radon measure on* Γ.

This theorem can be extended to inverse limits and countable direct limits of LCA groups. It was also extended to some other classes of abelian topological groups: nuclear locally convex spaces (the Minlos theorem), Hausdorff quotient groups of such spaces (Yang [Y]), locally convex spaces over p-adic fields (Mądrecki [M]). Trying to give a common generalization of the corresponding results, the author introduced in [B1] the so-called nuclear groups, a variety of abelian topological groups containing LCA groups and nuclear locally convex spaces (the definition and basic properties of nuclear groups are given in Section 5 below). It was proved in [B1, (12.1)] that every nuclear group G satisfies an analogue of (α).

The Lévy continuity theorem may be formulated in the following way:

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(β) *A family of Radon probability measures on Γ is tight if and only if the corresponding family of inverse Fourier transforms is equicontinuous on G;*

(γ) *A sequence $(\mu_n)_{n=1}^{\infty}$ of Radon probability measures on G is weakly convergent to a Radon probability measure μ if and only if the corresponding sequence of Fourier transforms is pointwise convergent to the Fourier transform of μ.*

An analogue of (β) for nuclear groups was obtained in [B1, (12.5)]. Boulicaut [Bo] proved that every nuclear locally convex space G satisfies (γ). The aim of the present paper is to complete the picture by proving that every nuclear group G satisfies (γ) (Theorem 5.3 below). The main idea of the proof is similar to that of [Bo], with vector spaces replaced by their subgroups and quotient groups. Another proof of Theorem 5.3 will be given in [BT].

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2. Notation, terminology and preliminary lemmas. Let G be an abelian topological group (which we abbreviate to a.t. group). Following Hejcman [H], we say that a subset X of G is *bounded* if, for each neighbourhood U of zero in G, one can find a positive integer n and a finite subset F of G such that

$$X \subset F + U + \overset{n}{\ldots} + U.$$

For instance, every precompact subset is bounded. If G is locally compact, then X is bounded if and only if it is precompact. If G is a locally convex space, then X is bounded if and only if it is bounded in the usual sense, i.e. absorbed by every neighbourhood of zero.

By a *character* of G we mean a homomorphism of G into the multiplicative group of complex numbers with modulus 1. The value of a character χ at a point $g \in G$ will be denoted by $\chi(g)$ or $\langle g, \chi\rangle$. The group of all continuous characters of G is denoted by $G^\wedge$. It is is usually endowed with the compact-open topology, but we shall also consider other topologies on $G^\wedge$.

Let $\mathfrak{S}$ be a family of subsets of G which satisfies the following conditions:

(i) if $X \in \mathfrak{S}$, then $-X \in \mathfrak{S}$;
(ii) if $X \in \mathfrak{S}$ and $Y \subset X$, then $Y \in \mathfrak{S}$;
(iii) if $X, Y \in \mathfrak{S}$, then $X \cup Y \in \mathfrak{S}$;
(iv) if $X, Y \in \mathfrak{S}$, then $X + Y \in \mathfrak{S}$;
(v) all finite subsets belong to $\mathfrak{S}$

(such a family is sometimes called a *boundedness* on G). Typical examples are the families of finite, compact, precompact or bounded subsets. It is a standard fact that there exists a unique group topology on $G^\wedge$ for which the family of sets of the form

$$\{\chi \in G^\wedge : |1 - \chi(g)| < \varepsilon \text{ for each } g \in X\} \quad (\varepsilon > 0,\ X \in \mathfrak{S})$$

is a base at zero (only (iii) is needed here). We call it the *topology of uniform convergence on elements of* $\mathfrak{S}$ and denote by $\tau_{\mathfrak{S}}$. Condition (v) implies that $\tau_{\mathfrak{S}}$ is Hausdorff. Condition (iv) implies easily that the family of sets of the form

$$\{\chi \in G^\wedge : \operatorname{Re}\chi(g) \geq 0 \text{ for each } g \in X\} \quad (X \in \mathfrak{S})$$

is also a base at zero for $\tau_{\mathfrak{S}}$. The topology of uniform convergence on finite, compact or bounded subsets of G is called the *topology of pointwise*, *compact* or *bounded convergence*, respectively; the corresponding character groups will be denoted by $G_{\mathrm{p}}^\wedge$, $G_{\mathrm{c}}^\wedge$ and $G_{\mathrm{b}}^\wedge$. If G is Hausdorff, then the topology of compact convergence coincides with the compact-open topology on $G^\wedge$. Note that the identity mappings $G_{\mathrm{b}}^\wedge \to G_{\mathrm{c}}^\wedge \to G_{\mathrm{p}}^\wedge$ are continuous.

Let H be another a.t. group and $\psi : G \to H$ an algebraic homomorphism. We say that ψ is *bounded* if the image of every bounded subset of G is a bounded subset of H. We say that ψ is *bounding* if every bounded subset of H is the image of some bounded subset of G. If $\psi : G \to H$ is continuous, then it is bounded, which implies that the dual homomorphism $\psi^\wedge : H_{\mathrm{b}}^\wedge \to G_{\mathrm{b}}^\wedge$ given by $\psi^\wedge(\chi) = \chi \circ \psi$ is continuous.

Now, let H be a closed subgroup of G and $\psi : G \to G/H$ the canonical homomorphism. It is not hard to see that if G has some bounded neighbourhood of zero, then ψ is bounding. Such a situation occurs if, for instance, G is a normed space or an LCA group.

LEMMA 2.1. *Let G, H be a.t. groups and let $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ be the canonical projections.*

(a) *A subset X of $G \times H$ is bounded if and only if $\pi_G(X)$ and $\pi_H(X)$ are bounded subsets of G and H, respectively.*

(b) *The group $(G \times H)_{\mathrm{b}}^\wedge$ is canonically topologically isomorphic to $G_{\mathrm{b}}^\wedge \times H_{\mathrm{b}}^\wedge$.*

Proof. Part (a) is a direct consequence of the definitions. It is a standard fact that the formula

$$\langle (g, h), \sigma(\chi, \kappa)\rangle = \langle g, \chi\rangle \cdot \langle h, \kappa\rangle \quad (g \in G,\ h \in H,\ \chi \in G^\wedge,\ \kappa \in H^\wedge)$$

defines an algebraic isomorphism $\sigma : G^\wedge \times H^\wedge \to (G \times H)^\wedge$, and (a) implies that σ is a homeomorphism between $G_{\mathrm{b}}^\wedge \times H_{\mathrm{b}}^\wedge$ and $(G \times H)_{\mathrm{b}}^\wedge$. ■

By $\mathcal{M}(G)$ we denote the family of finite positive Radon measures on G, and $\mathcal{P}(G) \subset \mathcal{M}(G)$ is the family of Radon probability measures. By the *weak topology* on $\mathcal{M}(G)$ we mean the topology induced by all functions of the form

$$\mathcal{M}(G) \ni \mu \mapsto \int_G f(g)\,d\mu(g) \in \mathbb{C}$$

where f is a bounded continuous complex-valued function on G. If a net (μ_α) in $\mathcal{M}(G)$ is weakly convergent to $\mu \in \mathcal{M}(G)$, then we write $\mu_\alpha \xrightarrow{\mathrm{w}} \mu$. A family $\mathcal{S} \subset \mathcal{M}(G)$ is said to be *tight* if to each $\varepsilon > 0$ there corresponds a compact subset X of G such that $\mu(G \setminus X) \le \varepsilon$ for each $\mu \in \mathcal{S}$.

By the Fourier transform of a measure $\mu \in \mathcal{M}(G)$ we mean the p.d. (positive-definite) function $\mathcal{F}\mu : G^\wedge \to \mathbb{C}$ given by

$$\mathcal{F}\mu(\chi) = \int_G \chi(g)\,d\mu(g) \quad (\chi \in G^\wedge).$$

Since μ is a Radon measure, $\mathcal{F}\mu$ is continuous in the compact-open topology on $G^\wedge$.

Let (φ_α) be a net of p.d. functions on $G^\wedge$. If it is pointwise convergent to some function φ, then we write $\varphi_\alpha \xrightarrow{\mathrm{p}} \varphi$. If (μ_α) is a net in $\mathcal{M}(G)$ with $\mu_\alpha \xrightarrow{\mathrm{w}} \mu \in \mathcal{M}(G)$, then, by definition, $\mathcal{F}\mu_\alpha \xrightarrow{\mathrm{p}} \mathcal{F}\mu$. The converse, in general, is not true, even if nets are replaced by usual sequences.

Let τ be a group topology on $G^\wedge$ such that the functions $\chi \mapsto \chi(g)$, $g \in G$, are continuous, and let $\nu \in \mathcal{M}(G^\wedge_\tau)$. By the *inverse Fourier transform* of ν we mean the p.d. function $\mathcal{F}^{-1}\nu : G \to \mathbb{C}$ given by

$$\mathcal{F}^{-1}\nu(g) = \int_{G^\wedge} \overline{\chi(g)}\,d\nu(\chi) \quad (g \in G).$$

Let E be a topological vector space (all vector spaces occurring are assumed to be real). We may treat E as an (additive) abelian topological group. By E^* we denote the dual space of all continuous linear functionals on E, and E^*_{b} is the space E^* endowed with the topology of uniform convergence on bounded sets. If E is a normed space, then E^*_{b} is just the dual space with the norm topology.

LEMMA 2.2. *Let E be a topological vector space. Then the formula*

$$\langle x, \alpha(f)\rangle = \exp\{2\pi i f(x)\} \quad (x \in E,\ f \in E^*)$$

defines a topological isomorphism $\alpha : E^*_{\mathrm{b}} \to E^\wedge_{\mathrm{b}}$.

Proof. It is a standard fact that α is an algebraic isomorphism (see e.g. [HR, (23.32)]), and it is easy to see that α and α^{-1} are both continuous. ∎

Let X, Y be symmetric convex subsets of a vector space E. Suppose that X is *absorbed* by Y, i.e. that $X \subset tY$ for some $t > 0$ (we write $X \prec Y$). The *Kolmogorov diameters* of X with respect to Y are given by

$$d_k(X, Y) = \inf_M \inf\{t > 0 : X \subset tY + M\} \quad (k = 1, 2, \ldots)$$

where the first infimum is taken over all linear subspaces M of E with $\dim M < k$.

Let p be a seminorm on E. We set

$$B_p = \{x \in E : p(x) \le 1\}.$$

We say that p is a *pre-Hilbert* seminorm if

$$p(x+y)^2 + p(x-y)^2 = 2p(x)^2 + 2p(y)^2 \quad (x, y \in E).$$

Let E be a Hilbert space. By B_E we denote the closed unit ball of E. Let F be another Hilbert space and $T : E \to F$ a bounded linear operator. Then the formula $p(x) = \|Tx\|$, $x \in E$, defines a pre-Hilbert seminorm p on E. The *operator numbers* of T are given by

$$d_k(T) = d_k(T(B_E), B_F) = d_k(B_E, B_p) \quad (k = 1, 2, \ldots).$$

We say that T is *α-approximable*, $0 < \alpha < \infty$, if $\sum_{k=1}^\infty d_k(T)^\alpha < \infty$.

LEMMA 2.3. *Let p, q be pre-Hilbert seminorms on a vector space E, with $B_q \prec B_p$, such that $\sum_{k=1}^\infty d_k(B_q, B_p)^2 < \infty$. Let Q be an arbitrary subgroup of E and let ν be a Borel probability measure on $Q^\wedge_{\mathrm{p}}$ such that* $\operatorname{Re} \mathcal{F}^{-1}\nu(x) \ge 1 - \varepsilon$ *for each $x \in Q \cap B_p$, where $\varepsilon > 0$. Define*

$$Z = \{\chi \in Q^\wedge : \operatorname{Re}\chi(x) \ge 0 \textit{ for each } x \in Q \cap \tfrac14 B_q\}.$$

Then

$$\nu(Q^\wedge \setminus Z) \le 2\varepsilon + \sum_{k=1}^\infty d_k(B_q, B_p)^2.$$

This is a direct consequence of [B3, Lemma 3.4]. For details of the proof see [A, Cor. 22.8].

LEMMA 2.4. *Let p, q be pre-Hilbert seminorms on a vector space E, with $B_p \prec B_q$, such that $\sum_{k=1}^\infty d_k(B_p, B_q) < \infty$. Let Q be a subgroup of E and χ a character of Q such that* $\operatorname{Re}\chi(x) \ge 0$ *for each $x \in Q \cap B_q$. Then there exists a bounded linear functional f on E such that* $\exp\{2\pi i f(x)\} = \chi(x)$ *for each $x \in Q$, and*

$$\|f\| \le 4\sum_{k=1}^\infty d_k(B_p, B_q).$$

This is a consequence of [B2, Thm. 3.1(i)]. For details of the proof see [A, Lemma 19.13(ii)].

3. The property (PL). Let G be an a.t. group. Following [Bo], we say that G has the *property* (PL) if (γ) is satisfied, i.e. if $\mathcal{F}\mu_n \xrightarrow{\mathrm{p}} \mathcal{F}\mu$ implies that $\mu_n \xrightarrow{\mathrm{w}} \mu$, for any $\mu \in \mathcal{P}(G)$ and any sequence $(\mu_n)_{n=1}^\infty$ in $\mathcal{P}(G)$.

LEMMA 3.1. *Let G be an a.t. group with the property* (PL) *and let H be an arbitrary subgroup of G. Then H also has the property* (PL).

Proof. Let $\mu \in \mathcal{P}(H)$ and let $(\mu_n)_{n=1}^\infty$ be a sequence in $\mathcal{P}(H)$ such that $\mathcal{F}\mu_n \xrightarrow{\mathrm{p}} \mathcal{F}\mu$. Let $\iota : H \to G$ be the identity embedding and let $\mu' = \iota(\mu)$ and $\mu'_n = \iota(\mu_n)$, $n = 1, 2, \ldots$ Then $\mathcal{F}\mu' = \mathcal{F}\mu \circ \iota^\wedge$ and $\mathcal{F}\mu'_n = \mathcal{F}\mu_n \circ \iota^\wedge$ for every n, so that $\mathcal{F}\mu'_n \xrightarrow{\mathrm{p}} \mathcal{F}\mu'$. Since G has the property (PL), it follows that $\mu'_n \xrightarrow{\mathrm{w}} \mu'$, which means that $\mu_n \xrightarrow{\mathrm{w}} \mu$ (see Lemma 2.1 of [Bo]). ∎

Let $\pi : G \to H$ be a continuous homomorphism of a.t. groups. Consider the following two conditions:

(∗) if $\mu \in \mathcal{P}(G)$ and if $(\mu_n)_{n=1}^\infty$ is a sequence in $\mathcal{P}(G)$ such that $\mathcal{F}\mu_n \xrightarrow{\mathrm{p}} \mathcal{F}\mu$, then $\pi(\mu_n) \xrightarrow{\mathrm{w}} \pi(\mu)$;

(∗∗) if $\mathcal{S} \subset \mathcal{P}(G)$ is a family of measures such that the family $\{\mathcal{F}\mu\}_{\mu\in\mathcal{S}}$ is equicontinuous on $G_{\mathrm{b}}^\wedge$, then $\{\pi(\mu)\}_{\mu\in\mathcal{S}}$ is a tight family of measures on H.

If (∗) is satisfied, then we say that the homomorphism π has the *property* (PL). If (∗∗) is satisfied, then we say that π is *tightening*.

LEMMA 3.2. *Let $(I, \leq)$ be a directed set and let G be the limit of an inverse system $\{G_i, \pi_{ij}, I\}$ of a.t. groups and continuous homomorphisms. Suppose that to each $i \in I$ there corresponds some $j \geq i$ such that the homomorphism $\pi_{ij} : G_j \to G_i$ has the property* (PL). *Then the group G has the property* (PL).

Proof. Let $\pi_i : G \to G_i$, $i \in I$, be the canonical homomorphisms. Let $\mu \in \mathcal{P}(G)$ and let $(\mu_n)_{n=1}^\infty$ be a sequence in $\mathcal{P}(G)$ such that $\mathcal{F}\mu_n \xrightarrow{\mathrm{p}} \mathcal{F}\mu$. Fix an arbitrary $i \in I$ and choose $j \geq i$ such that $\pi_{ij} : G_j \to G_i$ has the property (PL). We have $\mathcal{F}\pi_j(\mu) = \mathcal{F}\mu \circ \pi_j^\wedge$ and $\mathcal{F}\pi_j(\mu_n) = \mathcal{F}\mu_n \circ \pi_j^\wedge$ for every n, which means that $\mathcal{F}\pi_j(\mu_n) \xrightarrow{\mathrm{p}} \mathcal{F}\pi_j(\mu)$. Therefore $\pi_{ij}\pi_j(\mu_n) \xrightarrow{\mathrm{w}} \pi_{ij}\pi_j(\mu)$, i.e. $\pi_i(\mu_n) \xrightarrow{\mathrm{w}} \pi_i(\mu)$. Since $i \in I$ was arbitrary, it follows that $\mu_n \xrightarrow{\mathrm{w}} \mu$ (see Lemma 2.3 of [Bo]). ∎

LEMMA 3.3. *Let φ be a p.d. function on a (not necessarily abelian) group G, with $\varphi(0) = 1$. Let $\varepsilon \in (0, 1)$ and let $g_1, g_2 \in G$ be such that* $\operatorname{Re}\varphi(g_i) \geq 1 - \varepsilon$, $i = 1, 2$. *Then* $\operatorname{Re}\varphi(g_1 - g_2) \geq 1 - 4\varepsilon + 2\varepsilon^2 > 1 - 4\varepsilon$.

This follows easily from elementary properties of p.d. functions.

The next proposition may be treated as an analogue of the equicontinuity principle for p.d. functions.

PROPOSITION 3.4. *Let G be a (not necessarily abelian) Čech-complete group (or even a Baire group) and let $(\varphi_n)_{n=1}^\infty$ be a pointwise convergent sequence of p.d. functions on G such that the limit function is continuous. Then the sequence (φ_n) is equicontinuous.*

Proof. Denote the limit function by φ. We may assume that $\varphi(0) = \varphi_n(0) = 1$. Fix $\varepsilon \in (0, 1)$ and consider the closed subsets

$$X_m = \bigcap_{n\geq m} \{g \in G : \operatorname{Re}\varphi_n(g) \geq 1 - \varepsilon\} \quad (m = 1, 2, \ldots).$$

Since $\varphi_n \xrightarrow{\mathrm{p}} \varphi$, it follows that

$$V := \{g \in G : \operatorname{Re}\varphi(g) \geq 1 - \varepsilon/2\} \subset \bigcup_{m=1}^\infty X_m.$$

We have $\operatorname{Int} V \neq \emptyset$ because φ is continuous. Now, a standard category argument shows that there is an index m such that $U := \operatorname{Int} X_m \neq \emptyset$. Then $U - U$ is a neighbourhood of zero in G and, by the previous lemma, we have $\operatorname{Re}\varphi_n(g) > 1 - 4\varepsilon$ for every $g \in U - U$ and $n \geq m$. ∎

An a.t. group G is said to be *dually separated* if $G^\wedge$ separates the points of G. If K is a subgroup of a topological vector space E, then it follows easily from Lemma 2.2 that E/K is a dually separated group if and only if K is weakly closed in E (cf. [B1, (2.5)]).

LEMMA 3.5. *Let G be a dually separated group and let $\mu_1, \mu_2 \in \mathcal{P}(G)$. If $\mathcal{F}\mu_1 = \mathcal{F}\mu_2$, then $\mu_1 = \mu_2$.*

This is a standard fact. See e.g. Theorem 2.2 of Chapter IV in [VTCh].

LEMMA 3.6. *Let G be a dually separated group. Let $\mu \in \mathcal{P}(G)$ and let (μ_α) be a net in $\mathcal{P}(G)$ such that $\mathcal{F}\mu_\alpha \xrightarrow{\mathrm{p}} \mathcal{F}\mu$. If the family $\{\mu_\alpha\}$ is tight, then $\mu_\alpha \xrightarrow{\mathrm{w}} \mu$.*

Proof. Suppose the contrary, i.e. that $\mu_\alpha \not\xrightarrow{\mathrm{w}} \mu$. Then there is a finer net (μ'_β) for which μ is not a weak cluster point. Being tight, the family $\{\mu_\alpha\}$ is weakly relatively compact in $\mathcal{P}(G)$ (see e.g. Theorem 3.6 of Chapter I in [VTCh]). So, there is a net (μ''_γ) finer than (μ'_β) which converges to some $\mu'' \in \mathcal{P}(G)$. We have $\mu'' \neq \mu$, otherwise μ would be a cluster point of (μ'_β). Then the net $(\mathcal{F}\mu'_\gamma)$ is pointwise convergent to $\mathcal{F}\mu$ and $\mathcal{F}\mu''$; hence $\mathcal{F}\mu = \mathcal{F}\mu''$. By Lemma 3.5, we obtain $\mu = \mu''$, which is a contradiction. ∎

LEMMA 3.7. *Let $\pi : G \to H$ be a continuous homomorphism of a.t. groups. Suppose that G is dually separated and $G_{\mathrm{b}}^\wedge$ is Čech-complete. If π is tightening, then it has the property* (PL).

Proof. Let $\mu \in \mathcal{P}(G)$ and let $(\mu_n)_{n=1}^\infty$ be a sequence in $\mathcal{P}(G)$ with $\mathcal{F}\mu_n \xrightarrow{\mathrm{p}} \mathcal{F}\mu$. By Lemma 3.4, $\{\mathcal{F}\mu_n\}_{n=1}^\infty$ is an equicontinuous family of functions on $G_{\mathrm{b}}^\wedge$ (the function $\mathcal{F}\mu$ is continuous on $G_{\mathrm{c}}^\wedge$ and hence on $G_{\mathrm{b}}^\wedge$). If π is tightening, then $\{\pi(\mu_n)\}_{n=1}^\infty$ is a tight family of measures on H. We have $\mathcal{F}\pi(\mu) = \mathcal{F}\mu \circ \pi^\wedge$ and $\mathcal{F}\pi(\mu_n) = \mathcal{F}\mu_n \circ \pi^\wedge$ for every n, which implies that $\mathcal{F}\pi(\mu_n) \xrightarrow{\mathrm{p}} \mathcal{F}\pi(\mu)$. Hence, by Lemma 3.6, $\pi(\mu_n) \xrightarrow{\mathrm{w}} \pi(\mu)$. ∎

LEMMA 3.8. *Let G, D, H be a.t. groups with D discrete. Identify G with the open subgroup $G \times \{0\}$ of $G \times D$. Let $\pi : G \times D \to H$ be a continuous homomorphism such that the restriction $\pi_{|G} : G \to H$ is tightening. Then π is also tightening.*

Proof. Let $\mathcal{S} \subset \mathcal{P}(G \times D)$ be a family of measures such that $\{\mathcal{F}\mu\}_{\mu \in \mathcal{S}}$ is an equicontinuous family of functions on $(G \times D)^\wedge_{\mathrm{b}}$. For each $d \in D$, let $G_d = G \times \{d\}$ be the corresponding coset modulo G. For $\mu \in \mathcal{S}$ and $d \in D$, let $\mu_d \in \mathcal{M}(G)$ be the measure given by $\mu_d(A) = \mu(A \cap G_d)$ for Borel subsets $A \subset G$ (i.e. μ_d is the restriction of μ to G_d). Then we may write $\mu = \sum_{d \in D} \mu_d$ for $\mu \in \mathcal{S}$. To prove that the family $\{\pi(\mu)\}_{\mu \in \mathcal{S}}$ is tight, it is enough to show the following two assertions:

(I) To each $\varepsilon > 0$ there corresponds a finite subset $I \subset D$ such that $\mu(G \times I) \geq 1 - \varepsilon$ for each $\mu \in \mathcal{S}$.

(II) For each $d \in D$, the family $\{\pi(\mu_d)\}_{\mu \in \mathcal{S}}$ is tight.

Let $\psi_G : G \times D \to G$ and $\psi_D : G \times D \to D$ be the canonical projections. Consider the dual homomorphisms $\psi_G^\wedge : G^\wedge_{\mathrm{b}} \to (G \times D)^\wedge_{\mathrm{b}}$ and $\psi_D^\wedge : D^\wedge_{\mathrm{b}} \to (G \times D)^\wedge_{\mathrm{b}}$. We have $\mathcal{F}\psi_G(\mu) = \mathcal{F}\mu \circ \psi_G^\wedge$ and $\mathcal{F}\psi_D(\mu) = \mathcal{F}\mu \circ \psi_D^\wedge$ for $\mu \in \mathcal{S}$. Therefore $\{\mathcal{F}\psi_G(\mu)\}_{\mu \in \mathcal{S}}$ and $\{\mathcal{F}\psi_D(\mu)\}_{\mu \in \mathcal{S}}$ are equicontinuous families of functions on $G^\wedge_{\mathrm{b}}$ and $D^\wedge_{\mathrm{b}}$, respectively. The Lévy theorem for discrete groups implies that $\{\psi_D(\mu)\}_{\mu \in \mathcal{S}}$ is a tight family of measures on $D^\wedge_{\mathrm{b}} = D^\wedge_{\mathrm{p}}$, which is equivalent to (I).

Let $\sigma : G \to H$ be the restriction of π to G. Since $\{\mathcal{F}\psi_G(\mu)\}_{\mu \in \mathcal{S}}$ is equicontinuous and σ is tightening, it follows that

(III) the family $\{\sigma\psi_G(\mu)\}_{\mu \in \mathcal{S}}$ is tight.

To prove (II), fix $d \in D$ and let $\tau : H \to H$ be the shift $h \mapsto h + \pi(d)$. A direct verification shows that $\pi(\mu_d) = \tau\sigma\psi_G(\mu_d)$ for $\mu \in \mathcal{S}$. Therefore it is enough to show that the family $\{\sigma\psi_G(\mu_d)\}_{\mu \in \mathcal{S}}$ is tight. This, however, follows immediately from (III), because $\mu_d \leq \mu$ and thus $\sigma\psi_G(\mu_d) \leq \sigma\psi_G(\mu)$ for $\mu \in \mathcal{S}$. ■

4. Subgroups and quotients of Hilbert spaces.

Let E be a (real) Hilbert space. The scalar product of vectors $x, y \in E$ is denoted by (x, y) or just by xy. It follows from Lemma 2.2 that the formula

$$\langle y, \zeta(x) \rangle = \exp\{2\pi ixy\} \quad (x, y \in E)$$

defines a topological isomorphism $\zeta : E \to E^\wedge_{\mathrm{b}}$. Next, let K be a closed additive subgroup of E. Define

$$Q = \{x \in E : (x, y) \in \mathbb{Z} \text{ for each } y \in K\}. \tag{1}$$

It is clear that Q is a weakly closed subgroup of E. Let $\psi : E \to E/K$ be the canonical mapping. If $x \in Q$, then $\zeta(x)$ is a continuous character of E trivial on K; it induces a continuous character $\xi(x)$ of E/K by the formula

$$\langle \psi(y), \xi(x) \rangle = \exp\{2\pi ixy\} \quad (x \in Q,\ y \in E).$$

It is clear that the mapping $\xi : Q \to (E/K)^\wedge$ thus defined is an algebraic isomorphism. In fact, $\xi : Q \to (E/K)^\wedge_{\mathrm{b}}$ is a topological isomorphism (ξ is continuous because ψ is bounding; ξ^{-1} is continuous because ψ is bounded). Let $\iota : Q \to E$ be the identity embedding. The composition $E \xrightarrow{\zeta} E^\wedge_{\mathrm{b}} \xrightarrow{\iota^\wedge} Q^\wedge_{\mathrm{b}}$ is a continuous homomorphism trivial on K, therefore it induces a continuous homomorphism $\eta : E/K \to Q^\wedge_{\mathrm{b}}$ given by

$$\langle x, \eta(\psi(y)) \rangle = \exp\{2\pi ixy\} \quad (x \in Q,\ y \in E).$$

Observe that η is injective if and only if K is weakly closed in E. If $\mu \in \mathcal{P}(E/K)$, then $\nu = \eta(\mu) \in \mathcal{P}(Q^\wedge_{\mathrm{b}})$ and $\mathcal{F}^{-1}\nu(x) = \overline{\mathcal{F}\mu(\xi(x))}$ for each $x \in Q$, which can be verified directly. In what follows, by the *canonical* homomorphisms $Q \to (E/K)^\wedge$ and $E/K \to Q^\wedge$ we mean the homomorphisms ξ and η defined above.

Now, suppose we are given two Hilbert spaces E_1, E_2 with weakly closed subgroups K_1, K_2, respectively, and a bounded linear operator $T : E_1 \to E_2$ with $T(K_1) \subset K_2$. Let $\psi_i : E_i \to E_i/K_i$, $i = 1, 2$, be the canonical mappings. Then the formula $\pi\psi_1 = \psi_2 T$ defines a continuous homomorphism $\pi : E_1/K_1 \to E_2/K_2$, as shown in the following diagram:

$$\begin{tikzcd} E_1 \arrow[r, "T"] \arrow[d, "\psi_1"] & E_2 \arrow[d, "\psi_2"] \\ E_1/K_1 \arrow[r, "\pi"] & E_2/K_2 \end{tikzcd} \tag{2}$$

We say that the homomorphism π is *induced* by T.

Under these assumptions, the following is true:

LEMMA 4.1. (a) *Let $\mu_1 \in \mathcal{P}(E_1/K_1)$ and let $\mu_2 = \pi(\mu_1) \in \mathcal{P}(E_2/K_2)$. Let ε and r be some fixed positive numbers and let $A = 16r^{-1}\varepsilon^{-1/2}B_{E_2}$. Suppose that*

$$\sum_{k=1}^{\infty} d_k(T)^{2/3} \leq 1, \tag{3}$$

$$\operatorname{Re} \mathcal{F}\mu_1(\xi_1(x)) \geq 1 - \varepsilon \quad \textit{for each } x \in Q_1 \cap rB_{E_1}. \tag{4}$$

Then $\mu_2(\psi_2(A)) \geq 1 - 3\varepsilon$.

(b) *If the operator T is 2/3-approximable, then π is tightening.*

REMARK. Condition (3) may be replaced by $\sum_{k=1}^{\infty} d_k(T) \leq c$ where c is some universal constant. Similarly, 2/3-approximable operators in (b) may be replaced by 1-approximable. The proofs of these assertions need certain additional preparations and will be given elsewhere.

Proof. (a) For $i=1,2$, define

$$Q_i=\{x\in E_i:(x,y)\in\mathbb{Z}\text{ for each }y\in K_i\}$$

and let $\xi_i:Q_i\to(E_i/K_i)^\wedge$ and $\eta_i:E_i/K_i\to Q_i^\wedge$ be the corresponding canonical homomorphisms. Let $T^*:E_2\to E_1$ be the adjoint operator given by

$$(x,T^*y)=(Tx,y)\quad(x\in E_1,\ y\in E_2).$$

Then $T^*(Q_2)\subset Q_1$ and we obtain the following commutative diagrams of continuous homomorphisms:

$$\begin{tikzcd} Q_2 \arrow[r, "T^*|_{Q_2}"] \arrow[d, "\xi_2"] & Q_1 \arrow[d, "\xi_1"] \\ (E_2/K_2)^\wedge_{\mathrm{b}} \arrow[r, "\pi^\wedge"] & (E_1/K_1)^\wedge_{\mathrm{b}} \end{tikzcd}\qquad \begin{tikzcd} E_1/K_1 \arrow[r, "\pi"] \arrow[d, "\eta_1"] & E_2/K_2 \arrow[d, "\eta_2"] \\ (Q_1)^\wedge_{\mathrm{b}} \arrow[r, "(T^*|_{Q_2})^\wedge"] & (Q_2)^\wedge_{\mathrm{b}} \end{tikzcd}$$

Consider the measure $\nu_2=\eta_2(\mu_2)\in\mathcal{P}((Q_2)^\wedge_{\mathrm{b}})$. We have

$$\overline{\mathcal{F}^{-1}\nu_2}=\mathcal{F}\mu_2\circ\xi_2=\mathcal{F}\mu_1\circ\pi^\wedge\circ\xi_2=\mathcal{F}\mu_1\circ\xi_1\circ T^*|_{Q_2}. \tag{5}$$

Let p be the continuous pre-Hilbert seminorm on E_2 given by

$$p(x)=r^{-1}\|T^*x\|\quad(x\in E_2).$$

Then

$$d_k(B_{E_2},B_p)=r^{-1}d_k(T^*)\quad(k=1,2,\ldots).$$

A standard argument based on the polar decomposition of T^* shows that there exists another continuous pre-Hilbert seminorm q on E_2 with $B_{E_2}\prec B_q\prec B_p$ and such that

$$d_k(B_q,B_p)=\varepsilon^{1/2}d_k(T^*)^{1/3},$$
$$d_k(B_{E_2},B_q)=r^{-1}\varepsilon^{-1/2}d_k(T^*)^{2/3}\quad(k=1,2,\ldots).$$

Since $d_k(T^*)=d_k(T)$, from (3) we obtain

$$\sum_{k=1}^{\infty}d_k(B_q,B_p)^2\le\varepsilon, \tag{6}$$

$$\sum_{k=1}^{\infty}d_k(B_{E_2},B_q)\le r^{-1}\varepsilon^{-1/2}. \tag{7}$$

If $x\in Q_2\cap B_p$, then $T^*x\in Q_1\cap rB_{E_1}$. Hence, by (5) and (4), we have

$$\operatorname{Re}\mathcal{F}^{-1}\nu_2(x)\ge1-\varepsilon\quad\text{for each }x\in Q_2\cap B_p. \tag{8}$$

Set

$$Z=\{\chi\in Q_2^\wedge:\operatorname{Re}\chi(x)\ge0\ \text{ for each }x\in Q_2\cap\tfrac14B_q\}.$$

Applying Lemma 2.3, (8) and (6), we obtain

$$\nu_2(Q_2^\wedge\setminus Z)\le3\varepsilon. \tag{9}$$

Now, take an arbitrary $\chi\in Z$. By Lemma 2.4, there exists a bounded linear functional f on E_2 such that

$$\exp\{2\pi if(x)\}=\chi(x)\quad\text{for each }x\in Q_2, \tag{10}$$

$$\|f\|\le4\sum_{k=1}^{\infty}d_k(B_{E_2},\tfrac14B_q). \tag{11}$$

Let $y\in E_2$ be given by $f(x)=(x,y)$ for $x\in E_2$. By (11) and (7), we have

$$\|y\|=\|f\|\le16\sum_{k=1}^{\infty}d_k(B_{E_2},B_q)\le16r^{-1}\varepsilon^{-1/2}.$$

Thus $y\in A$. Condition (10) means that $\eta_2\psi_2(y)=\chi$. Since $\chi\in Z$ was arbitrary, it follows that $Z\subset\eta_2\psi_2(A)$. Hence

$$\nu_2(Z)=\eta_2(\mu_2)(Z)=\mu_2(\eta_2^{-1}(Z))\subset\mu_2(\eta_2^{-1}(\eta_2\psi_2(A)))=\mu_2(\psi_2(A))$$

because η_2 is injective (K_2 was assumed to be weakly closed in E_2). In view of (9), this completes the proof of (a).

(b) Let $\mathcal{S}\subset\mathcal{P}(E_1/K_1)$ be a family of measures such that $\{\mathcal{F}\mu\}_{\mu\in\mathcal{S}}$ is an equicontinuous family of functions on $(E_1/K_1)^\wedge_{\mathrm{b}}$. Suppose that $\sum_{k=1}^{\infty}d_k(T)^{2/3}<\infty$. Using the polar decomposition of T etc., we can find a Hilbert space E_2' and bounded linear operators $T':E_1\to E_2'$ and $T'':E_2'\to E_2$ with $T=T''T'$ such that $\sum_{k=1}^{\infty}d_k(T')^{2/3}\le1$ and T'' is compact. Let K_2' be the weak closure of $T'(K_1)$ in E_2'. It is not hard to see that $T''(K_2')\subset K_2$. We obtain the canonical commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "T'"] \arrow[d, "\psi_1"] & E_2' \arrow[r, "T''"] \arrow[d, "\psi_2'"] & E_2 \arrow[d, "\psi_2"] \\ E_1/K_1 \arrow[r, "\pi'"] & E_2'/K_2' \arrow[r, "\pi''"] & E_2/K_2 \end{tikzcd}$$

where $\pi''\pi'=\pi$.

Take an arbitrary $\varepsilon>0$. Since $\xi_1:Q_1\to(E_1/K_1)^\wedge_{\mathrm{b}}$ is a topological isomorphism, $\{\mathcal{F}\mu\circ\xi_1\}_{\mu\in\mathcal{S}}$ is an equicontinuous family of functions on Q_1. So, there is some $r>0$ such that $\operatorname{Re}\mathcal{F}\mu(\xi_1(x))\ge1-\varepsilon$ for every $x\in Q_1\cap rB_{E_1}$ and $\mu\in\mathcal{S}$. Let $A=16r^{-1}\varepsilon^{-1/2}B_{E_2'}$ and let $X=\psi_2(\overline{T''(A)})$. Then X is a compact subset of E_2/K_2. Now, take any $\mu\in\mathcal{S}$. By (a), we have $\pi'(\mu)(\psi_2'(A))\ge1-3\varepsilon$. Hence

$$\pi(\mu)(X)\supset\pi(\mu)(\psi_2(T''(A)))=\pi''(\pi'(\mu))(\pi''(\psi_2'(A)))$$
$$=\pi'(\mu)((\pi'')^{-1}(\pi''(\psi_2'(A)))\ge\pi'(\mu)(\psi_2'(A))\ge1-3\varepsilon.\ \blacksquare$$

By an *EKD-group* we mean a group of the form $(E/K)\times D$ where D is a discrete abelian group and K is a weakly closed subgroup of a Hilbert space E. We shall identify E/K with the corresponding subgroup of $(E/K)\times D$.

LEMMA 4.2. *Let $G=(E/K)\times D$ be an EKD-group. Then the group $G^\wedge_{\mathrm{b}}$ is Čech-complete.*

Proof. By Lemma 2.1(b), the group $G^\wedge_{\mathrm{b}}$ is topologically isomorphic to $(E/K)^\wedge_{\mathrm{b}}\times D^\wedge_{\mathrm{b}}$. Let Q be defined as in (1). Since the canonical mapping $\xi: Q\to (E/K)^\wedge_{\mathrm{b}}$ is a topological isomorphism, the group $(E/K)^\wedge_{\mathrm{b}}$ is Čech-complete. Hence $G^\wedge_{\mathrm{b}}$ is Čech-complete because $D^\wedge_{\mathrm{b}}$ is compact. ∎

Let $G_1=(E_1/K_1)\times D_1$ and $G_2=(E_2/K_2)\times D_2$ be EKD-groups and let $\pi: G_1\to G_2$ be a continuous homomorphism with $\pi(E_1/K_1)\subset E_2/K_2$. We say that π is *α-approximable*, $0<\alpha<\infty$, if the restriction $\pi_{|E_1/K_1}: E_1/K_1\to E_2/K_2$ is induced by an α-approximable operator $T: E_1\to E_2$ (see diagram (2)).

LEMMA 4.3. *Every 2/3-approximable homomorphism of EKD-groups has the property* (PL).

Proof. Let $\pi:(E_1/K_1)\times D_1\to (E_2/K_2)\times D_2$ be a 2/3-approximable homomorphism of EKD-groups. Then the restriction $\sigma: E_1/K_1\to E_2/K_2$ of π to E_1/K_1 is induced by a 2/3-approximable operator $T: E_1\to E_2$. Lemma 4.1(b) says that σ is tightening. Hence π is tightening according to Lemma 3.8. It is now enough to apply Lemmas 3.7 and 4.2. ∎

5. Nuclear groups. Nuclear groups were defined in [B1, (7.1)] (an equivalent definition is given by Lemma 5.1 below). They form a class of a.t. groups with the following properties:

(1) every LCA group is nuclear;
(2) a topological vector space G is nuclear if and only if G is a nuclear locally convex space;
(3) every subgroup of a nuclear group is nuclear;
(4) every Hausdorff quotient group of a nuclear group is nuclear;
(5) the product of an arbitrary family of nuclear groups is nuclear;
(6) the direct sum of a countable family of nuclear groups is nuclear.

The proofs of these assertions are given in [B1, Sect. 7]. Moreover, if G is a Čech-complete nuclear group, then the group $G^\wedge_{\mathrm{c}}$ is nuclear [A, (20.36)].

Let F be a vector space and τ a topology on F such that F_τ is an additive topological group. We say that F_τ is a *locally convex vector group* if it is separated and has a base at zero consisting of symmetric convex sets. A locally convex vector group F is called a *nuclear vector group* if to each symmetric convex neighbourhood U of zero in F there corresponds another symmetric convex neighbourhood V with $d_k(V,U)\le k^{-1}$ for every k.

LEMMA 5.1. *An a.t. group G is nuclear if and only if it is topologically isomorphic to a group of the form H/K, where H is a subgroup of a nuclear vector group F and K is a closed subgroup of H.*

This follows from [B1, (9.4) and (9.6)].

LEMMA 5.2. *Let K be a closed subgroup of a nuclear vector group F. Then the quotient group F/K is topologically isomorphic to a dense subgroup of the limit of an inverse system $\{G_i,\pi_{ij},I\}$ of EKD-groups with the following property: to each $i\in I$ there corresponds some $j\ge i$ such that the homomorphism $\pi_{ij}: G_j\to G_i$ is 1/2-approximable.*

This is a reformulation of Theorem 3.4 of Galindo [G]. The number 1/2 may be replaced here by an arbitrary $\alpha\in(0,\infty)$.

THEOREM 5.3. *Every nuclear group has the property* (PL).

Proof. Let G be a nuclear group. By Lemma 5.1, there exist a nuclear vector group F and a closed subgroup K of F such that G is topologically isomorphic to a subgroup of F/K. By Lemma 3.1, we may assume that $G=F/K$. That F/K has the property (PL) follows from Lemmas 5.2, 3.1, 3.2 and 4.3. ∎

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