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Interpolation of real method spaces via some ideals of operators

by

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Abstract. Certain operator ideals are used to study interpolation of operators between spaces generated by the real method. Using orbital equivalence a new reiteration formula is proved for certain real interpolation spaces generated by ordered pairs of Banach lattices of the form $(X, L_\infty(w))$. As an application we extend Ovchinnikov's interpolation theorem from the context of classical Lions–Peetre spaces to a larger class of real interpolation spaces. A description of certain abstract $\mathcal{J}$-method spaces is also presented.

0. Introduction. The Riesz–Thorin–Marcinkiewicz interpolation theorems are important tools in classical and modern analysis. Recall that the Riesz–Thorin theorem states that if a linear operator T is bounded from L_{p_j} into L_{q_j} for $j = 0, 1$ then T is bounded from L_p into L_q, where $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_0$ and $0 < \theta < 1$. It is natural to ask if under the same assumptions we can improve the conclusion: for example we ask if it is possible to find a smaller range space Y such that T is bounded from L_p into Y. It was known for a long time that if $q_0 < p_0$ or $q_1 < p_1$ the result is not sharp. Finally in [13] Ovchinnikov obtained a sharp version of the Riesz–Thorin–Marcinkiewicz theorem: under the same assumptions of the classical Riesz–Thorin–Marcinkiewicz theorem we can conclude that T maps continuously L_p into the Lorentz space $L_{q,r}$ with $1/r = (1-\theta)\max\{1/q_0, 1/p_0\} + \theta\max\{1/q_1, 1/p_1\}$. The proof of this remarkable result is based on the application of a factorization theorem of Bennett [1], which states that the inclusion map $\ell_p \hookrightarrow \ell_\infty$ is a $(p,1)$-summing operator, to prove a new interpolation theorem for operators acting on weighted sequence ℓ_p-spaces modelled on the set $\mathbb{Z}$ of integers. A simple application of the reiteration theorem allows Ovchinnikov to prove his general interpolation theorem for Lions–Peetre scales.

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The fact that in the majority of applications of interpolation theory to classical analysis we have $p_j \le q_j$, $j = 0, 1$, probably explains why the result remained unnoticed for so long. Of course the factorization theorems lying at the heart of the result already point to importance of studying operators that are not "improving". Moreover, another area of possible applications is the theory of weighted norm inequalities for classical operators. We hope to return to this last point and its connections with the theory of commutator estimates elsewhere.

The main purpose of this paper is to extend Ovchinnikov's theorem from the context of classical Lions–Peetre spaces to more general spaces generated by abstract real method spaces. This extension is not completely straightforward and demands the introduction of appropriate tools which could be of interest in their own right.

Let us outline briefly the content of the paper. In Section 1 we define new operator ideals, which include the ideals of (q,p)-summing operators, in order to prove certain basic interpolation results. The main result of this section is Theorem 1.4. It is of interest that the conditions on the spaces are expressed in terms of multiplicators of Banach sequence lattices, modelled on $\mathbb{Z}$.

Section 2 contains the main result of the paper. In order to prove it we need among other things a new reiteration formula for certain real method spaces generated by any ordered pair $(X, L_\infty(w))$ of Banach lattices. The results obtained are then applied to prove our generalization of Ovchinnikov's theorem.

In the last section the new ideals of operators are applied to give a description of abstract $\mathcal{J}$-method spaces.

In the paper we use freely the standard definitions and notation of interpolation theory as can be found in [2], [3] and [13].

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1. Summability and basic interpolation results. We start by recalling that a *quasi-normed space* is a vector space X whose topology is given by a quasi-norm $x \mapsto \|x\|$ satisfying

(i) $\|x\| > 0$ for $x \ne 0$,
(ii) $\|\alpha x\| = |\alpha| \cdot \|x\|$ for $\alpha \in \mathbb{R}$, $x \in X$,
(iii) there is $C > 0$ such that $\|x+y\| \le C(\|x\| + \|y\|)$ for $x, y \in X$.

If $\|\cdot\|$ is a quasi-norm on X defining a complete metrizable topology, then X is called a *quasi-Banach space*.

Let (Ω, μ) be a measure space with μ complete σ-finite, and let $L^0(\mu)$ denote, as usual, the space of all equivalence classes of measurable functions on Ω with the topology of convergence in measure on μ-finite sets.

If a quasi-normed subspace E of $L^0(\mu)$ is such that there exists $u \in L^0(\mu)$ with $u > 0$ a.e. and

(iv) $\|x\| \le \|y\|$ whenever $|x| \le |y|$ a.e.,

then we say that E is a quasi-normed *lattice* on (Ω, μ). If in addition the unit ball $B_E = \{x : \|x\|_E \le 1\}$ is closed in $L^0(\mu)$, so that E has the *Fatou property*, then E is a quasi-Banach space which is called a *maximal* quasi-Banach lattice.

A quasi-normed lattice modelled on the set $\mathbb{Z}$ of integers is called a quasi-normed *sequence lattice* on $\mathbb{Z}$.

If X is a quasi-Banach lattice and $w \in L^0$ with $w > 0$ a.e., we define the *weighted space* $X(w)$ by $\|x\|_{X(w)} := \|xw\|_X$.

Let E be a quasi-Banach sequence lattice on $\mathbb{Z}$ and let X be a Banach space. The vector sequence $x = \{x_n\}_{n=-\infty}^{\infty}$ in X is *strongly E-summable* if the corresponding scalar sequence $\{\|x_n\|_X\}$ is in E. We denote by $E(X)$ the set of all such sequences in X. This is a quasi-Banach space under pointwise operations, and a natural quasi-norm given by

$$\|x\|_{E(X)} := \|\{\|x_n\|_X\}\|_E.$$

The vector sequence $\{x_n\}$ in X is *weakly E-summable* if the scalar sequences $\{x^*(x_n)\}$ are in E for every x^* in the dual X^* of X. Note that if E has *order continuous norm* (i.e., $x_n \downarrow 0$ implies $\|x_n\| \to 0$) then for any weakly E-summable sequence $\{x_n\}$ in X the associated finite rank operators $u_k : X^* \to E$ given by $u_k(x^*) = \sum_{n=-k}^{k} x^*(x_n) e_n$ for $x^* \in X^*$, where $\{e_n\}$ is the standard unit vector basis in c_0, satisfy

$$\lim_{k\to\infty} \|u_k(x^*) - \{x^*(x_n)\}\|_E = 0$$

for every $x^* \in X^*$. Consequently, by the Banach–Steinhaus theorem, the linear map $u : X^* \to E$ defined by $u(x^*) := \{x^*(x_n)\}$ for $x^* \in X^*$ is bounded, and thus

$$\|u\|_{X^* \to E} = w_{E,X}(\{x_n\}) := \sup\{\|\{x^*(x_n)\}\|_E : \|x^*\|_{X^*} \le 1\} < \infty.$$

In what follows the quasi-normed space of all weakly E-summable sequences $\{x_n\}$ of a Banach space X such that $w_{E,X}(\{x_n\}) < \infty$ will be denoted by $E^w(X)$.

Let E and F be two quasi-Banach sequence lattices on $\mathbb{Z}$, let X and Y be Banach spaces, and let $T : X \to Y$ be a linear operator. We shall say that T is *(F,E)-summing* if the induced operator $\widehat{T}$ defined on $E^w(X)$ by

$$\widehat{T}\{x_n\} := \{Tx_n\} \quad \text{for } \{x_n\} \in E^w(X)$$

is bounded from $E^w(X)$ into $F(Y)$. In this case we write $\pi_{F,E}(T) := \|\widehat{T}\|_{E^w(X)\to F(Y)}$. We denote by $\Pi_{F,E}(X,Y)$ the space of all (F,E)-summing operators with the quasi-norm $\pi_{F,E}$.

Note that by taking $\{x_n\} = \{\xi x\}$ with $\|x\|_X = 1$ and $\{\xi_n\} \in E$, it follows that $\Pi_{F,E}(X,Y) \neq \{0\}$ only in the case $E \hookrightarrow F$.

In what follows, we always assume that $E \hookrightarrow F$. Under some additional assumptions on E and F, $\Pi_{F,E}$ is a quasi-Banach operator ideal in the sense of Pietsch; we refer to [14] for the basic definitions and more detailed information.

PROPOSITION 1.1. *Let E, F be Banach sequence lattices on $\mathbb{Z}$ such that the norm of the inclusion map $F \hookrightarrow E$ equals 1 and let $\|e_n\|_E = \|e_n\|_F = 1$ for some $n \in \mathbb{Z}$. Then $[\Pi_{F,E}, \pi_{F,E}]$ is a Banach operator ideal in the sense of Pietsch.*

Proof. Let X and Y be Banach spaces. By the continuous inclusion $\Pi_{F,E}(X,Y) \hookrightarrow \mathcal{L}(X,Y)$, it follows easily that $\Pi_{F,E}(X,Y)$ is a Banach space under the norm $\pi_{F,E}$. Here as usual $\mathcal{L}(X,Y)$ denotes the Banach space of all bounded linear operators from X into Y, equipped with its natural norm. Clearly the rank one operators are in $\Pi_{F,E}$. Moreover it is not hard to check that, under the assumptions above, for the rank one operator $T = x^* \otimes y \in \mathcal{L}(X,Y)$ with $x^* \in X^*$ and $y \in Y$, we have

$$\pi_{F,E}(T) = \|x^*\|_{X^*}\|y\|_Y.$$

It is also readily seen that if $T \in \Pi_{F,E}(X,Y)$, and X_0, Y_0 are Banach spaces, then $UTV \in \Pi_{F,E}(X_0,Y_0)$ and

$$\pi_{F,E}(UTV) \le \|U\|\pi_{F,E}(T)\|V\|$$

for any $U \in \mathcal{L}(Y,X_0)$ and $V \in \mathcal{L}(X_0,X)$. ∎

We now show some elementary properties of this ideal. Note that for $E = \ell_p$ and $F = \ell_q$ with $1 \le p \le q < \infty$, $\Pi_{F,E}$ is the well known ideal of (q,p)-summing operators.

Recall that if X and Y are quasi-normed lattices on a measure space (Ω,μ), the space $M(X,Y)$ of *mulitplicators* from X into Y is the space of all measurable functions $x \in L^0(\mu)$ such that the associated multiplication operator $X \ni y \mapsto xy$ is bounded from X into Y. The space $M(X,Y)$ is equipped with the quasi-norm

$$\|x\|_{M(X,Y)} = \sup\{\|xy\|_Y : \|y\|_X \le 1\}.$$

Note that if X is a Banach lattice then $M(X,L^1)$ is the usual Köthe dual space X' of X.

Before proceeding we will need a technical lemma.

LEMMA 1.2. *Let X and Y be Banach spaces and let E be a Banach sequence lattice on $\mathbb{Z}$. Every (F,ℓ_1)-summing operator $T : X \to Y$ is $(M(E,F),E')$-summing with $\pi_{M(E,F),E'}(T) \le \pi_{F,\ell_1}(T)$.*

Proof. Since $\ell_1 \hookrightarrow F$, we easily see that $E' \hookrightarrow M(E,F)$ for any Banach sequence lattice E. Let $x = \{x_n\} \in (E')^w(X)$. Then $\xi x := \{\xi_n x_n\} \in \ell_1^w(X)$ for every $\xi = \{\xi_n\} \in E$. This yields, by the assumption that $T : X \to Y$ is (F,ℓ_1)-summing, $\{\xi_n\|Tx_n\|_Y\} \in F$ and

$$\begin{aligned}\|\{\xi_n\|Tx_n\|_Y\}\|_E &\le C\sup\Big\{\sum_{n=-\infty}^{\infty}|\xi_n x^*(x_n)| : \|x^*\|_{X^*} \le 1\Big\}\\ &\le C\|\xi\|_E \sup\{\|\{x^*(x_n)\}\|_{E'} : \|x^*\|_{X^*} \le 1\}\end{aligned}$$

with $C = \pi_{E,\ell_1}(T)$. In consequence, $\{Tx_n\} \in M(F,E)(X)$ and the induced map $\widehat{T} : (E')^w(X) \to M(E,F)(Y)$ is bounded with $\|\widehat{T}\| \le C$. ∎

Throughout the paper a pair $\overline{\Phi} = (\Phi_0,\Phi_1)$ of quasi-Banach sequence lattices on $\mathbb{Z}$ is called a *parameter of the $\mathcal{J}$-method* if $\Phi_0 \cap \Phi_1 \subset \ell_1$. The *$\mathcal{J}$-method space* $\mathcal{J}_{\overline{\Phi}}(\overline{X}) = \mathcal{J}_{\Phi_0,\Phi_1}(\overline{X})$ consists of all $x \in X_0 + X_1$ which can be represented in the form

$$x = \sum_{n=-\infty}^{\infty} u_n \quad (\text{convergence in } X_0 + X_1) \tag{1}$$

with $u = \{u_n\} \in \Phi_0(X_0) \cap \Phi_1(X_1)$. Similarly to the case of Banach sequence lattices Φ_0, Φ_1, we easily show that $\mathcal{J}_{\overline{\Phi}}(\overline{X})$ is a quasi-Banach space under the quasi-norm

$$\|x\|_{\mathcal{J}_{\overline{\Phi}}(\overline{X})} = \inf\max\{\|u\|_{\Phi_0(X_0)}, \|u\|_{\Phi_1(X_1)}\},$$

where the infimum is taken over all representations (1) (cf. [3], [9]). If $\overline{E} = (E_0,E_1)$ is a pair of quasi-Banach sequence lattices on $\mathbb{Z}$ so that $(\Phi_0,\Phi_1) = (E_0(1/\varrho(2^n)), E_1(2^n/\varrho(2^n)))$ is a parameter of the $\mathcal{J}$-method for a quasi-concave function $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ (i.e., $\varrho(s) \le \max\{1, s/t\}\varrho(t)$ for all $s,t > 0$), then the space $\mathcal{J}_{\overline{\Phi}}(\overline{X})$ (resp., $\mathcal{J}_{\Phi_0,\Phi_1}(\overline{X})$) is denoted by $\mathcal{J}_{\varrho,\overline{E}}(\overline{X})$ (resp., $\mathcal{J}_{\varrho,E_0,E_1}(\overline{X})$) and $\mathcal{J}_{\theta,\overline{E}}(\overline{X})$ (resp., $\mathcal{J}_{\theta,E_0,E_1}(\overline{X})$), whenever $\varrho(t) = t^\theta$, $0 \le \theta \le 1$. Note that if E is a Banach sequence lattice on $\mathbb{Z}$ and $E_0 = E_1 = E$, then $\mathcal{J}_{0,E,E}(\overline{X})$ is the classical abstract $\mathcal{J}$-space which is denoted by $\mathcal{J}_E(\overline{X})$ (see [5], [9]).

The following interpolation result is an immediate consequence of the definition of the $\mathcal{J}$-method spaces.

PROPOSITION 1.3. *Suppose E_j, F_j for $j = 0,1$ are quasi-Banach sequence lattices on $\mathbb{Z}$. If $T : \overline{X} \to \overline{Y}$ is an operator such that $T : X_j \to Y_j$ is*

(F_j, E_j)*-summing for* $j = 0, 1$, *then* T *is bounded from* $\mathcal{J}_{\overline{E}}(\overline{X})$ *into* $\mathcal{J}_{\overline{F}}(\overline{Y})$ *and*

$$\|T\|_{\mathcal{J}_{\overline{E}}(\overline{X})\to\mathcal{J}_{\overline{F}}(\overline{Y})} \le \max\{\pi_{F_0,E_0}(T), \pi_{F_1,E_1}(T)\}.$$

Proof. Let $x \in \mathcal{J}_{\overline{E}}(\overline{X})$ with $\|x\|_{\mathcal{J}_{\overline{E}}(\overline{X})} < 1$. Then there exists $u = \{u_n\} \in E_0(X_0) \cap E_1(X_1)$ such that

$$x = \sum_{n=-\infty}^{\infty} u_n \quad \text{(convergence in } X_0 + X_1)$$

and $\|u\|_{E_j(X_j)} \le 1$, $j = 0, 1$. Since $T : \overline{X} \to \overline{Y}$,

$$Tx = \sum_{n=-\infty}^{\infty} Tu_n \quad \text{(convergence in } Y_0 + Y_1).$$

Obviously $E_j(X_j) \hookrightarrow E_j^w(X_j)$, $j = 0, 1$. Thus, as $T : X_j \to Y_j$ is (F_j, E_j)-summing, we have

$$\|\{Tu_n\}\|_{F_j(Y_j)} \le C_j w_{E_j,X_j}(u_n) \le C_j\|u\|_{E_j(X_j)} \le C_j,$$

where $C_j = \pi_{F_j,E_j}(T)$, $j = 0, 1$. We conclude that $Tx \in \mathcal{J}_{\overline{F}}(\overline{Y})$ and

$$\|T\|_{\mathcal{J}_{\overline{E}}(\overline{X})\to\mathcal{J}_{\overline{F}}(\overline{Y})} \le \max\{C_0, C_1\}. \ \blacksquare$$

In what follows for a given Banach lattice X the largest ideal consisting of all elements with order continuous norm will be denoted by X_a. Clearly

$$X_a = \{x \in X : |x| \ge x_n \downarrow 0 \text{ implies } \|x_n\| \to 0\}.$$

THEOREM 1.4. *Let* $\overline{\Phi} = (\Phi_0, \Phi_1)$ *be a parameter of the* $\mathcal{J}$*-method and let* $\overline{E}$, $\overline{F}$ *be pairs of Banach sequence lattices on* $\mathbb{Z}$ *so that* $M(E_j, F_j) \overset{C}{\hookrightarrow} M(E, \Phi_j)$ *for* $j = 0, 1$ *and some quasi-Banach sequence lattice* $E \hookrightarrow (E_0 + E_1)_a$. *If* $\overline{Y}$ *is a Banach pair and* $T : \overline{E} \to \overline{Y}$ *is an operator such that* $T : E_j \to Y_j$ *is* (F_j, ℓ_1)*-summing for* $j = 0, 1$, *then* T *is bounded from* E *into* $\mathcal{J}_{\overline{\Phi}}(\overline{Y})$ *and*

$$\|T\|_{E\to\mathcal{J}_{\overline{\Phi}}(\overline{Y})} \le C \max\{\pi_{F_0,\ell_1}(T), \pi_{F_1,\ell_1}(T)\}.$$

Proof. The unit vectors e_n form an unconditional basis in $(E_0 + E_1)_a$. Since $E \hookrightarrow (E_0 + E_1)_a$, for any $\xi = \{\xi_n\} \in E$ we have

$$\xi = \sum_{n=-\infty}^{\infty} \xi_n e_n \quad \text{(convergence in } E_0 + E_1).$$

Thus, from $T : \overline{E} \to \overline{Y}$, it follows that

$$T\xi = \sum_{n=-\infty}^{\infty} \xi_n Te_n \quad \text{(convergence in } Y_0 + Y_1).$$

By Lemma 1.2 we see that $T : E_j \to Y_j$ is $(M(E_j, F_j), E_j')$-summing, and consequently also $(M(E, \Phi_j), E_j')$-summing since $M(E_j, F_j) \hookrightarrow M(E, \Phi_j)$, $j = 0, 1$. It is clear that the sequence $\{e_n\}$ is F'-summing in each Banach sequence lattice F on $\mathbb{Z}$ and $w_{F',F}(\{e_n\}) \le 1$. Combining the above, we conclude that for $u_n = \xi_n Te_n$ and $j = 0, 1$,

$$\|\{u_n\}\|_{\Phi_j(Y_j)} \le \|\{\|Te_n\|_{Y_j}\}\|_{M(E,\Phi_j)}\|\xi\|_E \le C\pi_{F_j,\ell_1}(T)\|\xi\|,$$

where C is a constant depending on the embedding of $M(E_j, F_j)$ in $M(E, \Phi_j)$. Since $u_n \in Y_0 \cap Y_1$ and $T\xi = \sum_{n=-\infty}^{\infty} u_n$ (convergence in $Y_0 + Y_1$), we have $T\xi \in \mathcal{J}_{\overline{\Phi}}(\overline{Y})$ with norm at most $C \max\{\pi_{F_0,\ell_1}(T), \pi_{F_1,\ell_1}(T)\}$. $\blacksquare$

In what follows a quasi-concave function ϱ is said to be of class $\mathcal{P}^{+-}$ if the *dilatation indices* of ϱ defined by $\alpha_\varrho := \lim_{t\to 0}(\ln s_\varrho(t)/\ln t)$, $\beta_\varrho := \lim_{t\to\infty}(\ln s_\varrho(t)/\ln t)$ are non-trivial, i.e., $0 < \alpha_\varrho \le \beta_\varrho < 1$. Here $s_\varrho(t) = \sup_{u>0} \varrho(tu)/\varrho(u)$ for $t > 0$.

PROPOSITION 1.5. *Suppose that* E_j, F_j, G_j *for* $j = 0, 1$ *are Banach sequence lattices on* $\mathbb{Z}$. *Suppose further that* Φ_j, $j = 0, 1$, *and* E *are quasi-Banach sequence lattices on* $\mathbb{Z}$ *satisfying the following conditions*:

(i) $\ell_1 \hookrightarrow E_j$, $\ell_1 \hookrightarrow F_j$, $\Phi_j \hookrightarrow \ell_\infty$ *and* $E \hookrightarrow \ell_\infty$,
(ii) $M(E_j, F_j) \hookrightarrow M(E, \Phi_j)$,
(iii) *the inclusion map* $F_j \hookrightarrow G_j$ *is an* (F_j, ℓ_1)*-summing operator.*

If a T *maps* $(E_0, E_1(2^{-n}))$ *into* $(F_0, F_1(2^{-n}))$, *then* T *is bounded from* $E(1/\varrho(2^n))$ *into* $(G_0, G_1(2^{-n}))_{\varrho,\overline{\Phi}}$ *for any* $\varrho \in \mathcal{P}^{+-}$.

Proof. Since $E \hookrightarrow \ell_\infty$ and $E(1/\varrho(2^n)) \hookrightarrow \ell_1 + \ell_1(2^{-n})$ for any $\varrho \in \mathcal{P}^{+-}$, it follows that $E(1/\varrho(2^n)) \hookrightarrow (E_0 + E_1(2^{-n}))_a$, because $\ell_1 \hookrightarrow E_j$ for $j = 0, 1$. Now observe that $M(X(u), Y(v)) = M(X, Y(v/u))$ with equality of norms for any weights u, v. Hence

$$M(E_0, F_0) \hookrightarrow M(E(1/\varrho(2^n)), \Phi_0(1/\varrho(2^n)))$$

and

$$M(E_1(2^{-n}), F_1) \hookrightarrow M(E(1/\varrho(2^n)), \Phi_1(2^n/\varrho(2^n))).$$

Since the inclusion map $F_1(2^{-n}) \hookrightarrow G_1(2^{-n})$ has an obvious factorization

$$F_1(2^{-n}) \xrightarrow{T_0} F_1 \xrightarrow{i} G_1 \xrightarrow{T_1} G_1(2^{-n})$$

with $T_0\{\xi_n\} = \{\xi_n 2^{-n}\}$, $T_1\{\xi_n\} = \{2^n \xi_n\}$ and $i\{\xi_n\} = \{\xi_n\}$, we conclude that $T : E_j(2^{-jn}) \to G_j(2^{-jn})$ is (F_j, ℓ_1)-summing, $j = 0, 1$. Thus Theorem 1.4 allows us to conclude. $\blacksquare$

2. Interpolation between real method spaces. In this section we present applications of the previous general results. Recall that if E is a *parameter of the K-method*, i.e., E is an interpolation space with respect to $(\ell_\infty, \ell_\infty(2^{-n}))$, then the *K-space* $\overline{X}_E := (X_0, X_1)_E$ consists of all $x \in X_0 + X_1$ with $\{K(2^n, x; \overline{X})\} \in E$. It is well known that $\overline{X}_E$ equipped with the norm

$$\|x\| := \|\{K(2^n, x; \overline{X})\}\|_E$$

is an exact interpolation space with respect to $\overline{X}$.

Throughout the paper a mapping $\mathcal{F}$ from the category of compatible pairs of Banach spaces into the category of quasi-Banach spaces is said to be an *interpolation functor* (or an *interpolation method*) if, for any Banach pair $\overline{X}$, $\mathcal{F}(\overline{X})$ is a quasi-Banach space intermediate with respect to $\overline{X}$ (i.e., $X_0 \cap X_1 \hookrightarrow \mathcal{F}(\overline{X}) \hookrightarrow X_0 + X_1$), and $T : \mathcal{F}(\overline{X}) \to \mathcal{F}(\overline{Y})$ for all Banach pairs $\overline{X}$, $\overline{Y}$ and every $T : \overline{X} \to \overline{Y}$.

Let $\overline{E} = (E_0, E_1)$ be a pair of parameters of the K-method. An interpolation functor $\mathcal{F}$ is said to be *stable* on $\overline{X}$ with respect to $\overline{E}$ if the following *reiteration formula* holds:

$$\mathcal{F}(\overline{X}_{E_0}, \overline{X}_{E_1}) = \overline{X}_{\mathcal{F}(E_0, E_1)}. \tag{2}$$

It is well known that if E_0 and E_1 are parameters of the real method then the Banach pair $(\overline{X}_{E_0}, \overline{X}_{E_1})$ is a partial retract of (E_0, E_1) and the reiteration formula above holds for any interpolation functor and any Banach pair $\overline{X}$ (cf. [13], [3]).

Recall (cf. [13]) that a sequence Banach lattice Φ is said to be a *parameter of the real method* if $\ell_\infty \cap \ell_\infty(2^{-n}) \hookrightarrow \Phi \hookrightarrow \ell_1 + \ell_1(2^{-n})$ and $T : \Phi \to \Phi$ for any operator $T : \overline{\ell}_1 \to \overline{\ell}_\infty$, where $\overline{\ell}_p := (\ell_p, \ell_p(2^{-n}))$ for $1 \le p \le \infty$. It is easily seen that Φ is a parameter of the real method if and only if the Calderón operator $\mathcal{P}$ defined by

$$(\mathcal{P}\xi)_n = \sum_{k=-\infty}^{\infty} \min\{1, 2^{n-k}\}\xi_k$$

is bounded in Φ. For example, if E is any *translation invariant* Banach lattice on $\mathbb{Z}$ (i.e., $\|\{\xi_{n-k}\}\|_E = \|\{\xi_n\}\|_E$), then $\Phi = E(1/\varrho(2^n))$ is a parameter of the real method for any $\varrho \in \mathcal{P}^{+-}$. This follows readily from

$$\|\{P\xi_n\}\|_\Phi = \Big\|\Big\{\sum_{k=-\infty}^{\infty} \min\{1, 2^k\}\xi_{n-k}\Big\}\Big\|_\Phi \le C\|\xi\|_\Phi$$

for all $\xi = \{\xi_n\} \in \Phi$, where $C(\varrho) = \sum_{k=-\infty}^{\infty} \min\{1, 2^k\}/\varrho(2^k) < \infty$.

In the sequel we will need the definition of a partial retract. Recall that if $\overline{X} = (X_0, X_1)$ and $\overline{Y} = (Y_0, Y_1)$ are Banach pairs then the elements $x \in X_0 + X_1$ and $y \in Y_0 + Y_1$ are said to be *orbitally equivalent* with respect to the pairs $\overline{X}$ and $\overline{Y}$ if there exist linear operators $T : \overline{X} \to \overline{Y}$ and $S : \overline{Y} \to \overline{X}$ such that $Tx = y$ and $Sy = x$. A pair $\overline{X}$ is called a *partial retract* of $\overline{Y}$ if every $x \in X_0 + X_1$ is orbitally equivalent to some $y \in Y_0 + Y_1$.

Following the ideas of the proof of Theorem 7.3.1 in [13] (in the case when both E_0 and E_1 are parameters of the real method) and [6], we will prove that for certain Banach pairs $\overline{X}$ the reiteration formula (2) holds without the assumption that both E_0 and E_1 are parameters of the real method. This provides new examples of real method spaces for which the general "commutativity" formula is valid (cf. [3], 4.7.3).

PROPOSITION 2.1. *Let $\overline{X} = (X_0, X_1)$ be an ordered pair of Banach lattices on a measure space (Ω, μ) with $X_j = L_\infty(\omega)$ for $j = 0$ or $j = 1$. Then:*

(i) *For any real parameter E_{1-j} and $E_j = \ell_\infty(2^{-jn})$ with corresponding $j = 0$ or 1 for which $X_j = L_\infty(\omega)$ the Banach pair $(\overline{X}_{E_0}, \overline{X}_{E_1})$ is a partial retract of the pair (E_0, E_1).*

(ii) *Any interpolation functor $\mathcal{F}$ is stable on $\overline{X}$ with respect to the pair $\overline{E}$ defined as in* (i).

Proof. Fix $0 \ne x \in \overline{X}_{E_0} + X_1$. Since $\overline{X}$ is an ordered pair, $x \in X_0^\circ + X_1^\circ$. Thus, by the fundamental lemma (see [2], Lemma 3.3.2, p. 35)

$$x = \sum_{n=-\infty}^{\infty} u_n \quad \text{(convergence in } X_0 + X_1)$$

and

$$J(2^n, u_n; \overline{X}) \le CK(2^n, x; \overline{X}),$$

with a universal constant $C < 4$. Since $\overline{X}$ is a pair of Banach lattices, the terms u_n can be chosen to have disjoint supports (see [6]), with possibly a new universal constant also denoted by C. This implies that the operator S defined by

$$S\xi = \sum_{u_n \ne 0} \frac{\xi_n u_n}{K(2^n, u_n; \overline{X})}$$

for $\xi = \{\xi_n\} \in \ell_1 + \ell_1(2^{-n})$ maps the pair $\overline{\ell}_1$ into $\overline{X}$, and $S(a_x) = x$, where $a_x = \{K(2^n, x; \overline{X})\}$.

Without loss of generality we can assume that $X_1 = L_\infty(\omega)$. Then by the disjointness of the supports of u_n, we have

$$\Big\|\sum_{k=-N}^{N} \frac{\xi_n u_n}{K(2^n, u_n; \overline{X})}\Big\|_{X_1} \le \|\xi\|_{E_1} \sup_{-N\le k\le N} \Big\|\frac{2^n u_n}{K(2^n, x; \overline{X})}\Big\|_{X_1} \le C\|\xi\|_{E_1}$$

for any $\xi = \{\xi_n\} \in E_1 = \ell_\infty(2^{-n})$ and $N \in \mathbb{N}$. Hence by the Fatou property of X_1, we get

$$\|T\xi\|_{X_1} \le C\|\xi\|_{E_1}.$$

Thus T is a bounded operator from E_1 into X_1. Now we show that x is orbitally equivalent to a_x. By the Hahn–Banach theorem there exists an operator $S:\overline{X}\to\ell_\infty$ (cf. the proof of Theorem 7.1.1 in [13]) such that

$$Sx=\{K(2^n,x;\overline{X})\}_{n=-\infty}^{\infty}.$$

Since E_0 is a parameter of the real method, we have by interpolation $S:(\overline{X}_{E_0},X_1)\to(E_0,\ell_\infty(2^{-n}))$. We also have $T:(E_0,E_1)\to(\overline{X}_{E_0},X_1)$. Since $ST(x)=S(a_x)=x$ and $\overline{X}_{E_1}=X_1$, by the Fatou property, the proof of (i) is finished.

The reiteration formula follows from the fact that if $x\in\mathcal{F}(\overline{X}_{E_0},\overline{X}_{E_1})$, then the orbitally equivalent element $a_x=\{K(2^n,x;\overline{X})\}_{n=-\infty}^{\infty}$ belongs to $\mathcal{F}(E_0,E_1)$, i.e., $x\in\overline{X}_{\mathcal{F}(\overline{E})}$. On the other hand, if $a_x=\{K(2^n,x;\overline{X})\}\in\mathcal{F}(\overline{X})$ then the orbitally equivalent element x belongs to $\mathcal{F}(\overline{X}_{E_0},\overline{X}_{E_1})$. ∎

As noted before, $\Phi=E(1/\varrho(2^n))$ is a real parameter for any translation invariant Banach lattice E and any $\varrho\in\mathcal{P}^{+-}$. In what follows for such Φ the K-space $\overline{X}_\Phi$ will be denoted by $\overline{X}_{\varrho,E}$.

PROPOSITION 2.2. *Let E_0, E_1 be translation invariant Banach lattices on $\mathbb{Z}$ and let $\varphi_0,\varphi_1\in\mathcal{P}^{+-}$ be such that for some $q>1$, $\{(\varphi_0/\varphi_1)(q^n)\}\asymp\{2^{-n}\}$. Then for any Banach pair $\overline{X}$ the pair $(\overline{X}_{\varphi_0,E_0},\overline{X}_{\varphi_1,E_1})$ is a partial retract of the pair $(E_0,E_1(2^{-n}))$.*

Proof. Recall from [8] that a real sequence $\{t_k\}$ is called *uniformly sparse* for a quasi-concave function ϱ if for any $A,B>0$ the number of k's for which $A\le\varrho(t_k)\le B$ or $A\le\varrho(t_k)/t_k\le B$ is finite and has a bound depending only on B/A.

We first observe that the sequence $\{q^k\}_{k=-\infty}^{\infty}$ is uniformly sparse for any $\varphi\in\mathcal{P}^{+-}$. To see this note that since $\varphi\in\mathcal{P}^{+-}$, for any $\varepsilon>0$ there exists $C(\varepsilon)>0$ such that

$$s_\varphi(t)\le C(\varepsilon)\max\{t^{\alpha_\varphi-\varepsilon},t^{\beta_\varphi+\varepsilon}\}.$$

This implies that for all $s,t>0$, we have

$$C(\varepsilon)/\min\{s^{\alpha_\varphi-\varepsilon},s^{\beta_\varphi+\varepsilon}\}\le\varphi(st)/\varphi(t).\tag{3}$$

Let $U=\{k\in\mathbb{Z}:A\le\varphi(q^k)\le B\}$. Since $\varphi\in\mathcal{P}^{+-}$, there exist $n\in\mathbb{Z}$ and $m\in\mathbf{N}$ such that

$$\varphi(q^n)\le A\le\varphi(q^{n+1})\quad\text{and}\quad\varphi(q^{n+m-1})\le B\le\varphi(q^{n+m}).$$

Hence by quasi-concavity of φ we have

$$\varphi(q^{n+m})/\varphi(q^n)\le 4B/A.$$

Take $\varepsilon>0$ such that $\alpha_\varphi-\varepsilon>0$. Combining the above, we obtain

$$q^{m(\alpha_\varphi-\varepsilon)}\le 4C(\varepsilon)B/A.$$

Consequently, from $\mathrm{card}(U)\le m+1$, we obtain

$$\mathrm{card}(U)\le\log_q(4C(\varepsilon)B/A)+1.$$

Since $\beta_\varphi<1$, we can show similarly that $\mathrm{card}\{k\in\mathbb{Z}:A\le\varphi(q^k)/q^k\le B\}$ has a bound depending only on B/A.

Now assume that φ_0 and φ_1 both satisfy the conditions of the proposition. Since the sequence $\{q^k\}$ is uniformly sparse for both φ_0 and φ_1, the functors

$$\mathcal{F}_j(\overline{X})=\{x\in X_0+X_1:\{K(q^n,x;\overline{X})/\varphi_j(q^n)\}\in E_j\}$$

do not depend on q (see [12], Section 7.6), and for any translation invariant Banach lattice E on $\mathbb{Z}$ and any $\varphi\in\mathcal{P}^{+-}$, the operator

$$(S\xi)_n=\sum_{k=-\infty}^{\infty}\min\{1,q^{n-k}\}\xi_k$$

is bounded in $E(1/\varphi(q^n))$. Therefore by the proof of Theorem 7.3.1 in [12], $(\mathcal{F}_0(\overline{X}),\mathcal{F}_1(\overline{X}))$ is a partial retract of $(E_0(1/\varphi_0(q^n)),E_1(1/\varphi_1(q^n)))$. Clearly this last pair is isometrically isomorphic to $(E_0,E_1((\varphi_0/\varphi_1)(q^n)))$. Thus if $\{(\varphi_0/\varphi_1)(q^n)\}\asymp\{2^{-n}\}$, we are done. ∎

COROLLARY 2.3. *Let E_0, E_1 be translation invariant Banach lattices on $\mathbb{Z}$ and let φ be a quasi-concave function. Then for any $0<\theta_j<1$, $j=0,1$, with $\theta_0\ne\theta_1$ and any Banach pair $\overline{X}$ the pair $(\overline{X}_{\varphi_0,E_0},\overline{X}_{\varphi_1,E_1})$ with $\varphi_j(t)=t^{\theta_j}\varphi(t)$ for $t>0$ is a partial retract of $(E_0,E_1(2^{-n}))$.*

Proof. We may assume that $\theta_0<\theta_1$. Clearly both φ_0 and φ_1 belong to $\mathcal{P}^{+-}$. Since $\{(\varphi_0/\varphi_1)(q^n)\}=\{2^{-n}\}$ for $q=2^{1/\theta_1-\theta_0}$, Proposition 2.2 applies. ∎

We are now finally able to state our main result of this section.

THEOREM 2.4. *Let E_j,F_j be translation invariant Banach lattices on $\mathbb{Z}$, and let the quasi-Banach sequence lattices Φ_j be such that $\Phi_j\hookrightarrow\ell_\infty$ and $M(E_j,F_j)\hookrightarrow M(E,\Phi_j)$, $j=0,1$. Assume that T is an operator such that $T:(\overline{X}_{\alpha_0,E_0},\overline{X}_{\alpha_1,E_1})\to(\overline{Y}_{\beta_0,F_0},\overline{Y}_{\beta_1,F_1})$ with $0<\alpha_j<1$, $0<\beta_j<1$, $\alpha_0\ne\alpha_1$ and $\beta_0\ne\beta_1$. Furthermore suppose that the Banach sequence lattices G_0 and G_1 on $\mathbb{Z}$ are such that the inclusion maps $F_j\hookrightarrow G_j$ are (F_j,ℓ_1)-summing and $(F_0,F_1(2^{-n}))_{\theta,\overline{\Phi}}=(G_0,G_1(2^{-n}))_{\theta,\overline{\Phi}}$ for $0<\theta<1$. Then T is bounded from $\overline{X}_{\alpha,E}$ into $\overline{Y}_F$, where $\alpha=(1-\theta)\alpha_0+\theta\alpha_1$, $0<\theta<1$ and $F=(F_0(2^{-n\beta_0}),F_1(2^{-n\beta_1}))_{\theta,\overline{\Phi}}$.*

Proof. By the reiteration theorem, we have

$$(\overline{Y}_{\beta_0,F_0},\overline{Y}_{\beta_1,F_1})_{\theta,\overline{\Phi}}=\overline{Y}_F,\tag{4}$$

where $F:=(F_0(2^{-n\beta_0}),F_1(2^{-n\beta_1}))_{\theta,\overline{\Phi}}$ and

$$(\overline{X}_{\alpha_0,E_0},\overline{X}_{\alpha_1,E_1})_{\theta,E}=\overline{X}_{(E_0(2^{-n\alpha_0}),E_1(2^{-n\alpha_1}))_{\theta,E}}.\tag{5}$$

Since E_j are translation invariant Banach lattices, we have $\ell_1 \hookrightarrow E_j \hookrightarrow \ell_\infty$ and thus
$$\ell_1(2^{-n\theta}) \hookrightarrow E_j(2^{-n\theta}) \hookrightarrow \ell_\infty(2^{-n\theta})$$
for any $0 \le \theta \le 1$. For any $0 \le \theta_j \le 1$, $j = 0,1$, we have the well known formula
$$K(t,\xi;\ell_1(2^{-n\theta_0}),\ell_1(2^{-n\theta_1})) = \sum_{k=-\infty}^{\infty} \min\{2^{-n\theta_0}, t2^{-n\theta_1}\}|\xi_k|.$$
This readily implies that if $\theta_0 \ne \theta_1$, then the following continuous inclusion holds with $\lambda = (1-\theta)\theta_0 + \theta\theta_1$:
$$E(2^{-n\lambda}) \hookrightarrow (\ell_1(2^{-n\theta_0}),\ell_1(2^{-n\theta_1}))_{\theta,E}.$$
On the other hand, by the well known equivalence
$$K(t,\xi;\ell_\infty(2^{-n\theta_0},\ell_\infty(2^{-n\theta_1})) \asymp \sup_{n\in\mathbb{Z}} \min\{2^{-n\theta_0}, t2^{-n\theta_1}\},$$
we obtain
$$(\ell_\infty(2^{-n\theta_0}),\ell_\infty(2^{-n\theta_1}))_{\theta,E} \hookrightarrow E(2^{-n\lambda}).$$
Combining the above, we find that for $\lambda = (1-\theta)\theta_0 + \theta\theta_1$,
$$(E_0(2^{-n\theta_0}), E_1(2^{-n\theta_1}))_{\theta,E} = E(2^{-n\lambda}).$$
Therefore, by (2), we obtain
$$(\overline{X}_{\alpha_0,E_0}, \overline{X}_{\alpha_1,E_1})_{\theta,E} = \overline{X}_{\alpha,E},$$
where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

Now let $T : (\overline{X}_{\alpha_0,E_0}, \overline{X}_{\alpha_1,E_1}) \to (\overline{Y}_{\beta_0,F_0}, \overline{Y}_{\beta_1,F_1})$ and let $x \in \overline{X}_{\alpha,E}$. By Proposition 2.2, $(\overline{X}_{\alpha_0,E_0}, \overline{X}_{\alpha_1,E_1})$ is a partial retract of $(E_0, E_1(2^{-n}))$. Thus there exists $\xi_x \in E_0 + E_1(2^{-n})$ orbitally equivalent to x such that $\xi_x = Ax$ for some operator $A : (\overline{X}_{\alpha_0,E_0}, \overline{X}_{\alpha_1,E_1}) \to (E_0, E_1(2^{-n}))$. From the above, we have
$$\xi_x \in (E_0, E_1(2^{-n}))_{\theta,E} = E(2^{-n\theta}).$$

Let ξ_y be an element in $F_0 + F_1(2^{-n})$ orbitally equivalent to $y = Tx$. By the orbital equivalence, there are operators
$$B : (E_0, E_1(2^{-n})) \to (\overline{X}_{\alpha_0,E_0}, \overline{X}_{\alpha_1,E_1})$$
and
$$C : (\overline{Y}_{\beta_0,F_0}, \overline{Y}_{\beta_1,F_1}) \to (F_0, F_1(2^{-n}))$$
such that $B(\xi_x) = x$ and $C(y) = \xi_y$. By assumption the inclusion maps $F_j \hookrightarrow G_j$ are (F_j,ℓ_1)-summing, thus by Proposition 1.5, the operator $U = CTB$ maps continuously $E(2^{-n\theta})$ into $(G_0, G_1(2^{-n}))_{\theta,\overline{\Phi}}$. Since $U(\xi_x) = \xi_y$ and $(F_0, F_1(2^{-n}))_{\theta,\overline{\Phi}} = (G_0, G_1(2^{-n}))_{\theta,\overline{\Phi}}$, we have
$$\xi_y \in (F_0, F_1(2^{-n}))_{\theta,\overline{\Phi}}.$$

But ξ_y is orbitally equivalent to $y = Tx$ in $(\overline{Y}_{\beta_0,F_0}, \overline{Y}_{\beta_1,F_1})$, thus there exists $V : (F_0, F_1(2^{-n})) \to (\overline{Y}_{\beta_0,F_0}, \overline{Y}_{\beta_1,F_1})$ such that $V(\xi_y) = y$. Hence by (2), we have
$$V : (F_0, F_1(2^{-n}))_{\theta,\overline{\Phi}} \to \overline{Y}_F$$
with $F = (F_0(2^{-n\beta_0}), F_1(2^{-n\beta_1}))_{\theta,\overline{\Phi}}$. Combining, we conclude that $y = Tx = V(\xi_y) \in \overline{Y}_F$. Consequently, by the closed graph theorem, T is bounded from $\overline{X}_{\alpha,E}$ into $\overline{Y}_F$. ∎

Let us now indicate how Ovchinnikov's result follows from ours (see [13]).

THEOREM 2.5. *Let $\overline{X}$ and $\overline{Y}$ be any Banach pairs and let*
$$T : (\overline{X}_{\alpha_0,p_0}, \overline{X}_{\alpha_1,p_1}) \to (\overline{Y}_{\beta_0,q_0}, \overline{Y}_{\beta_1,q_1}),$$
$0 < \alpha_j < 1$, $0 < \beta_j < 1$, $\alpha_0 \ne \alpha_1$, $\beta_0 \ne \beta_1$ and $1 \le p_i \le \infty$, $1 \le q_j \le \infty$, $j = 0,1$. Then T is bounded from $\overline{X}_{\alpha,p}$ into $\overline{Y}_{\beta,q}$, where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\beta = (1-\theta)\beta_0 + \theta\beta_1$, $0 < p \le \infty$, and $1/q = 1/p + (1-\theta)(1/q_0 - 1/p_0)_+ + \theta(1/q_1 - 1/p_1)_+$, where $x_+ := \max\{0,x\}$ for $x \in \mathbb{R}$.

Proof. Let $E_j = \ell_{p_j}$ and $F_j = \ell_{q_j}$ for $j = 0,1$. By Hölder's inequality we have
$$M(E_j, F_j) = M(\ell_{p_j}, \ell_{q_j}) = \ell_{r_j}$$
with equality of norms, where $1/r_j = (1/q_j - 1/p_j)_+$ for $j = 0,1$. Now if we take $E = \ell_p$ and $\Phi_j = \ell_{t_j}$ with $1/t_j = 1/p + 1/r_j$, we have $1/r_j = 1/t_j - 1/p$, $j = 0,1$. Hence
$$M(E, \Phi_j) = M(\ell_p, \ell_{t_j}) = \ell_{r_j}$$
with equality of norms. This shows that $M(E_j, F_j) = M(E, \Phi_j)$ with equality of norms, $j = 0,1$. By Bennett's result (see [1]), the inclusion map $\ell_p \hookrightarrow \ell_\infty$ is $(p,1)$-summing for any $1 \le p \le \infty$. Now Theorem 2.4 with $G_0 = \ell_\infty$ and $G_1 = \ell_\infty$ shows that the operator T is bounded from $\overline{X}_{\alpha,p}$ into $\overline{Y}_F$, where
$$F = (\ell_{q_0}(2^{-n\beta_0}), \ell_{q_1}(2^{-n\beta_1}))_{\theta,\overline{\Phi}}.$$
From Theorem 5 of [15] (see also [13]) it follows that for any Banach pair (A_0, A_1) and $0 < p_0, p_1 \le \infty$, $0 < \theta < 1$, we have
$$\mathcal{J}_{\theta,\ell_{p_0},\ell_{p_1}}(A_0, A_1) = (A_0, A_1)_{\theta,p},$$
where $1/p = (1-\theta)/p_0 + \theta/p_1$. It is well known (see [2], Theorem 5.6.1) that for all $0 < r \le \infty$,
$$(\ell_{q_0}(2^{-n\beta_0}), \ell_{q_1}(2^{-n\beta_1}))_{\theta,r} = (\ell_\infty(2^{-n\beta_0}), \ell_\infty(2^{-n\beta_1}))_{\theta,r} = \ell_r(2^{-n\beta}),$$
where $\beta = (1-\theta)\beta_0 + \theta\beta_1$. Since $1/q = (1-\theta)/t_0 + \theta/t_1$, we conclude that $F = \ell_q(2^{-n\beta})$. Thus T is bounded from $\overline{X}_{\alpha,p}$ into $\overline{Y}_{\beta,q}$. ∎

REMARK. Note that from the proof of Theorem 2.4, it follows that the statement of the theorem is true for Banach pairs $(\overline{X}_{\alpha_0,E_0}, \overline{X}_{\alpha_1,E_1})$ and

$(\overline{Y}_{\beta_0,F_0}, \overline{Y}_{\beta_1,F_1})$ with $0 \le \alpha_j \le 1$, $0 \le \beta_j \le 1$, $\alpha_0 \ne \alpha_1$, $\beta_0 \ne \beta_1$ satisfying the following conditions:

(i) $(\overline{X}_{\alpha_0,E_0}, \overline{X}_{\alpha_1,E_1})$ is a partial retract of $(E_0, E_1(2^{-n}))$,
(ii) $(\overline{Y}_{\beta_0,F_0}, \overline{Y}_{\beta_1,F_1})$ is a partial retract of $(F_0, F_1(2^{-n}))$.

THEOREM 2.6. *Let $\overline{X} = (X_0, X_1)$ and $\overline{Y} = (Y_0, Y_1)$ be ordered pairs of Banach lattices with $X_i = L_\infty(u)$ for $i = 0$ or $i = 1$ and $Y_k = L_\infty(v)$ for $k = 0$ or $k = 1$. Let $0 < \alpha_i < 1$, $\alpha_{1-i} = 0$, $1 \le p_i \le \infty$, $p_{1-i} = \infty$ for $i = 0$ or $i = 1$ where i is such that $X_i = L_\infty(u)$ and let $0 < \beta_j < 1$, $\beta_{1-j} = 0$, $1 \le q_j \le \infty$, $q_{1-j} = \infty$ where j is such that $Y_j = L_\infty(v)$. If $T : (\overline{X}_{\alpha_0,p_0}, \overline{X}_{\alpha_1,p_1}) \to (\overline{Y}_{\beta_0,q_0}, \overline{Y}_{\beta_1,q_1})$, then T is bounded from $\overline{X}_{\alpha,p}$ into $\overline{Y}_{\beta,q}$, where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\beta = (1-\theta)\beta_0 + \theta\beta_1$, $0 < p \le \infty$, and $1/q = 1/p + (1-\theta)(1/q_0 - 1/p_0)_+ + \theta(1/q_1 - 1/p_1)_+$.*

Proof. From Sparr's result [16], we know that $(\ell_p(2^{-ns}), \ell_\infty)$ and $(\ell_p, \ell_\infty(2^{-n}))$ are relative Calderón pairs for any $0 < s < 1$ and $1 \le p \le \infty$. This implies that these pairs are orbitally equivalent. Combining Proposition 2.1, the remark above and the proof of Theorem 2.5, we obtain the required result. ∎

3. Special $\mathcal{J}$-method spaces.

The description of abstract $\mathcal{J}$-method spaces is in general a difficult problem. We recall that if $\overline{\Phi} = (\Phi_0, \Phi_1)$ is a pair of Banach sequence lattices on $\mathbb{Z}$ which is a parameter for the $\mathcal{J}$-method, then by Theorem 4.2.33 of [3],

$$\mathcal{J}_{\overline{\Phi}}(\overline{X}) = \mathcal{J}_E(\overline{X})$$

for any Banach pair $\overline{X}$, where $E = \mathcal{J}_{\overline{\Phi}}(\ell_1, \ell_1(2^{-n}))$.

We now show that under certain conditions on $\overline{\Phi}$ we can identify the space $\mathcal{J}_{\overline{\Phi}}(\ell_1, \ell_1(2^{-n}))$. In what follows we work with Calderón–Lozanovskiĭ spaces. Recall that if $\overline{X} = (X_0, X_1)$ is a pair of Banach lattices on (Ω, μ) and $\mathcal{U}$ denotes the set of all concave and positive functions $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that $\psi(0,0) = 0$, then the *Calderón–Lozanovskiĭ space* $\psi(\overline{X}) = \psi(X_0, X_1)$ consists of all $x \in L^0(\mu)$ such that $|x| = \lambda\psi(|x_0|, |x_1|)$ a.e. for some $x_j \in X_j$, $j = 0, 1$. The space $\psi(\overline{X})$ is a Banach lattice equipped with the norm

$$\|x\| = \inf\{\max\{\|x_0\|_{X_0}, \|x_1\|_{X_1}\} : |x| = \psi(|x_0|, |x_1|),\ x_0 \in X_0,\ x_1 \in X_1\}$$

(see [10]). In the case of the power function $\psi_\theta(s,t) = s^{1-\theta}t^\theta$ with $0 \le \theta \le 1$, $\psi_\theta(\overline{X})$ is the well known Calderón space $X_0^{1-\theta}X_1^\theta$ (see [4]).

The subset of functions in $\mathcal{U}$ for which $\psi(s,1) \to 0$ and $\psi(1,t) \to 0$ as $s \to 0$ and $t \to 0$ is denoted by $\mathcal{U}_0$. If $\psi \in \mathcal{U}_0$ and $\psi^* \in \mathcal{U}_0$ where $\psi^*(s,t) = 1/\psi(1/s, 1/t)$ for $s, t > 0$, then ψ is called *non-degenerate*.

THEOREM 3.1. *Let $E_j \subset \ell_\infty$ for $j = 0, 1$ be Banach sequence lattices on $\mathbb{Z}$ and let $\psi \in \mathcal{U}$ be a non-degenerate function. There exists a sequence $\{t_n\}_{n\in\mathbb{Z}}$ of positive numbers such that for any pair (X_0, X_1) of Banach lattices and any operator $T : (X_0, X_1) \to (Y_0, Y_1)$ such that $T : X_j \to Y_j$ is (E_j, ℓ_1)-summing for $j = 0, 1$, the operator T is bounded from $\psi(X_0, X_1)$ into $\mathcal{J}_{\overline{\Phi}}(Y_0, Y_1)$, where $\Phi_0 = E_0(1/\varrho(t_{2n+1}))$, $\Phi_1 = E_1(t_{2n+1}/\varrho(t_{2n+1}))$ and $\varrho = \psi(1, \cdot)$.*

Proof. Let $q > 1$ be fixed and let $t_0 = 1$. Since ψ is non-degenerate, it follows by Proposition 3.3.5 of [3] that there exists a sequence $\{t_k\}_{k=-\infty}^{\infty}$ of positive numbers such that $\varrho(t_{2k})/t_{2k} = q\varrho(t_{2k+1})/t_{2k+1}$, $\varrho(t_{2k}) = q\varrho(t_{2k-1})$ for any $k \in \mathbb{Z}$ and $\{\varrho(t_{2n+1})\} \in \ell_1 + \ell_1(1/t_{2n+1})$. Since $E_j \hookrightarrow \ell_\infty$ for $j = 0, 1$, we get

$$\ell_\infty \hookrightarrow E_0'(\varrho(t_{2n+1})) + E_1'(\varrho(t_{2n+1})/t_{2n+1}).$$

Therefore, by Köthe duality, $\Phi_0 \cap \Phi_1 \hookrightarrow \ell_1$. Thus $\overline{\Phi} = (\Phi_0, \Phi_1)$ is a parameter of the $\mathcal{J}$-method.

Assume that X_0 and X_1 are Banach lattices defined on (Ω, μ). Let $0 \le x \in \psi(\overline{X})$ and $\|x\|_{\psi(\overline{X})} < 1$. Then $x = \psi(x_0, x_1)$ for some $0 \le x_j \in X_j$ such that $\|x_j\|_{X_j} < 1$, $j = 0, 1$. Since $\psi \in \mathcal{U}_0$, the support of x is contained in the intersection of the supports of x_0 and x_1. Hence without loss of generality we may suppose that x, x_0, x_1 are not zero on Ω.

Define for any $k \in \mathbb{Z}$ the measurable set

$$A_k = \{\omega \in \Omega : t_{2k} \le x_1(\omega)/x_0(\omega) < t_{2k+2}\}$$

and put

$$y_k = x\chi_{A_k}, \quad u_k = x_0\chi_{A_k}, \quad v_k = x_1\chi_{A_k}.$$

Clearly $\{A_k\}_{k\in\mathbb{Z}}$ is a sequence of pairwise disjoint subsets with union Ω, $y_k \in X_0 \cap X_1$ and $\sum_{k=-\infty}^{\infty} y_k = x$. We now show that

$$x = \sum_{k=-\infty}^{\infty} y_k \quad \text{(convergence in } X_0 + X_1\text{)}.$$

In fact for any $n \in \mathbb{Z}$, we have

$$y_k \le q\varrho(t_{2k+1})u_k \quad \text{and} \quad y_k \le \frac{\varrho(t_{2k})}{t_{2k}}x_1\chi_{A_k} \le q\frac{\varrho(t_{2k+1})}{t_{2k+1}}v_k.$$

This implies that for any $n \in \mathbb{Z}$, we have the estimates

$$0 \le \sum_{k=-\infty}^{-n} y_k \le \varrho(t_{-2n+1}) \sum_{k=-\infty}^{-n} \frac{y_k}{\varrho(t_{2k+1})} \le q\psi(1, t_{-2n+1})x_0,$$

$$0 \le \sum_{k=-\infty}^{-n} y_k \le \frac{\varrho(t_{2n+1})}{t_{2n+1}} \sum_{k=n}^{\infty} t_{2k+1}y_k\varrho(t_{2k+1}) \le \psi(t_{2n+1}, 1)x_1.$$

Since ψ is non-degenerate, $\psi(t_n,1)\to 0$ and $\psi(1,t_{-n})\to 0$ because $t_n\to 0$ and $t_{-n}\to 0$ as $n\to\infty$. Therefore,

$$\Big\|x-\sum_{k=-n+1}^{n-1} y_k\Big\|_{X_0+X_1}\le\Big\|\sum_{k=-\infty}^{-n} y_k\Big\|_{X_0}+\Big\|\sum_{k=-\infty}^{-n} y_k\Big\|_{X_1}$$
$$\le q\psi(1,t_{-2n+1})+q\psi(t_{2n+1},1)\to 0.$$

Further, for any $N\in\mathbb{N}$, by the previous discussion we have

$$0\le\sum_{|k|\le N}\frac{y_k}{\varrho(t_{2k+1})}\le q\sum_{|k|\in N} u_k\le qx_0,$$
$$0\le\sum_{|k|\in N}\frac{t_{2k+1}y_k}{\varrho(t_{2k+1})}\le q\sum_{|k|\in N} v_k\le qx_1.$$

Since in a Banach lattice every order bounded disjoint sequence is weakly ℓ_1-summable, it follows from the estimates above that the sequences $\{y_k/\varrho(t_{2k+1})\}$ and $\{t_{2k+1}y_k/\varrho(t_{2k+1})\}$ are weakly ℓ_1-summable.

Suppose that $T:\overline{X}\to\overline{Y}$ and $T:X_j\to Y_j$ is (E_j,ℓ_1)-summing, $j=0,1$. Then

$$Tx=\sum_{n=-\infty}^{\infty} Ty_n\quad(\text{convergence in } Y_0+Y_1)$$

and

$$\|\{\|Ty_n\|_{Y_0}/\varrho(t_{2n+1})\}\|_{E_0}\le\pi_{E_0,\ell_1}(T)\sup_{\varepsilon_n=\pm1}\Big\|\sum_{n=-\infty}^{\infty}\varepsilon_n\frac{y_n}{\varrho(t_{2n+1})}\Big\|_{X_0}$$
$$\le q\pi_{E_1,\ell_1}(T),$$
$$\|\{t_{2n+1}\|Ty_n\|_{Y_1}/\varrho(t_{2n+1})\}\|_{E_1}\le\pi_{E_0,\ell_1}(T)\sup_{\varepsilon_n=\pm1}\Big\|\sum_{n=-\infty}^{\infty}\varepsilon_n\frac{y_n}{\varrho(t_{2n+1})}\Big\|_{X_1}$$
$$\le q\pi_{E_0,\ell_1}(T).$$

This yields $Tx\in\mathcal{J}_{\overline{\Phi}}(\overline{Y})$ and $\|Tx\|_{\mathcal{J}_{\overline{\Phi}}(\overline{Y})}\le q\max\{\pi_{E_0,\ell_1}(T),\pi_{E_1,\ell_1}(T)\}$. ■

COROLLARY 3.2. *Suppose that E_j for $j=0,1$ are Banach sequence lattices on $\mathbb{Z}$ such that the inclusion maps $E_j\hookrightarrow\ell_\infty$ are (E_j,ℓ_1)-summing. Then for any positive sequences w_0 and w_1 on $\mathbb{Z}$ and $0<\theta<1$,*

$$E_0(w_0)^{1-\theta}E_1(w_1)^\theta\hookrightarrow\mathcal{J}_{\theta,\overline{E}}(\ell_\infty(w_0),\ell_\infty(w_1)).$$

Proof. It is easily seen that if $0<\theta<1$ and $\psi(s,t)=s^{1-\theta}t^\theta$ for $s,t\ge 0$, then the proof of Theorem 3.1 works for the sequence $\{t_k\}=\{2^k\}$. In fact since $E_j\hookrightarrow\ell_\infty$ is (E_j,ℓ_1)-summing, so is $E_j(w_j)\hookrightarrow\ell_\infty(w_j)$, for $j=0,1$. ■

PROPOSITION 3.3. *Suppose that E_j for $j=0,1$ are maximal translation invariant Banach lattices on $\mathbb{Z}$ such that the inclusion maps $E_j'\hookrightarrow\ell_\infty$ are (E_j',ℓ_1)-summing. Then for any Banach pair $\overline{X}$ and $0<\theta<1$,*

$$\mathcal{J}_{\theta,E_0,E_1}(\overline{X})\hookrightarrow\mathcal{J}_{E_0^{1-\theta}E_1(2^{-n})^\theta}(\overline{X}).$$

Proof. Observe that the method of proof of the duality result of Dmitriev [7] yields that for the $\mathcal{J}$-method spaces (cf. [9], Theorem 2.16), we have

$$\mathcal{J}_{\theta,\overline{E}}(\ell_1,\ell_1(2^{-n}))=\mathcal{J}_{\theta,E_0'',E_1''}((\ell_\infty)',\ell_\infty(2^n)')\hookrightarrow\mathcal{J}_{\theta,E_0',E_1'}(\ell_\infty,\ell_\infty(2^n))'.$$

From Corollary 3.2 we have

$$E_0'^{\,1-\theta}E_1'(2^n)^\theta\hookrightarrow\mathcal{J}_{\theta,E_0',E_1'}(\ell_\infty,\ell_\infty(2^n)).$$

Combining this with Lozanovskiĭ's result (see [10]) for the Köthe dual of Calderón–Lozanovskiĭ spaces, we get

$$\mathcal{J}_{\theta,\overline{E}}(\ell_1,\ell_1(2^{-n}))\hookrightarrow E_0^{1-\theta}E_1(2^{-n})^\theta.$$

Now the result follows since for any parameter $\overline{\Phi}$ of the $\mathcal{J}$-method we have the formula

$$\mathcal{J}_{\overline{\Phi}}(\overline{X})=\mathcal{J}_{\mathcal{J}_{\overline{\Phi}}(\overline{\ell}_1)}(\overline{X})$$

for any Banach pair $\overline{X}$ (see [3], Theorem 4.2.33). ■

THEOREM 3.4. *Suppose that E_j are maximal translation invariant Banach lattices on $\mathbb{Z}$ for $j=0,1$ such that the inclusion maps $E_j\hookrightarrow\ell_\infty$ and $E_j'\hookrightarrow\ell_\infty$ are (E_j,ℓ_1)-summing and (E_j',ℓ_1)-summing respectively. If $E_0^{1-\theta}E_1(2^{-n})^\theta$ is a parameter of the real method for some $0<\theta<1$, then for any Banach pair $\overline{X}$,*

$$\mathcal{J}_{\theta,E_0,E_1}(\overline{X})=\overline{X}_{E_0^{1-\theta}E_1(2^{-n})^\theta}.$$

Proof. Suppose that $E_0^{1-\theta}E_1(2^{-n})^\theta$ is a real parameter of the real method. Then for any Banach pair $\overline{X}$, we have

$$\mathcal{J}_{(E_0)^{1-\theta}E_1(2^{-n})^\theta}(\overline{X})=\overline{X}_{E_0^{1-\theta}E_1(2^{-n})^\theta}.$$

Combining this with Proposition 3.3, we get

$$\mathcal{J}_{\theta,\overline{E}}(\ell_1,\ell_1(2^{-n}))\hookrightarrow(\ell_1,\ell_1(2^{-n}))_{E_0^{1-\theta}E_1(2^{-n})^\theta}=E_0^{1-\theta}E_1(2^{-n})^\theta.$$

By Corollary 3.2, we have $E_0^{1-\theta}E_1(2^{-n})^\theta\hookrightarrow\mathcal{J}_{\theta,\overline{E}}(\ell_\infty,\ell_\infty(2^{-n}))$. We conclude that

$$\mathcal{J}_{\theta,\overline{E}}(\ell_1,\ell_1(2^{-n}))=\mathcal{J}_{\theta,\overline{E}}(\ell_\infty,\ell_\infty(2^{-n}))=E_0^{1-\theta}E_1(2^{-n})^\theta.$$

Since $\mathcal{J}_{\overline{\Phi}}=\mathcal{J}_{\mathcal{J}_{\overline{\Phi}}(\overline{\ell}_1)}$, an application of the minimality of the $\mathcal{J}$-method functor with respect to the pair $(\ell_1,\ell_1(2^{-n}))$ and the maximality of the K-method functor with respect to $(\ell_\infty,\ell_\infty(2^{-n}))$ (see [3], [12]) completes the proof. ■

The following known result (see [2], Theorem 3.12.1) is an immediate consequence of Theorem 3.4, Bennett's result [1] on $(q,1)$-summability of the inclusion map $\ell_q \hookrightarrow \ell_\infty$, $1 \le q < \infty$, and the easily verified relation

$$(\ell_{p_0})^{1-\theta}\ell_{p_1}(2^{-n})^{\theta} = \ell_p(2^{-n\theta}),$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$ for $0<\theta<1$ and $1 \le p_j < \infty$, $j=0,1$ (see [4]).

COROLLARY 3.5. *Let $1 \le p_j < \infty$ for $j=0,1$ and let $1/p = (1-\theta)/p_0 + \theta/p_1$ for $0<\theta<1$. Then for any Banach pair $\overline{X}$,*

$$\mathcal{J}_{\theta,\ell_{p_0},\ell_{p_1}}(\overline{X}) = \overline{X}_{\theta,p}.$$

We remark that in [11] a generalization of Bennett's result on $(r,1)$-summability of the inclusion map $\ell_p \hookrightarrow \ell_q$, $1 \le p \le q < \infty$, is proven. From these results it follows that there are reflexive Orlicz sequence spaces ℓ_φ generated by convex functions not equivalent to power functions such that the inclusion map $\ell_\varphi \hookrightarrow \ell_\infty$ is (ℓ_φ,ℓ_1)-summing and $\ell_{\varphi_*} \hookrightarrow \ell_\infty$ is $(\ell_{\varphi_*},\ell_1)$-summing, where φ_* is Young's complementary function to φ.

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