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Induced stationary process and structure of locally square integrable periodically correlated processes

by

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Abstract. A one-to-one correspondence between locally square integrable periodically correlated (PC) processes and a certain class of infinite-dimensional stationary processes is obtained. The correspondence complements and clarifies Gladyshev's known result [3] describing the correlation function of a continuous periodically correlated process. In contrast to Gladyshev's paper, the procedure for explicit reconstruction of one process from the other is provided. A representation of a PC process as a unitary deformation of a periodic function is derived and is related to the correspondence mentioned above. Some consequences of this representation are discussed.

1. Introduction. Periodically correlated processes were introduced by Gladyshev in [3] and were defined as continuous functions $x:\mathbb{R}\to\mathcal{H}$, from the set of real numbers $\mathbb{R}$ into a complex separable Hilbert space $\mathcal{H}$, for which there exists a number $T>0$ (called the *period*) such that

$$(1)\qquad (x(t),x(s))=(x(t+T),x(s+T))\quad \text{for all } s,t\in\mathbb{R}.$$

In the paper we will call such processes *Continuous Periodically Correlated* and abbreviate them CPC. If $x(t)$ is a CPC process then the mapping $V: x(t)\to x(t+T)$, $t\in\mathbb{R}$, extends linearly to a unitary operator V, called the *T-shift operator*, from the space $M_x=\overline{\operatorname{span}}\{x(t):t\in\mathbb{R}\}$ onto itself. If now W^t is any *continuous unitary representation* of $\mathbb{R}$ in some $\mathcal{K}\supseteq M_x$ such that $W^T=V$ on M_x, then the function $p(t)=W^{-t}x(t)$ is a continuous periodic function in $\mathcal{K}$ and $x(t)=W^tp(t)$. This gives the following theorem, which we will refer to as the *Structure Theorem for CPC Processes* (see also [8]).

THEOREM 1.1 (Structure Theorem for CPC Processes). *Let $T>0$. A function $x:\mathbb{R}\to\mathcal{H}$ is a CPC process with period T iff there are a Hilbert space $\mathcal{K}\supseteq\mathcal{H}$, a unitary representation W^t of $\mathbb{R}$ in $\mathcal{K}$, and a T-periodic*

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continuous function $p:\mathbb{R}\to\mathcal{K}$ such that

(2) $$x(t) = W^t p(t) \quad \textit{for every } t\in\mathbb{R}.$$

If $\widehat{p}(n) = T^{-1}\int_0^T e^{-2\pi int/T}p(t)\,dt$, $n\in\mathbb{Z}$, denotes the Fourier transform of a periodic function p, then neglecting the meaning of convergence for the moment, the representation (2) yields that

(3) $$x(t) = \sum_k e^{2\pi ikt/T}W^k(t)$$

where $W^k(t) = W^t\widehat{p}(k)$, $k\in\mathbb{Z}$, is a sequence of jointly stationary (in t) processes.

In other words, the Structure Theorem associates an infinite-dimensional stationary process with each CPC process. The weakness of this construction is that the triple $(\mathcal{K}, W^t, p(t))$ in the representation (2), and hence the process $W^k(t)$ in (3), is highly non-unique. Additionally, the natural choice for W^t suggested by Gladyshev in [2], $W^t = \int_0^{2\pi} e^{ist/T}\,E_V(ds)$, where E_V is the spectral resolution of V, always leads to a deterministic infinite-dimensional stationary process and hence fails to reflect prediction properties of $x(t)$. Because of possible future applications in prediction, there is a need for more concrete representation of the form (2).

Another way to assign an infinite-dimensional stationary process to a PC process is via its correlation function. This was done for CPC processes by Gladyshev in [3] and is described in the following theorem, which we will call the *Correspondence Theorem*. Recall that the *correlation function* of a process $x(t)$ is a function of two variables defined by $K(s,t) = (x(s), x(t))$, and the correlation function of an infinite-dimensional stationary process $(X^k(t))$ is an infinite matrix-valued function of one variable whose (k,j)th entry (k is a row and j is a column index) is $K^{k,j}(t) = (X^k(t), X^j(0))$.

THEOREM 1.2 (Correspondence Theorem for CPC Processes). *Let $K(s,t)$ be a continuous function of two variables such that $K(s,t) = K(s+T, t+T)$ for every $s,t\in\mathbb{R}$. Let*

$$a_j(t) = \frac{1}{T}\int_0^T e^{-2\pi iju/T}K(t+u,u)\,du.$$

Then the function K is the correlation function of a CPC process iff the infinite matrix-valued function

$$K(t) = [K^{k,j}(t)]_{k,j\in\mathbb{Z}} = [a_{k-j}(t)e^{-2\pi ikt/T}]_{k,j\in\mathbb{Z}}$$

is the correlation function of an infinite-dimensional stationary process.

In [5] the Correspondence Theorem was extended to a certain class of discontinuous PC processes. The Correspondence Theorem assigns a unique (in the sense of correlation function) infinite-dimensional stationary process, say $(X^k(t))$, to a CPC process $x(t)$. The original proof of the theorem, as well as the proof given in [5], employed direct computation to show that $K(s,t)$ is positive definite iff $K(t)$ is positive definite in a proper sense, and required some regularity conditions to be imposed on K to assure Cesàro summability. The proof did not provide an explicit construction of $X^k(t)$ from $x(t)$ or vice versa, and hence common approximation properties of these processes were not recoverable from Gladyshev's proof. An explicit one way construction of $(X^k(t))$ from $x(t)$ was then described in [11]; however, a way to recover $x(t)$ from $(X^k(t))$ remained unknown. In consequence the question whether each infinite-dimensional stationary process with correlation matrix

$$[K^{k,j}(t)]_{j,k\in\mathbb{Z}} = [a_{k-j}(t)e^{-2\pi ikt/T}]_{k,j\in\mathbb{Z}}$$

corresponds to some PC process in the sense described in Theorem 1.2 remained open, and will be answered in this paper. Note that Gladyshev's Correspondence Theorem gives an affirmative answer only under the additional assumption that $a_j(t)$'s are the Fourier coefficients of $K(t+\cdot,\cdot)$ for some continuous function K of two variables. Without this assumption it is not even clear whether the sequence $(a_j(t))$ is square summable for each t.

Although the Structure and Correspondence Theorems form foundations of the theory of PC processes, their mutual relationship was not clear. One of the main goals of the paper is also to link these two theorems.

We deal with functions defined almost everywhere w.r.t. Lebesgue measure (abbr. a.e.). A function f defined a.e. on $\mathbb{R}$ is called *T-periodic a.e.* if $f(u) = f(u+T)$ du-a.e. In this paper we define a periodically correlated process as follows.

DEFINITION 1.1. A *periodically correlated* (*PC*) *process* with period $T>0$ is a function $x:\mathbb{R}\to\mathcal{H}$, where $\mathcal{H}$ is a complex Hilbert space, which is Bochner square integrable over each compact interval and such that for every $t\in\mathbb{R}$,

(4) $$(x(t+u), x(u)) = (x(t+T+u), x(T+u)) \quad du\text{-a.e.}$$

In other words, a locally square integrable function $x:\mathbb{R}\to\mathcal{H}$ is a PC process with period $T>0$ iff for every $t\in\mathbb{R}$ the function $K(t+u,u) = (x(t+u), x(u))$ is T-periodic a.e. in u.

In Section 2 the Correspondence Theorem for the class of PC processes defined above is established and a constructive procedure to recover a PC process from an associated infinite-dimensional process is provided. This is the main result of the paper and most of the subsequent results are its consequences.

In Section 3 the Structure Theorem for PC processes is proved with the help of the Correspondence Theorem. Note that since a PC process is defined only almost everywhere (Definition 1.1), the T-shift operator V

cannot be defined via the extension of the mapping $V : x(t) \to x(t+T)$, $t \in \mathbb{R}$, as in the CPC case, and a more delicate approach is necessary. The main result of Section 3 is, however, an explicit representation of $x(t)$ in the form $x(t) = W^t p(t)$ constructed on the base of the Correspondence Theorem. As a consequence, an explicit representation of any PC process $x(t)$ in the form (3), with $W^k(t)$ sharing the regularity properties of $x(t)$, is produced. Finally, an unexpected link between the Correspondence Theorem and Mackey's Imprimitivity Theorem from Abstract Harmonic Analysis is noted.

Discontinuous PC processes have already been studied in [5] and [11] (bounded PC processes) and in [8] ($L_2[0,T)$-PC processes). *A bounded PC process* is a PC process which is norm bounded. An $L_2[0,T)$-*PC process* is a PC process that has a version for which the mapping $V : x(t) \to x(t+T)$ extends to a unitary operator. In Section 3 we prove that every PC process is an $L_2[0,T)$-PC process. An overview of the theory of PC processes on the line and on the set of integers can be found for example in [10].

The letters $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ will stand for the sets of real numbers, complex numbers, integers and positive integers, respectively. The symbols $\mathcal{H}$ and $\mathcal{K}$ will be reserved for complex Hilbert spaces. $L^2(I;\mathcal{H})$ will denote the Hilbert space of $\mathcal{H}$-valued functions on an interval I which are Bochner square integrable w.r.t. the Lebesgue measure over I.

Because of probabilistic background a Bochner measurable function $x : \mathbb{R} \to \mathcal{H}$ will be called a *stochastic process* or simply a *process*. A process $x(t)$ is said to be *locally square integrable* if $\int_I \|x(t)\|^2\,dt < \infty$ for each bounded interval I. If a process $x(t)$ is locally square integrable and $S \subseteq \mathbb{R}$ is open then $M_x(S)$ will stand for the space essentially spanned by the values of $x(t)$, $t \in S$, that is,

$$M_x(S) = \overline{\operatorname{span}}\Big\{ \int_{\mathbb{R}} \phi(t)x(t)\,dt : \phi \in B_0(S)\Big\}, \tag{5}$$

where $B_0(S)$ is the set of all bounded complex measurable functions which are 0 outside some compact $K \subset S$. For simplicity $M_x(\mathbb{R}) = M_x$. The function $K_x(s,t) = (x(s),x(t))$ is called the *correlation function* of $x(t)$. Two processes $x(t)$ and $y(t)$ are *unitarily equivalent* if there is a unitary operator $\Phi : M_x \to M_y$ such that $y(t) = \Phi x(t)$ dt-a.e.

LEMMA 1.1. *Two locally integrable processes $x(t)$ and $y(t)$ are unitarily equivalent iff $K_x(s,t) = K_y(s,t)$ $ds \times dt$-a.e.*

Proof. Assume first that $K_x(s,t) = K_y(s,t)$ $ds \times dt$-a.e. For each $\phi \in B_0(\mathbb{R})$ define $x(\phi) = \int_{-\infty}^{\infty} \phi(t)x(t)\,dt$ and similarly for $y(\phi)$. Then

$$(x(\phi),x(\psi)) = (y(\phi),y(\psi)), \qquad \phi,\psi \in B_0(\mathbb{R}).$$

and hence the mapping $\Phi x(\phi) = y(\phi)$, $\phi \in B_0(\mathbb{R})$, defines an isometry from M_x onto M_y. Since

$$\int_{-\infty}^{\infty} \phi(t)(y(t) - \Phi x(t))\,dt = y(\phi) - \Phi x(\phi) = 0$$

for all $\phi \in B_0(\mathbb{R})$, it follows that $y(t) = \Phi x(t)$ dt-a.e. The inverse implication is trivial. ∎

A *stationary process* is a continuous bounded process such that $K_x(s,t) = K_x(s-t,0)$ for all $s,t \in \mathbb{R}$. If $x(t)$ is stationary then its correlation function is a function of one variable defined by $K_x(t) = (x(t),x(0))$. An *infinite-dimensional stationary* (abbr. IDS) process $(X^k(t))$, $k \in \mathbb{Z}$, is a sequence of stationary processes such that for every $k,j \in \mathbb{Z}$, the crosscorrelation function $(X^k(t),X^j(s))$ depends only on $t-s$. The correlation function of an IDS process $(X^k(t))$, $k \in \mathbb{Z}$, is an infinite matrix function whose (k,j)th entry is $K_X^{k,j}(t) = (X^k(t),X^j(0))$. Since a stationary process is continuous, the space generated by the values of an IDS process $(X^k(t))$ can be defined without use of $B_0(S)$, namely $M_X(S) = \overline{\operatorname{span}}\{X^k(t) : t \in S,\ k \in \mathbb{Z}\}$. If $(X^k(t))$ is an IDS process then there is a continuous unitary representation U^t of $\mathbb{R}$ in M_X, called the *shift group* of the process $(X^k(t))$, such that $X^k(t) = U^t X^k(0)$ for every $k \in \mathbb{Z}$ and $t \in \mathbb{R}$ (simply define $U^t(X^k(s)) = X^k(s+t)$). Two IDS processes $(X^k(t))$ and $(Y^k(t))$ are unitarily equivalent if there is a unitary operator $\Phi : M_X \to M_Y$ such that $Y^k(t) = \Phi X^k(t)$ for every $t \in \mathbb{R}$ and $k \in \mathbb{Z}$; the latter holds iff $K_X^{k,j}(t) = K_Y^{k,j}(t)$ for every $t \in \mathbb{R}$ and $k,j \in \mathbb{Z}$.

2. Correspondence Theorem.

Recall that a *periodically correlated* (PC) process with period $T > 0$ is a locally square integrable process $x : \mathbb{R} \to \mathcal{H}$ such that for every $t \in \mathbb{R}$,

$$(x(t+u),x(u)) = (x(t+T+u),x(T+u)) \qquad du\text{-a.e.} \tag{6}$$

No assumption on continuity or existence of the T-shift operator V is imposed.

We split the Correspondence Theorem into two parts. The first part was proved by Makagon, Miamee and Salehi [11] for bounded PC processes and the proof carries over to the general case without any changes. For the future use we sketch the proof.

THEOREM 2.1 (Correspondence Theorem, Part 1). *Let $x(t)$ be a PC process with period T, K_x be the correlation function of $x(t)$ and let*

$$a_j(t) = \frac{1}{T}\int_0^T e^{-2\pi i j u/T} K_x(t+u,u)\,du.$$

Then the infinite matrix function $K(t) = [K^{k,j}(t)]_{j,k\in\mathbb{Z}}$ *whose* (k,j)*th entry is given by*

$$K^{k,j}(t) = a_{k-j}(t)e^{-2\pi ikt/T} \tag{7}$$

is the correlation function of an infinite-dimensional stationary process.

Proof. We construct an IDS process with correlation function $K(t)$. For every $k \in \mathbb{Z}$ and $t \in \mathbb{R}$ let $X^k(t)$ be the element of $L^2([0,T); M_x)$ defined by

$$X^k(t)(\cdot) = x(t+\cdot)e^{-2\pi ik(t+\cdot)/T}, \tag{8}$$

that is, $X^k(t)(\cdot)$ is the part of the trajectory $x(t+\cdot)e^{-2\pi ik(t+\cdot)/T}$ that is seen through the *window* $[0,T)$. Since x is locally square integrable and the translation group in $L^2(\mathbb{R};\mathcal{H})$ is continuous, each $X^k(t)$ is a continuous function of t. Moreover, since $K_x(t+u, s+u)$ is periodic a.e. in u, we obtain

$$\begin{aligned}
(X^k(t), X^j(s)) &= \frac{1}{T}\int_0^T e^{-2\pi ik(t+u)/T}e^{2\pi ij(s+u)/T}K_x(t+u, s+u)\,du \\
&= \frac{1}{T}\int_s^{s+T} e^{-2\pi ik(t-s+w)/T}e^{2\pi ijw/T}K_x(t-s+w, w)\,dw \\
&= e^{-2\pi ik(t-s)/T}\frac{1}{T}\int_0^T e^{-2\pi i(k-j)w/T}K_x(t-s+w, w)\,dw \\
&= e^{-2\pi k(t-s)/T}a_{k-j}(t-s).
\end{aligned}$$

Hence $(X^k(t))$ is an IDS process and $K_X^{k,j}(t) = e^{-2\pi ikt/T}a_{k-j}(t)$. ∎

The $L^2([0,T); M_x)$-valued process $(X^k(t))$ defined by (8) will be referred to as *induced* by the PC process $x(t)$. Note that $M_X = L^2([0,T); M_x)$. Indeed, if $f \in L^2([0,T); M_x)$ is such that

$$(f, X^k(t)) = \frac{1}{T}\int_0^T e^{2\pi iku/T}(f(u), x(t+u))\,du = 0$$

for every $k \in \mathbb{Z}$ and $t \in \mathbb{R}$, then $f(u) \perp M_x$ du-a.e., and hence $f = 0$ a.e. Also observe that from the Fubini theorem and Lemma 1.1 it follows that two PC processes are unitarily equivalent if and only if the processes induced by them are unitarily equivalent.

The roots of definition (8) can be traced back to Hurd [6] and Gardner and Franks [1]. In the discrete case, the definition of the multidimensional stationary sequence induced by a PC sequence was announced in the survey paper [10] and studied in [12] and [13]. Originally, the induced stationary sequence was defined in [12] for an arbitrary *periodically distributed* sequence with possibly infinite second moments.

Now we turn to the more difficult part of the Correspondence Theorem; namely, to showing that each IDS process $(Y^k(t))$ whose correlation is of the form $K^{k,j}(t) = e^{-2\pi ikt/T}a_{k-j}(t)$ is unitarily equivalent to an IDS process induced by some PC process $x(t)$. No assumption about the sequence $a_j(t)$ is made, even regarding the square summability of the series $\sum_j |a_j(t)|^2$.

The idea of the proof is to construct a process $x(t)$ by means of the process $(Y^k(t))$. Note that even if $Y^k(t) = X^k(t)$ is induced from $x(t)$ by formula (8), a straightforward recovery of $x(t)$ from $(X^k(t))$ is a rather complex matter. If $x : \mathbb{Z} \to \mathcal{H}$ is a PC sequence with period $T \in \mathbb{N}$, then $x(n)$ induces a T-dimensional stationary process $X^k(n)(\cdot) = e^{-2\pi ik(n+\cdot)/T}x(n+\cdot)$ with values in $\bigoplus_{j=0}^{T-1} M_x$ and one can take $x(n) = X^0(n)(0)$. This method does not work in the case of PC processes indexed by $\mathbb{R}$ because $X^k(t)(u)$ is an element of $L^2([0,T); M_x)$ and so it is defined du-a.e. while $x(t)$ lives on a single point $u \in [0,T)$, and hence the function $X^k(t)(0)$ may not be defined at all. The trick is to define a PC process in the subspace of constant functions in $L^2([0,T); M_x)$ rather than in M_x itself. The last remark will be clarified in Theorem 3.2 in the next section.

THEOREM 2.2 (Correspondence Theorem, Part 2). *If*

$$K^{k,j}(t) = e^{-2\pi ikt/T}a_{k-j}(t) \tag{9}$$

is the correlation matrix function of an IDS process, then there is a PC process $x(t)$ *such that*

$$a_j(t) = \frac{1}{T}\int_0^T e^{-2\pi iju/T}K_x(t+u, u)\,du$$

for every $j \in \mathbb{Z}$ *and* $t \in \mathbb{R}$.

Proof. Let $Y^k(t) \in \mathcal{K}$, $k \in \mathbb{Z}$, be an IDS process such that

$$(Y^k(t), Y^j(s)) = K^{k,j}(t-s) = e^{-2\pi ik(t-s)/T}a_{k-j}(t-s), \tag{10}$$

and let U^t be the shift group of $(Y^k(t))$. Define the operator S in M_Y as the linear extension of the mapping

$$S : Y^k(t) \to e^{2\pi it/T}Y^{k+1}(t). \tag{11}$$

Then from (10) it follows that

$$\begin{aligned}
\Big\| S\sum_p \alpha_p Y^{k_p}(t_p)\Big\|^2 &= \sum_p\sum_q \alpha_p\overline{\alpha}_q e^{2\pi ik_p(t_p-t_q)/T}a_{k_p-k_q}(t_p-t_q) \\
&= \Big\|\sum_p \alpha_p Y^{k_p}(t_p)\Big\|^2,
\end{aligned}$$

and hence S^n, $n\in\mathbb{Z}$, is a unitary representation of $\mathbb{Z}$ in M_Y. Since

$$S^nU^tY^k(s)=S^nY^k(s+t)=e^{2\pi i(t+s)n/T}Y^{k+n}(s+t)$$
$$=e^{2\pi itn/T}U^t(e^{2\pi isn/T}Y^{k+n}(s))=e^{2\pi itn/T}U^tS^nY^k(s),$$

the groups S^n, $n\in\mathbb{Z}$, and U^t, $t\in\mathbb{R}$, satisfy the commutation relation

$$S^nU^t=e^{2\pi itn/T}U^tS^n. \tag{12}$$

Define

$$\widetilde{p}(n)=\frac{1}{T}\int_0^T Y^n(s-T)\,ds,\qquad n\in\mathbb{Z}. \tag{13}$$

Since $Y^n(t)=e^{-2\pi int/T}S^nU^tY^0(0)$, we have

$$\widetilde{p}(n)=S^n\Big(\frac{1}{T}\int_0^T e^{-2\pi ins/T}U^{s-T}Y^0(0)\,ds\Big). \tag{14}$$

Therefore $\widetilde{p}(n)=S^n\widehat{f}(n)$ are isometric images of the Fourier coefficients of the $L^2([0,T);M_Y)$ function $f(\cdot)=U^{\cdot-T}Y^0(0)$, and hence

$$\sum_n\|\widetilde{p}(n)\|^2=\sum_n\|\widehat{f}(n)\|^2<\infty. \tag{15}$$

Let $p(t)=\sum_k\widetilde{p}(k)e^{2\pi ikt/T}$. Because of (15), p is a well defined element of $L^2([0,T);M_Y)$. Fix a version of p and extend it periodically to $\mathbb{R}$. For every $t\in\mathbb{R}$ define $x(t)=U^tp(t)$. Then $x(t+T)=U^Tx(t)$, $t\in\mathbb{R}$, and

$$K_x(t+u,u)=(x(t+u),x(u))=(U^tp(t+u),p(u)),\qquad u\in\mathbb{R},$$

is periodic in u and so $x(t)$ is PC.

To complete the proof we need to show that

$$\frac{1}{T}\int_0^T e^{-2\pi iku/T}K_x(t+u,u)\,du=a_k(t),\qquad t\in\mathbb{R},\ k\in\mathbb{Z}.$$

To do this observe first that from (14) and (12) it follows that

$$\widetilde{p}(k-n)=S^{k-n}\Big(\frac{1}{T}\int_0^T e^{-2\pi i(k-n)s/T}U^{s-T}Y^0(0)\,ds\Big)$$
$$=S^k\Big(\frac{1}{T}\int_0^T e^{-2\pi iks/T}U^{s-T}S^{-n}Y^0(0)\,ds\Big)$$
$$=S^k(U^{\cdot-T}S^{-n}Y^0(0))^\wedge(k).$$

Using the Parseval Identity twice and the property (12) we obtain

$$\frac{1}{T}\int_0^T e^{-2\pi iku/T}K_x(t+u,u)\,du$$
$$=(p(t+\cdot),U^{-t}p(\cdot)e^{2\pi ik\cdot/T})_{L^2[0,T)}$$
$$=\sum_j(e^{2\pi ijt/T}\widetilde{p}(j),U^{-t}\widetilde{p}(j-k))$$
$$=\sum_j e^{2\pi ijt/T}(U^tS^j(U^{\cdot-T}Y^0(0))^\wedge(j),S^j(U^{\cdot-T}S^{-k}Y^0(0))^\wedge(j))$$
$$=\sum_j(S^jU^t(U^{\cdot-T}Y^0(0))^\wedge(j),S^j(U^{\cdot-T}S^{-k}Y^0(0))^\wedge(j))$$
$$=\sum_j(U^t(U^{\cdot-T}Y^0(0))^\wedge(j),(U^{\cdot}S^{-k}Y^0(0))^\wedge(j))$$
$$=\frac{1}{T}\int_0^T(U^{s+t-T}Y^0(0),U^{s-T}S^{-k}Y^0(0))\,ds$$
$$=(U^tY^0(0),S_{-k}Y^0(0)=(Y^0(t),Y^{-k}(0))=a_k(t).\ \blacksquare$$

Since an infinite matrix-valued function with continuous entries $K^{j,k}(t)$ is the correlation matrix function of an IDS process iff it is a *positive definite matrix function* in the sense that for every positive integer n, complex numbers $c_1,\dots,c_n$, real numbers $t_1,\dots,t_n$ and integers $k_1,\dots,k_n$,

$$\sum_{i=1}^n\sum_{j=1}^n c_i\bar{c}_jK^{k_i,k_j}(t_i-t_j)\geq 0 \tag{16}$$

(see e.g. [4]), both theorems can be formulated in the language of positive definite functions.

3. Structure Theorem. In the proof of Theorem 2.2 we have already shown that every PC process is of the form $x(t)=W^tp(t)$, where p is a measurable periodic function and W^t a continuous representation of $\mathbb{R}$ (in a space possibly bigger than M_x).

THEOREM 3.1 (Structure Theorem for PC Processes). *For a locally square integrable process $x(t)$ the following three conditions are equivalent:*

(i) *$x(t)$ is PC with period T,*

(ii) *there is a Hilbert space $\mathcal{K}\supseteq M_x$, a continuous unitary representation W^t of $\mathbb{R}$ in $\mathcal{K}$ and a $\mathcal{K}$-valued, T-periodic a.e., process p such that $x(t)=W^tp(t)$ dt-a.e.,*

(iii) *there is a unitary operator V in M_x such that $x(t+T)=Vx(t)$ dt-a.e.*

Moreover, the group W^t above can be chosen to be unitarily equivalent to the shift group of the IDS process induced by $x(t)$.

Proof. The implication (i)⇒(ii) follows from Theorems 2.1 and 2.2. More precisely, for a given PC process $x(t)$ with correlation K_x, in the proof of Theorem 2.2 we constructed a PC process y of the form $y(t) = U^t p(t)$ dt-a.e. such that U^t is unitarily equivalent to the shift group of the IDS process induced by $x(t)$, and for every $t \in \mathbb{R}$ and $j \in \mathbb{Z}$,

$$\frac{1}{T}\int_0^T e^{-2\pi iju/T} K_y(t+u,u)\,du = a_j(t) = \frac{1}{T}\int_0^T e^{-2\pi iju/T} K_x(t+u,u)\,du.$$

The Fubini Theorem and Lemma 1.1 imply that the processes $x(t)$ and $y(t)$ are unitarily equivalent. This also takes care of the *moreover* part.

(ii)⇒(iii). If (ii) holds true then $x(t+T) = W^{T+t}p(t+T) = W^T(W^t p(t)) = W^T x(t)$ dt-a.e. This shows that W^T maps M_x onto itself and it is enough to define $V = W^T$ restricted to M_x.

(iii)⇒(i). Finally, if we assume that (iii) holds true, then the standard Gladyshev argument works. Namely, if we let W^t be a continuous representation of $\mathbb{R}$ in $\mathcal{K}$ such that $W^T = V$ on M_x and define $p(t) = W^{-t}x(t)$, then $p(t+T) = W^{-t}V^{-1}x(t+T) = W^{-t}x(t) = p(t)$ dt-a.e. ∎

The Structure Theorem was first proved by Gladyshev [2] for CPC sequences and his proof (see the proof of (iii)⇒(i) above) works provided the T-shift operator can be defined ([8], Propositions 1 and 9, [10], p. 160). In fact it also works in our case since from the fact that $x(t)$ and $x(t+T)$ have the same correlation function $ds \times dt$-a.e., and from Lemma 1.1, it immediately follows that $x(t+T) = Vx(t)$ dt-a.e. for some unitary V. We have used the Correspondence Theorem to derive the Structure Theorem instead, because that way explains the relationship between the two theorems and produces a specific realization of the triple $(\mathcal{K}, W^t, p(t))$. The details of this realization are spelled out in the next theorem. The symbol 1_A below stands for the indicator function of a set A.

THEOREM 3.2. *Let $x(t)$ be a PC process with period T. Let U^t be the shift group of the IDS process $X^k(t) \in L^2([0,T); M_x)$ induced by $x(t)$, and let $p : \mathbb{R} \to L^2([0,T); M_x)$ be the T-periodic function defined for $0 \le t < T$ by*

$$p(t)(u) = 1_{[0,t]}(u)x(t-T) + 1_{(t,T)}(u)x(t), \qquad u \in [0,T), \tag{17}$$

and then extended periodically to $\mathbb{R}$. Let $x_0(t) = U^t p(t)$, $t \in \mathbb{R}$. Then, except for t in a set of measure zero, the function $x_0(t)(u)$ is constant du-a.e. and this constant is equal to $x(t)$.

In other words, $x(t)$ can be written as $x(t) = U^t p(t)$, where p is periodic and given by (17) and U^t is the shift group of the IDS induced by $x(t)$.

Proof. Theorem 2.1 yields that the correlation of $(X^k(t))$ satisfies $K^{k,j}(t) = a_{k-j}(t)e^{-2\pi ikt/T}$, where $a_j(t) = T^{-1}\int_0^T e^{-2\pi iju/T}K_x(t+u,u)\,du$. We now reexamine the construction given in the proof of Theorem 2.2 using the process $(X^k(t))$ in place of $(Y^k(t))$. Recall that

$$X^k(t)(u) = x(t+u)e^{-2\pi k(t+u)/T}, \qquad u \in [0,T).$$

First observe that by definition (11), $SX^k(t) = e^{2\pi it/T}X^{k+1}(t)$, and hence S is multiplication by $e^{-2\pi i\cdot/T}$ in $L^2([0,T); M_x)$. From (14) we see that

$$\widetilde{p}(n)(u) = \frac{1}{T}\int_0^T e^{-2\pi in(s+u)/T}x(s+u-T)\,ds. \tag{18}$$

We now show that $\widetilde{p}(n)$ is the nth Fourier coefficient of the function (17), that is, the p from (17) is equal to the p from the proof of Theorem 2.2. Indeed, from

$$\begin{aligned}
\frac{1}{T}\int_0^T & e^{-2\pi int/T}p(t)(u)\,dt \\
&= \frac{1}{T}\int_0^T e^{-2\pi int/T}(1_{[0,t]}(u)x(t-T) + 1_{(t,T)}(u)x(t))\,dt \\
&= \frac{1}{T}\int_u^T e^{-2\pi int/T}x(t-T)dt + \frac{1}{T}\int_0^u e^{-2\pi int/T}x(t)\,dt,
\end{aligned}$$

by substituting $t = w+T$ in the first integral, relabeling $t = w$ in the second integral, and combining the integrals we obtain

$$\begin{aligned}
\frac{1}{T}\int_0^T e^{-2\pi int/T}p(t)(u)\,dt &= \frac{1}{T}\int_{u-T}^{u} e^{-2\pi inw/T}x(w)\,dw \\
&= \frac{1}{T}\int_0^T e^{-2\pi in(s+u)/T}x(s+u-T)\,ds = \widetilde{p}(n)(u).
\end{aligned}$$

Let $x_0(t) = U^t p(t)$, $t \in \mathbb{R}$. From the proof of Theorem 2.2 we conclude that $x_0(t)$ is an $L^2([0,T); M_x)$-valued PC process unitarily equivalent to $x(t)$.

We will show that except for t in a set of measure zero, the function $x_0(t)(u)$ is constant du-a.e. and this constant is equal to $x(t)$. To see this we need to describe the action of the group U^t. For every $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, let $\phi_{t,k}$ be the function with values in M_x defined on $\mathbb{R}$ by the formula

$$\phi_{t,k}(u) = e^{-2\pi ik(t+u)/T}x(t+u), \qquad u \in \mathbb{R}, \tag{19}$$

and let $\mathcal{M}$ be the set of all linear combinations of the functions $\phi_{t,k}$, $t \in \mathbb{R}$, $k \in \mathbb{Z}$. We emphasize that $\mathcal{M}$ consists of functions defined on the whole line. If we equip $\mathcal{M}$ with the norm $\|\phi\|^2_{\mathcal{M}} = (1/T)\int_0^T \|\phi(u)\|^2\,du$ and complete it (we keep the same symbol for the completion) then $\mathcal{M}$ becomes a Hilbert space isometric to $L^2([0,T);M_x)$ through the mapping $\Phi\phi = \phi_{|[0,T)}$ (the restriction of ϕ to $[0,T)$). From (8) it follows that the shift group U^t of $(X^k(t))$ is the translation group in $\mathcal{M}$ transported to $L^2([0,T);M_x)$ through Φ, that is, $U^t = \Phi T^t \Phi^{-1}$, where $(T^t\phi)(\cdot) = \phi(t+\cdot)$, $\phi \in \mathcal{M}$.

Note that every $u \in \mathbb{R}$ can be uniquely written as $u = [u] + mT$, where $[u]$ is the remainder of the division of u by T, $0 \le [u] < T$. If a sequence $\Phi(\sum_n a_n\phi_{t_n,k_n}) \to \psi$ in $L^2([0,T);M_x)$ and $\phi_{t,k}$'s are of the form (19), then $\sum_n a_n\phi_{t_n,k_n}$ converges in $\mathcal{M}$ to the function $\phi = \Phi^{-1}\psi$, which is given by $\phi(u) = \lim\sum_n a_n\phi_{t_n+mT,k_n}([u])$ du-a.e. In particular $(\Phi^{-1}p(t))(u) = p(t+mT)([u])$ if $mT \le u < (m+1)T$. This and definition (17) imply that for each fixed t, $\Phi^{-1}p(t)$ is a step function in u and

$$(\Phi^{-1}p(t))(u) = x(t) \quad \text{if } t \le u < t+T. \tag{20}$$

From $U^t = \Phi T^t\Phi^{-1}$ and (20) we conclude that $(T^t\Phi^{-1}p(t))(u) = x(t)$ for $u \in [0,T)$, and hence $U^t p(t)(u) = x(t)$ on $[0,T)$. ■

Any representation of the form $x(t) = W^t p(t)$ yields a representation

$$x(t) = \sum_k e^{2\pi ikt/T} W^k(t)$$

where $W^k(t) = W^t\widehat{p}(k)$, $k \in \mathbb{Z}$, is an IDS process, and so does the representation established in Theorem 3.2.

Recall that a process $z(t)$ is called *regular* if $\bigcap_t M_z((-\infty,t)) = \{0\}$.

THEOREM 3.3. *Let $x(t)$ be a PC process and let*

$$W^k(t)(\cdot) = \frac{1}{T}\int_{-T}^{0} e^{-2\pi ik(t+s+\cdot)/T} x(t+s+\cdot)\,ds. \tag{21}$$

Then $(W^k(t))$ is an $L^2([0,T);M_x)$-valued IDS process and

$$x(t) = \sum_k e^{2\pi ikt/T} W^k(t)(\cdot), \tag{22}$$

where the series converges in $L^2([-M,M];L^2([0,T);M_x))$ for every $M > 0$, that is,

$$\int_{-M}^{M}\int_0^T \Big\| \sum_{k=-n}^{n} e^{2\pi ikt/T} W^k(t)(u) - x(t)\Big\|^2 \xrightarrow{n} 0. \tag{23}$$

Moreover, if $(X^k(t))$ denotes the IDS process induced by $x(t)$, then $x(t)$ is regular iff $(W^k(t))$ is regular iff $(X^k(t))$ is regular.

Proof. Consider the representation $x(t) = U^t p(t)$, where U^t and $p(t)$ are defined in Theorem 3.2. Since the Fourier coefficients $\widehat{p}(n)$ of $p(t)$ are given by (18), after replacing $u-T$ by u in (18), we obtain $W^k(t) = U^t\widehat{p}(k)$. Hence $(W^k(t))$ is IDS and has the same shift group as the process $(X^k(t))$ induced by x.

Since $\sum_k e^{2\pi ikt/T}\widehat{p}(k)$ converges in $L^2([-M,M];L^2([0,T);M_x))$ for every $M > 0$, so does the series

$$\sum_k e^{2\pi ikt/T} W^k(t)(\cdot) = W^t\Big(\sum_k e^{2\pi ikt/T}\widehat{p}(k)(\cdot)\Big).$$

From Theorem 3.2 it follows that the sum of the latter is $W^t p(t)(\cdot) = x(t)$ dt-a.e. This proves (23).

To prove the *moreover* part note first that since we have $W^k(t) = T^{-1}\int_{-T}^0 X^k(t+s)\,ds$,

$$M_W((-\infty,t)) \subseteq M_X((-\infty,t)), \quad t \in \mathbb{R}, \tag{24}$$

and therefore the regularity of $X^k(t)$ implies the regularity of $W^k(t)$. Also it is not difficult to see that (cf. [11], Prop. 3.5)

$$M_X((-\infty,t)) = \{f \in L^2([0,T);M_x) : f(u) \in M_x((-\infty,t+u))\ du\text{-a.e.}\}. \tag{25}$$

Thus the regularity of $x(t)$ implies the regularity of $(X^k(t))$. Therefore it remains to prove that the regularity of $(W^k(t))$ implies the regularity of $x(t)$. Recall that for a fixed t, $x(t)$ is interpreted here as a constant function in $L^2([0,T);M_x)$ which equals $x(t)$ a.e. Since on compact sets L^2 convergence implies L^1 convergence, from (22) we have

$$x(\phi) = \sum_k \int_{-\infty}^{\infty} e^{2\pi ikt/T}\phi(t)W^k(t)\,dt \in M_W((-\infty,t))$$

for every $\phi \in B_0((-\infty,t))$. Therefore $M_x((-\infty,t)) \subseteq M_W((-\infty,t))$ and so if $(W^k(t))$ is regular then so is $x(t)$. ■

We conclude the paper with three remarks concerning Theorems 3.2 and 2.1.

REMARK 1. The facts that $p(t)$ is given by (17) and that $U^t p(t)(\cdot) = x(t)$ a.e. can also be seen from an easy, but less rigorous, argument described below. Note that

$$\sum_n e^{2\pi inz/T}\cdot\frac{1}{T}\int_0^T e^{-2\pi ins/T}x(r+s-T)\,ds \xrightarrow{L^2[-M,M]} x(r-T+[z]),$$

where $[z]$ is the remainder of division of z by T. This is because the integral above is the nth Fourier coefficient of the $L^2([0,T);M_x)$ function $x(r-T+\cdot)$

and $z = kT + [z]$. Extracting a subsequence of partial sums that converges a.e. we see that for a.e. $u \in [0, T)$,

$$\begin{aligned} p(t)(u) &= \sum_n e^{2\pi int/T} \cdot \frac{1}{T}\int_0^T e^{-2\pi in(s+u)/T} x(s+u-T)\,ds \\ &= \sum_n e^{2\pi in(t-u)/T} \cdot \frac{1}{T}\int_0^T e^{-2\pi ins/T} x(s+u-T)\,ds \\ &= x([t-u]+u-T) = \begin{cases} x(t-T) & \text{if } 0 \le u \le t < T, \\ x(t) & \text{if } 0 \le t \le u < T. \end{cases} \end{aligned}$$

Similarly

$$\begin{aligned} (U^t p(t))(u) &= \sum_n e^{2\pi int/T} \cdot \frac{1}{T}\int_0^T e^{-2\pi in(s+t+u)/T} x(s+t+u-T)\,ds \\ &= \sum_n e^{-2\pi inu/T} \cdot \frac{1}{T}\int_0^T e^{-2\pi ins/T} x(t+u+s-T)\,ds \\ &= x([-u]+t+u-T) = x(-u+T+t+u-T) = x(t). \end{aligned}$$

REMARK 2. The existence of the T-shift operator V for any PC process $x(t)$ allows us to describe the action of the shift group U^t of the process $(X^k(t))$ in a more direct way. Namely for every $f \in L^2([0,T); M_x)$,

$$(U^t f)(u) = V^{q(u+t)T} f([u+t]), \quad u \in [0,T), \tag{26}$$

where for any real $s \in \mathbb{R}$, $q(s)$ denotes the quotient and $[s]$ denotes the remainder of division of s by T (i.e. $s = q(s)T + [s]$ with $q(s) \in \mathbb{Z}$ and $[s] \in [0,T)$).

REMARK 3. Formula (26) and the commutation relation (12) indicate that there is a close link between Theorems 2.1 and 2.2 and the theory of induced representations (see for example [9]). In fact, although completely differently worded, the Correspondence Theorem is equivalent to the so-called Mackey Imprimitivity Theorem for the case $G = \mathbb{R}$ and $K = \{nT : n \in \mathbb{Z}\}$.

Mackey's Imprimitivity Theorem characterizes representations of groups induced from representations of their closed subgroups. Let $T > 0$ be fixed. Then $K = \{nT : n \in \mathbb{Z}\}$ is a closed subgroup of $\mathbb{R}$. Any representation D^k, $k \in K$, of K in a Hilbert space $\mathcal{H}$ induces a representation U^t of $\mathbb{R}$ in the space $L^2([0,T); \mathcal{H})$ by the formula

$$(U^t f)(u) = D^{k(u+t)} f([u+t]) \tag{27}$$

where for every $s \in \mathbb{R}$, $s = k(s) + [s]$ with $k(s) \in K$ and $[s] \in \mathbb{R}/K = [0,T)$ (compare with (26)). Such a representation of $\mathbb{R}$ is called *induced from the representation D^k of K*.

To see the connections with Mackey's Imprimitivity Theorem suppose that U^t, $t \in \mathbb{R}$, and S^λ, $\lambda \in \widehat{[0,T)} = \{2\pi ik/T : k \in \mathbb{Z}\}$, are continuous representations of $\mathbb{R}$ and of the dual of $\mathbb{R}/K = [0,T)$ in a Hilbert space $\mathcal{K}$, respectively. Assume also that $\mathcal{K} = \overline{\operatorname{span}}\{U^t S^\lambda z : t \in \mathbb{R}, \lambda \in \widehat{[0,T)}\}$ for some $z \in \mathcal{K}$ and that for every $t \in \mathbb{R}$ and $\lambda \in \widehat{[0,T)}$,

$$S^\lambda U^t = e^{i\lambda t} U^t S^\lambda. \tag{28}$$

Mackey's Imprimitivity Theorem for the pair $(\mathbb{R}, K)$ states that the condition (28) is necessary and sufficient for U^t to be unitarily equivalent to a representation induced from a representation of K.

This conclusion can be easily derived from Theorem 2.2. Define $Y^n(s) = U^s S^{\lambda_n} z$, where $\lambda_n = 2\pi n/T$, and observe that

$$(Y^k(t), Y^j(s)) = (U^t S^{\lambda_k} z, U^s S^{\lambda_j} z) = e^{-2\pi k(t-s)/T}(U^{t-s} z, S^{\lambda_j - k} z).$$

Hence the correlation of $(Y^k(t))$ is of the form (9). By Theorem 2.2 the process $(Y^n(t))$ is unitarily equivalent to an IDS induced by some PC process $x(t)$, and hence from (26) we conclude that U^t is unitarily equivalent to a representation induced from K.

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Compact endomorphisms of $H^\infty(D)$

by

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Abstract. Compact composition operators on $H^\infty(G)$, where G is a region in the complex plane, and the spectra of these operators were described by D. Swanton (*Compact composition operators on* $B(D)$, Proc. Amer. Math. Soc. 56 (1976), 152–156). In this short note we characterize all compact endomorphisms, not necessarily those induced by composition operators, on $H^\infty(D)$, where D is the unit disc, and determine their spectra.

Let D be the open unit disc and, as usual, let $H^\infty(D)$ be the algebra of bounded analytic functions on D with $\|f\| = \sup_{z\in D}|f(z)|$. With pointwise addition and multiplication, $H^\infty(D)$ is a well known uniform algebra. In this note we characterize the compact endomorphisms of $H^\infty(D)$ and determine their spectra.

We show that although not every endomorphism T of $H^\infty(D)$ has the form $T(f)(z) = f(\phi(z))$ for some analytic ϕ mapping D into itself, if T is compact, there is an analytic function $\psi : D \to D$ associated with T. In the case where T is compact, the derivative of ψ at its fixed point determines the spectrum of T.

The structure of the maximal ideal space M_{H^∞} is well known. Evaluation at a point $z \in D$ gives rise to an element in M_{H^∞} in the natural way. The remainder of M_{H^∞} consists of singleton Gleason parts and Gleason parts which are analytic discs. An analytic disc, $P(m)$, containing a point $m \in M_{H^\infty}$ is a subset of M_{H^∞} for which there exists a continuous bijection $L_m : D \to P(m)$ such that $L_m(0) = m$ and $\widehat{f}(L_m(z))$ is analytic on D for each $f \in H^\infty(D)$. Moreover, the map L_m has the form

$$L_m(z) = \mathrm{w}^* \lim \frac{z + z_\alpha}{1 + \overline{z}_\alpha z}$$

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