where $\psi_{ij} = \sum_{m=0}^{\infty} a_m z^{mn+i}$. Therefore,

$$(V_i L_{ij})(f) = \frac{\psi_{ij}}{z^j}(V_j f) \quad \text{for } f \in H.$$

Now using the same idea as in Theorem 2.2 we can complete the proof.

EXAMPLE 2.6. If $a_i = i+1$ in Theorem 2.5 then

$$K(z,w) = \frac{1}{(1-z)(1-\overline{w})(1-\overline{w}z)}$$

and $U = (U_{ij})_{i,j=0}^{\infty}$, where $U_{ij} = 0$ for $i > j$ and $U_{ij} = 1$ for $i \le j$. So $\{z^n/(1-z) : n \ge 0\}$ is an orthogonal basis for H.

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A class of l_1-preduals which are isomorphic to quotients of $C(\omega^\omega)$

by

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Abstract. For every countable ordinal α, we construct an l_1-predual X_α which is isometric to a subspace of $C(\omega^{\omega^{\omega^{\alpha+2}}})$ and isomorphic to a quotient of $C(\omega^\omega)$. However, X_α is not isomorphic to a subspace of $C(\omega^{\omega^\alpha})$.

1. Introduction. The study of quotients of $C(\alpha)$, for α a countable ordinal, is closely related to the problem of the isomorphic classification of the complemented subspaces of $C[0,1]$. Indeed, every complemented subspace of $C[0,1]$ is either isomorphic to a quotient of $C(\alpha)$ for some $\alpha < \omega_1$ (see [4]), or isomorphic to $C[0,1]$ (see [11]).

According to a result of Johnson and Zippin [8], every quotient of $C(\omega)$ is isomorphic to a subspace of $C(\omega)$. A natural question which arises then is if such a phenomenon occurs in $C(\alpha)$ for every $\alpha < \omega_1$. Alspach [1] gave a negative answer to this question by exhibiting a quotient of $C(\omega^\omega)$ which is not isomorphic to a subspace of $C(\alpha)$ for any $\alpha < \omega_1$.

Alspach's example left open the following question: Suppose X is isomorphic to a quotient of $C(\omega^\omega)$ and that there exists $\alpha < \omega_1$ with X isomorphic to a subspace of $C(\alpha)$. Is X isomorphic to a subspace of $C(\omega^\omega)$?

In this article, we answer this question in the negative by proving the following:

THEOREM 1.1. *For every countable ordinal α, there exists an l_1-predual space X_α with the following properties*:

1. *X_α is isomorphic to a quotient of $C(\omega^\omega)$.*

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2. X_α *is not isomorphic to a subspace of* $C(\omega^{\omega^\alpha})$.
3. X_α *is isometric to a subspace of* $C\big(\omega^{\omega^{\omega^\alpha+2}}\big)$.

We recall here that according to a result of Bessaga and Pełczyński [5], for every countable compact metric space K, there exists a unique countable ordinal α so that $C(K)$ is isomorphic to $C(\omega^{\omega^\alpha})$.

The spaces X_α are obtained by a general method of constructing l_1-preduals which are quotients of $C(\omega^\omega)$. (Definitions of the relevant concepts are given in Sections 2 and 3.) This method, Theorem 5.1, associates with every w^*-compact subset K of the probability measures on $[1,\omega]$ an l_1-predual space $X(K)$. Several choices of the set K give rise to l_1-preduals with interesting properties. For instance, if K is taken to be the w^*-closure of the set of the $(\omega^\alpha+1)$-averages (this concept is defined at the end of Section 2) of (δ_n), the sequence of point masses on $[1,\omega)$, then $X(K)$ is the space X_α satisfying the conclusion of Theorem 1.1.

If we choose K to be the set of all probability measures on $[1,\omega]$, then $X(K)$, which in this special case is denoted by X_∞, is isometric to the space constructed by Alspach in [1]. In fact, as shown in Corollary 6.1, X_∞ contains a contractively complemented subspace isometric to X_α for all $\alpha<\omega_1$. Therefore, as a corollary to our Theorem 1.1, X_∞ is isomorphic to a subspace of $C(\alpha)$ for no $\alpha<\omega_1$.

The proof of Theorem 1.1 is based on Theorems 4.3 and 1.2 which are proved in Sections 4 and 6 respectively. The former is a criterion for estimating the norming constant of a w^*-compact norming subset of B_{X^*}, in a Banach space X with separable dual X^*. The latter is a generalization of Alspach's [1] main lemma, which roughly says that if L is a w^*-compact countable subset of the closed unit ball of $l_1(\omega)$, then there exists a probability measure on $[1,\omega)$ which is "almost" mutually singular with respect to each member of L. More precisely, in Section 6 we show:

THEOREM 1.2. *Let K be a w^*-compact subset of $B_{l_1(\omega)}$ homeomorphic to $[1,\omega^\alpha n]$ for some $\alpha<\omega_1$ and $n\in\mathbb{N}$. (K is endowed with the w^*-topology and $[1,\omega^\alpha n]$ is given the order topology.) Let $\varepsilon>0$ and (y_i), a convex block subsequence of (δ_i), be given. There exists a convex block subsequence (z_i) of (y_i), consisting of $(\alpha+1)$-averages of (y_i), such that z_i is ε-disjoint from K for all $i\in\mathbb{N}$.*

We recall that if K is a compact metric space and $\varepsilon>0$, then two signed Borel measures $\mu,\nu\in B_{C(K)^*}$ are called *ε-disjoint* if there exist disjoint Borel measurable subsets A, B of K so that $K=A\cup B$ and $|\mu|(A)<\varepsilon$, $|\nu|(B)<\varepsilon$. If $M\subset B_{C(K)^*}$, then μ is called ε-disjoint from M provided that μ, ν are ε-disjoint for all $\nu\in M$. It is shown in [1] that if K is a countable w^*-compact subset of $B_{l_1(\omega)}$ and $\varepsilon>0$, then there is a probability measure μ on $[1,\omega)$ which is ε-disjoint from K. Theorem 1.2 generalizes this result.

Finally, we remark that none of the spaces X_α is isomorphic to a complemented subspace of $C[0,1]$, since each of them contains a contractively complemented subspace isometric to the Alspach–Benyamini l_1-predual [3].

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2. Preliminaries. We shall make use of standard Banach space facts and terminology. In this section we review some of the necessary concepts.

For a Banach space X, B_X denotes its closed unit ball. A subset B of B_{X^*} is called *δ-norming*, $\delta>0$, provided that δB_{X^*} is contained in the w^*-closure of the convex hull of $B\cup -B$. $\mathcal{L}(X)$ is the Banach space of bounded linear operators from X into itself under the operator norm.

l_1 denotes the Banach space of the absolutely summable sequences under the norm given by the sum of the absolute values of the coordinates. X is an l_1-predual if X^* is isometric to l_1.

If K is a compact metric space, A is a closed subset of K and $\alpha<\omega_1$, then we write $A^{(\alpha)}$ to denote the αth derived set of A. If $x\in K$, then δ_x stands for the point mass at x.

$C(K)$ denotes the Banach space of real-valued functions continuous on K equipped with the supremum norm. If α is an ordinal and $K=[1,\alpha]$, the space of ordinals not exceeding α, with the order topology, then we alternatively write $C(\alpha)$ to denote $C(K)$. $C_0(\alpha)$ is the subspace of $C(\alpha)$ consisting of the functions vanishing at α.

If L is a w^*-compact subset of B_{X^*} and $\alpha<\omega_1$, then we say that L is homeomorphic to $[1,\alpha]$ if there exists a map between L and $[1,\alpha]$ which is a homeomorphism when L is endowed with the w^*-topology and $[1,\alpha]$ is given the order topology.

If K is a countable compact metric space then $C(K)^*$ is isometrically isomorphic to l_1. We adopt the notation $l_1(K)$ for l_1 viewed as the dual of $C(K)$. If $K=[1,\alpha]$ for some $\alpha<\omega_1$, then we write $l_1(\alpha)$ instead of $l_1(K)$. By the w^*-topology of $l_1(K)$ we mean the $\sigma(l_1(K),C(K))$-topology. The *positive face* of the ball of $l_1(K)$ is the set $\{\sum_{a\in K}\lambda_a\delta_a:\lambda_a\ge 0,\ \sum_{a\in K}\lambda_a=1\}$.

In the sequel we will, without further comment, consider elements of the space $l_1(K)$ as either functions defined on K, or as measures on K.

Next we recall the definition of a convex block subsequence of a sequence (e_i) in X. Given finite subsets F, G of $\mathbb{N}$, we denote by $F<G$ the relation $\max F<\min G$. A sequence (x_i) is called a *convex block subsequence* of (e_i) if there exist sets $F_i\subset\mathbb{N}$ with $F_1<F_2<\ldots$ and a sequence of non-negative scalars (a_i) such that for every $i \in \mathbb{N}$, $x_i = \sum_{n\in F_i} a_n e_n$ and $\sum_{n\in F_i} a_n = 1$. By $s(x_i)$ we denote the support of x_i, i.e. the set $\{n \in F_i : a_n > 0\}$. We write $x_1 < x_2 < \ldots$ to indicate that $F_1 < F_2 < \ldots$

We now pass to the definition of an α-average of the sequence (e_i), where α is a countable ordinal. The set of the 0-averages of (e_i) is $\{e_i : i \in \mathbb{N}\}$. Assume that the set of the β-averages of (e_i) has been defined for every $\beta < \alpha$. Suppose first that α is a successor ordinal, say $\alpha = \beta + 1$. A vector $x \in X$ is an *α-average* of (e_i) if there exist $n \in \mathbb{N}$ and $x_1 < \ldots < x_n$, where x_j is a β-average of (e_i) for all $j \le n$, so that $x = (x_1 + \ldots + x_n)/n$.

If α is a limit ordinal, let (a_n) be a strictly increasing sequence of ordinals tending to α. A vector $x \in X$ is now called an *α-average* of (e_i) if there exists $n \in \mathbb{N}$ so that x is an α_n-average of (e_i) and $e_n \le x$. Clearly, an α-average of (e_i) is a finite convex combination of elements of (e_i).

3. Tree description of ordinal intervals. In this article we shall use trees in order to describe the ordinal intervals $[1, \omega^n]$, $n \in \mathbb{N}$. Recall that a tree $(\mathcal{T}, \leqslant)$ is a non-empty partially ordered set such that for all $x \in \mathcal{T}$, the set of the predecessors of x in $\mathcal{T}$ is well ordered.

$\mathcal{T}_\infty$ denotes the tree of all finite sequences of positive integers under the following partial order: Given $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$ in $\mathcal{T}_\infty$, we have $\alpha \leqslant \beta$ if and only if $n \le m$ and $a_i = b_i$ for $i \le n$.

We denote by $(\emptyset)$ the empty sequence and by $|\alpha|$ the level of α. That is, $|\alpha| = n$ if $\alpha = (a_1, \ldots, a_n)$.

We set $\mathcal{T}_n = \{\alpha \in \mathcal{T}_\infty : |\alpha| \le n\} \cup \{(\emptyset)\}$ for all $n \in \mathbb{N}$. If $\alpha \in \mathcal{T}_n$, then F_α denotes the set of the followers of α in $\mathcal{T}_n$. Thus, $F_\alpha = \{\beta \in \mathcal{T}_n : \alpha \leqslant \beta\}$. We also set $D_\alpha = \emptyset$ if $|\alpha| = n$, and $D_\alpha = \{(\alpha, i) : i \in \mathbb{N}\}$ if $|\alpha| < n$. Given a finite subset F of D_α, we set

$$\mathcal{U}_F = \{\beta \in \mathcal{T}_n : \gamma \leqslant \beta \text{ for some } \gamma \in D_\alpha \setminus F\}.$$

It is not difficult to show that the family $\{\mathcal{U}_F \cup \{\alpha\} : \alpha \in \mathcal{T}_n,\ F \subset D_\alpha \text{ finite}\}$ forms a basis for a Hausdorff topology in $\mathcal{T}_n$. It then follows that $\mathcal{T}_n$ endowed with this topology becomes a countable compact metric space homeomorphic to $[1, \omega^n]$.

We let U_n denote the *canonical w^*-continuous projection* of $l_1(\mathcal{T}_{2n})$ onto its subspace $[\delta_\alpha : \alpha \in \mathcal{T}_{2n},\ |\alpha| \le 1]$, that is, the linear projection induced by the relations

$$U_n(\delta_{(m,\alpha)}) = \delta_{(m)} \quad \text{for all } \alpha \in \mathcal{T}_{2n-1} \text{ and } m \in \mathbb{N}$$

and

$$U_n(\delta_\alpha) = \delta_\alpha \quad \text{for all } \alpha \in \mathcal{T}_{2n},\ |\alpha| \le 1.$$

Clearly, U_n is w^*-continuous and $\|U_n\| = 1$.

Finally we state the following lemma, whose proof is straightforward and therefore we omit it.

LEMMA 3.1. *Let x be a finitely supported normalized element of $l_1(\mathcal{T}_1)$ whose support lies in the set $\{\alpha \in \mathcal{T}_1 : |\alpha| = 1\}$. If (x_n) is a sequence in $l_1(\mathcal{T}_1)$ such that $\|x_n\| \le 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ (w^*), then $\lim_{n\to\infty} x_n = x$ in norm.*

4. Estimating norming constants. In this section we give a criterion, Theorem 4.3, for estimating the norming constant of a w^*-compact norming subset of B_{X^*} for a Banach space X with X^* separable. This criterion was motivated by the proof of the main result of [1].

DEFINITION 4.1. Let X be a Banach space and A, B subsets of X. Then A and B are *ε-almost l_1^2* for some $\varepsilon > 0$ if the following holds for every $x \in A$ and $y \in B$:

$$\|ax + by\| \ge |a|\|x\| + |b|\|y\| - \varepsilon(|a| + |b|) \quad \text{for all } a, b \in \mathbb{R}.$$

REMARK. If μ, ν are ε-disjoint, then $\{\mu\}$ and $\{\nu\}$ are 2ε-almost l_1^2.

DEFINITION 4.2. Let X be a Banach space, $B \subset B_X$, $x \in B_X$ and $\varepsilon > 0$. Suppose that $V \in \mathcal{L}(X)$ with $\|V\| \le 1$. We say that V satisfies $(*)$ for (ε, x, B) if it admits a decomposition

$$V = V_1 - V_2 \quad \text{where } V_i \in \mathcal{L}(X) \text{ and } \|V_i\| \le 1 \text{ for } i = 1, 2,$$

so that the following properties are satisfied:

1. $\|V_i(x)\| > 1 - \varepsilon$, for $i = 1, 2$.
2. $V_1(B)$ and $V_2(B)$ are ε-almost l_1^2.

We are now able to state our criterion.

THEOREM 4.3. *Let X be a Banach space with a separable dual X^*. Suppose that B is a w^*-compact, δ-norming subset of B_{X^*} and $n \in \mathbb{N}$. Assume that for every $\varepsilon > 0$, there exist $x^* \in B_{X^*}$ and $(T_i)_{i=1}^n \subset \mathcal{L}(X^*)$ so that:*

1. *T_i satisfies $(*)$ for (ε, x^*, B) and all $i \le n$.*
2. *$\sum_{i=1}^n \|T_i(y^*)\| \le 1$ for all $y^* \in B_{X^*}$.*

Then $\delta \le 1/(2n)$.

Proof. Let $\varepsilon > 0$. Choose $x^* \in B_{X^*}$ and $(T_i)_{i=1}^n \subset \mathcal{L}(X^*)$ according to the hypothesis. For every $i \le n$, T_i admits a decomposition $T_i = T_{i1} - T_{i2}$ so that:

(a) $T_{ij} \in \mathcal{L}(X^*)$ and $\|T_{ij}\| \le 1$ for $j \le 2$,
(b) $\|T_{ij}(x^*)\| > 1 - \varepsilon$ for $j \le 2$,
(c) $T_{i1}(B)$ and $T_{i2}(B)$ are ε-almost l_1^2.

Next, since X^* is separable and B is a w^*-compact subset of X^*, we observe that according to a result of Bessaga and Pełczyński [6], the norm closed convex hull of $B \cup -B$ coincides with the w^*-closed convex hull of $B \cup -B$. Since B is a δ-norming subset of B_{X^*}, it now follows that there exist $(z_i^*)_{i=1}^m \subset B$ and scalars $(\lambda_i)_{i=1}^m$ so that

$$\text{(d)} \qquad \Big\|x^* - \sum_{k=1}^m \lambda_k z_k^*\Big\| < \varepsilon \quad \text{and} \quad \sum_{k=1}^m |\lambda_k| \le \frac{1}{\delta}.$$

Fix now $i \le n$. By (c), we have

$$\|aT_{i1}(z_k^*) + bT_{i2}(z_k^*)\| \ge |a|\|T_{i1}(z_k^*)\| + |b|\|T_{i2}(z_k^*)\| - \varepsilon(|a| + |b|)$$

for all $k \le m$ and a, b in $\mathbb{R}$.

Applying the preceding inequality for $a = \lambda_k$, $b = -\lambda_k$, $k \le m$, we obtain

$$|\lambda_k|\|T_i(z_k^*)\| \ge |\lambda_k|\|T_{i1}(z_k^*)\| + |\lambda_k|\|T_{i2}(z_k^*)\| - 2\varepsilon|\lambda_k|.$$

Summing over k we get

$$\sum_{k=1}^m |\lambda_k|\|T_i(z_k^*)\| \ge \Big\|T_{i1}\Big(\sum_{k=1}^m \lambda_k z_k^*\Big)\Big\| + \Big\|T_{i2}\Big(\sum_{k=1}^m \lambda_k z_k^*\Big)\Big\| - 2\varepsilon \sum_{k=1}^m |\lambda_k|.$$

If we apply (a), (b), (d), the preceding inequality yields

$$\sum_{k=1}^m |\lambda_k|\|T_i(z_k^*)\| \ge 2(1-2\varepsilon) - 2\varepsilon \sum_{k=1}^m |\lambda_k|$$

for all $i \le n$. Summation over i now gives

$$\sum_{i=1}^n \sum_{k=1}^m |\lambda_k|\|T_i(z_k^*)\| \ge 2n(1-2\varepsilon) - 2\varepsilon n \sum_{k=1}^m |\lambda_k|,$$

or

$$\sum_{k=1}^m |\lambda_k| \sum_{i=1}^n \|T_i(z_k^*)\| \ge 2n(1-2\varepsilon) - 2\varepsilon n \sum_{k=1}^m |\lambda_k|.$$

But from our hypothesis we have

$$\sum_{i=1}^n \|T_i(z_k^*)\| \le 1 \quad \text{for all } k \le m,$$

and thus,

$$\sum_{k=1}^m |\lambda_k| \ge \frac{2n(1-2\varepsilon)}{1+2n\varepsilon}.$$

Hence,

$$\frac{1}{\delta} \ge \frac{2n(1-2\varepsilon)}{1+2n\varepsilon}.$$

Since ε was arbitrary, we obtain the desired estimate $\delta \le 1/(2n)$. ∎

5. A construction of l_1-preduals. The aim of this section is to prove the following

THEOREM 5.1. *Let K be a w^*-compact subset of the positive face of the ball of $l_1(\mathcal{T}_1)$. There exists a sequence (X_n) of l_1-preduals (depending on K) so that for every $n \in \mathbb{N}$ the following are satisfied:*

1. *The dual, Y_n, of X_n is a w^*-closed subspace of $l_1(\mathcal{T}_{2n})$ with a basis C_n consisting of disjointly supported finite convex combinations of elements of $\{\delta_\alpha : \alpha \in \mathcal{T}_{2n}\}$.*
2. *$C_1 = \{x_i : i \in \mathbb{N}\} \cup \{\delta_\alpha : \alpha \in \mathcal{T}_2,\ |\alpha| \le 1\}$, where the support of each x_i is contained in $\{\alpha \in \mathcal{T}_2 : |\alpha| = 2\}$, and K is equal to the set of the w^*-cluster points of the sequence (x_i) in $l_1(\mathcal{T}_2)$.*
3. *There exists a sequence $(C_{n-1,i})$ of pairwise disjoint subsets of C_n so that the support of each member of $C_{n-1,i}$ is contained in $\{\alpha \in \mathcal{T}_{2n} : |\alpha| > 1\}$ and*

$$C_n = \bigcup_{i=1}^\infty C_{n-1,i} \cup \{\delta_\alpha : \alpha \in \mathcal{T}_{2n},\ |\alpha| \le 1\}.$$

Moreover, for all $i \in \mathbb{N}$, $Y_{n-1,i} = [C_{n-1,i}]$ is w^*-closed in Y_n and w^*-isometric to Y_{n-1}. (We take $C_{0,i} = \{x_i\}$ for all $i \in \mathbb{N}$.)

Proof. We first let $\{z_i : i \in \mathbb{N}\}$ be a countable norm dense subset of K. Then, for each $i \in \mathbb{N}$, we choose a sequence (y_{ij}) of finitely supported convex combinations of elements of $\{\delta_{(n)} : n \in \mathbb{N}\} \cup \{\delta_{(\emptyset)}\}$ such that

$$\lim_{j\to\infty} \|y_{ij} - z_i\| = 0.$$

Let (M_i) be an infinite partition of $\mathbb{N}$ into pairwise disjoint infinite subsets. Set

$$x_j = \sum_{l=1}^\infty y_{ij}((l))\delta_{(l,j)} + y_{ij}((\emptyset))\delta_{(j,j)} \quad \text{for all } j \in M_i \text{ and } i \in \mathbb{N}.$$

Clearly, (x_i) consists of disjointly supported convex combinations of elements of $\{\delta_\alpha : \alpha \in \mathcal{T}_2,\ |\alpha| = 2\}$ and the set of its w^*-cluster points in $l_1(\mathcal{T}_2)$ equals K.

Set $Y_1 = [C_1]$, where

$$C_1 = \{x_i : i \in \mathbb{N}\} \cup \{\delta_\alpha : \alpha \in \mathcal{T}_2,\ |\alpha| \le 1\}.$$

Since every w^*-cluster point of C_1 is contained in Y_1, we deduce, by Lemma 1 of [2], that Y_1 is w^*-closed in $l_1(\mathcal{T}_2)$.

Assume that $n \ge 2$ and that $Y_{n-1} = [C_{n-1}]$ has been constructed satisfying 1 and 3. We first consider the set $\mathcal{T}_{2n}^{(2n-2)} = \{\alpha \in \mathcal{T}_{2n} : |\alpha| \le 2\}$, which is order isomorphic to $\mathcal{T}_2$. Let $(x_i) \subset l_1(\mathcal{T}_{2n}^{(2n-2)})$ be a sequence chosen as in the case $n = 1$. There exists a sequence (E_i) of pairwise disjoint finite subsets of $\{\alpha \in \mathcal{T}_{2n} : |\alpha| = 2\}$ so that for all $i \in \mathbb{N}$,

$$x_i = \sum_{\alpha\in E_i} \lambda_\alpha \delta_\alpha, \quad \lambda_\alpha \ge 0 \text{ for all } \alpha \in E_i, \quad \sum_{\alpha\in E_i} \lambda_\alpha = 1.$$

Fix $i \in \mathbb{N}$ and $\alpha \in E_i$. Then F_α, the set of the followers of α in $\mathcal{T}_{2n}$, is order isomorphic to $\mathcal{T}_{2n-2}$. Let

$$\phi_\alpha : \mathcal{T}_{2n-2} \to F_\alpha$$

be an order isomorphism. Define $\theta_{n-1,i} : l_1(\mathcal{T}_{2n-2}) \to l_1(\mathcal{T}_{2n})$ by

$$\theta_{n-1,i}(y) = \sum_{\alpha\in E_i} \lambda_\alpha \phi^*_\alpha(y) \quad \text{for all } y \in l_1(\mathcal{T}_{2n-2}),$$

where by ϕ^*_α we denote the adjoint of the natural isometry from $C(F_\alpha)$ onto $C(\mathcal{T}_{2n-2})$ induced by ϕ_α. Clearly, $\theta_{n-1,i}$ is a w^*-continuous isometric embedding. Set

$$Y_{n-1,i} = \theta_{n-1,i}(Y_{n-1}) \quad \text{and} \quad C_{n-1,i} = \theta_{n-1,i}(C_{n-1}).$$

Then $Y_{n-1,i} = [C_{n-1,i}]$ is w^*-closed in $l_1(\mathcal{T}_{2n})$, w^*-isometric to Y_{n-1}, and $C_{n-1,i}$ consists of disjointly supported finite convex combinations of elements of the set $\{\delta_\beta : \beta \in \bigcup_{\alpha\in E_i} F_\alpha\}$. Note that $\theta_{n-1,i}(\delta_{(\emptyset)}) = x_i$, and thus $x_i \in C_{n-1,i}$.

Finally, set $Y_n = [C_n]$, where

$$C_n = \bigcup_{i=1}^{\infty} C_{n-1,i} \cup \{\delta_\alpha : \alpha \in \mathcal{T}_{2n},\ |\alpha| \le 1\}.$$

Assume now that (μ_k) is a sequence of elements in C_n such that $\lim_k \mu_k = \mu$ (w^*). Assume further that there exist integers $i_1 < i_2 < \ldots$ so that $\mu_k \in C_{n-1,i_k}$ for all $k \in \mathbb{N}$. We then observe that $\lim_k x_{i_k} = \mu$ (w^*) as well. It follows that every w^*-cluster point of C_n is contained in Y_n and hence Y_n is w^*-closed in $l_1(\mathcal{T}_{2n})$, by Lemma 1 of [2].

Let now X_n be the quotient of $C(\mathcal{T}_{2n})$ modulo the annihilator of Y_n in $C(\mathcal{T}_{2n})$. This is the desired l_1-predual which, by the results of [8], is isomorphic to c_0. The inductive construction of the sequence (X_n) is now complete. ∎

NOTATION. We let $X(K)$ denote the c_0-sum of the sequence (X_n) of l_1-preduals constructed in Theorem 5.1. Clearly, this space is an l_1-predual isometric to a quotient of $C_0(\omega^\omega)$. In the sequel we shall refer to $X(K)$ as the *l_1-predual corresponding to K*.

REMARKS. 1. $U_n(y) = U_n(x_i)$, for all $y \in C_{n-1,i}$, where U_n is the canonical w^*-continuous projection of $l_1(\mathcal{T}_{2n})$ onto $[\delta_\alpha : \alpha \in \mathcal{T}_{2n},\ |\alpha| \le 1]$.

2. The basis projection $Q_{n-1,i} : Y_n \to Y_{n-1,i}$, i.e. the linear map which is the identity on $C_{n-1,i}$ and vanishes on $C_n \setminus C_{n-1,i}$, is w^*-continuous. This is so since $\bigcup_{\alpha\in E_i} F_\alpha$ is a clopen subset of $\mathcal{T}_{2n}$ containing the support of each member of $C_{n-1,i}$, and intersecting the support of none of the members of $C_n \setminus C_{n-1,i}$.

3. The space $X(K)$ is, up to isometry, independent of the choice of the sequence (x_i) in $l_1(\mathcal{T}_2)$ whose set of w^*-cluster points is equal to K.

6. Proofs of the main results. We first give the proof of Theorem 1.1 using Theorem 1.2 and our previously obtained results, and then pass to the proof of Theorem 1.2 itself.

Proof of Theorem 1.1. We denote by $A(\alpha)$, $\alpha < \omega_1$, the set of the α-averages of the sequence $(\delta_{(n)})$. Also, let K_α be the w^*-closure of $A(\omega^\alpha + 1)$ in $l_1(\mathcal{T}_1)$.

We fix $\alpha < \omega_1$. For every $n \in \mathbb{N}$ let Y_n, C_n, $(C_{n-1,i})$ be as in the conclusion of Theorem 5.1 applied to K_α. Let also X_α be the l_1-predual corresponding to K_α. We are going to show by induction on $n \in \mathbb{N}$ the following

CLAIM. *For every w^*-compact subset B of B_{Y_n} homeomorphic to $[1, \omega^{\omega^\alpha}]$, and for all $\varepsilon > 0$, there exist $i_0 \in \mathbb{N}$, $x^* \in C_{n-1,i_0}$ and $(T_i)_{i=1}^n \subset \mathcal{L}(Y_n)$ so that:*

(1) *T_i satisfies $(*)$ for (ε, x^*, B) and all $i \le n$.*
(2) *$\sum_{i=1}^n \|T_i(y)\| \le 1$ for all $y \in B_{Y_n}$.*

Once the claim is established, we deduce from Theorem 4.3 that for every $n \in \mathbb{N}$, the norming constant of any w^*-compact norming subset of B_{Y_n} homeomorphic to $[1, \omega^{\omega^\alpha}]$ is at most $1/(2n)$. Therefore, X_α is not isomorphic to a subspace of $C(\omega^{\omega^\alpha})$.

We now proceed to establish our claim. First we treat the case $n = 1$. Let $U_{1|Y_1}$ be the restriction of the canonical w^*-continuous projection to Y_1. Because $U_1(B)$ is homeomorphic to $[1, \omega^\beta k]$ for some $\beta \le \omega^\alpha$ and $k \in \mathbb{N}$, Theorem 1.2 with $(y_i) = (\delta_{(i)})$ yields a $(\beta+1)$-average μ of $(\delta_{(i)})$ such that $\{\mu\}$ and $U_1(B)$ are $\varepsilon/4$-almost l_1^2. Since the $(\beta+1)$-averages are contained in the $(\omega^\alpha+1)$-averages, we deduce that $\mu \in K_\alpha$. By the choice of the sequence $(x_i) \subset C_1$, μ is a w^*-cluster point of (x_i). Since U_1 is w^*-continuous and $U_1(\mu) = \mu$, Lemma 3.1 yields the existence of $i_0 \in \mathbb{N}$ such that

$$\|U_1(x_{i_0}) - \mu\| < \varepsilon/4.$$

It now follows that

(3) $\quad \{U_1(x_{i_0})\}$ and $U_1(B)$ are ε-almost l_1^2 and $\|U_1(x_{i_0})\| > 1 - \varepsilon$.

(Actually, $\|U_1(x_{i_0})\| = 1$.)

Let $R_{i_0} : Y_1 \to [x_{i_0}]$ be the basis projection onto $[x_{i_0}]$. Set

$$T_1 = U_{1|Y_1} - U_{1|Y_1} R_{i_0} \quad \text{and} \quad x^* = x_{i_0}.$$

It follows that $\|T_1\| \le 1$ and that T_1 satisfies $(*)$ for (ε, x^*, B). Indeed, $T_1(x_{i_0}) = 0$ and $T_1(y) = U_1(y)$ for all $y \in C_1$, $y \neq x_{i_0}$. Thus $\|T_1\| \le 1$. We also find, by (3), that $U_1(B)$ and $U_1 R_{i_0}(B)$ are ε-almost l_1^2, since for all $b \in B$, there exists $\lambda \in \mathbb{R}$ with $|\lambda| \le 1$ so that $U_1 R_{i_0}(b) = \lambda U_1(x_{i_0})$.

Thus the claim holds for $n = 1$. Assuming the claim is proved for $n-1$, let $U_{n|Y_n}$ be the restriction of the canonical w^*-continuous projection onto Y_n. As in the first part of the argument given in the case $n = 1$, we obtain $i_0 \in \mathbb{N}$ such that

(4) $\{U_n(x_{i_0})\}$ and $U_n(B)$ are ε-almost l_1^2 and $\|U_n(x_{i_0})\| > 1 - \varepsilon$.

(Recall that by our construction of the space Y_n, we have $x_{i_0} \in C_{n-1,i_0}$ and thus it belongs to Y_n.)

Let now $Q_{n-1,i_0} : Y_n \to Y_{n-1,i_0}$ be the basis projection. As was pointed out in the Remarks of Section 5, Q_{n-1,i_0} is w^*-continuous. We may thus apply our induction hypothesis to Y_{n-1,i_0} (which is of course w^*-isometric to Y_{n-1} via the map θ^{-1}_{n-1,i_0} that sends C_{n-1,i_0} onto C_{n-1}) and the w^*-compact set $Q_{n-1,i_0}(B)$ in order to obtain $x^* \in C_{n-1,i_0}$ and $(T_i')_{i=1}^{n-1} \subset \mathcal{L}(Y_{n-1,i_0})$ with $\|T_i'\| \le 1$ for $i \le n-1$, so that

(5) T_i' satisfies $(*)$ for $(\varepsilon, x^*, Q_{n-1,i_0}(B))$ and all $i \le n-1$.

(6) $\sum_{i=1}^{n-1} \|T_i'(y)\| \le 1$ for all $y \in C_{n-1,i_0}$.

We set $T_i = T_i' Q_{n-1,i_0}$ for all $i \le n-1$. Evidently, T_i satisfies $(*)$ for (ε, x^*, B) and all $i \le n-1$. Let now $R_{x_{i_0}} : Y_n \to [x_{i_0}]$ be the projection which takes the constant value x_{i_0} on C_{n-1,i_0} and vanishes on $C_n \setminus C_{n-1,i_0}$. It is clear that $\|R_{x_{i_0}}\| = 1$. Finally, set

$$T_n = U_{n|Y_n} - U_{n|Y_n} R_{x_{i_0}}.$$

Note that T_n vanishes on C_{n-1,i_0}, since $U_n(y) = U_n(x_{i_0})$ for every $y \in C_{n-1,i_0}$, by the Remarks after the proof of Theorem 5.1. On the other hand, T_i vanishes on $C_n \setminus C_{n-1,i_0}$ for all $i \le n-1$. This fact combined with (6) yields that (2) of our claim holds.

To complete the inductive proof of the claim, it remains to be shown that T_n satisfies $(*)$ for (ε, x^*, B). We need only observe that $U_n(x_{i_0}) = U_n(x^*)$ and repeat the argument given in the case $n = 1$, using (4).

Finally, we wish to show that X_α is isometric to a subspace of $C(\omega^{\omega^{\omega^\alpha+2}})$. To this end, we note that a routine transfinite induction argument (details appear in [7]) shows that the w^*-closure of $A(\alpha)$ in $l_1(\mathcal{T}_1)$ is homeomorphic to $[1, \omega^{\omega^\alpha}]$. It now follows, by induction on $n \in \mathbb{N}$, that the w^*-closure of C_n in $l_1(\mathcal{T}_{2n})$ is homeomorphic to $[1, \omega^{\omega^{\omega^\alpha+1}n}]$. The proof of Theorem 1.1 is now complete. ∎

Proof of Theorem 1.2. Let us say that K *α-works* provided K is a countable w^*-compact subset of $B_{l_1(\omega)}$ homeomorphic to $[1, \omega^\alpha n]$ for some $n \in \mathbb{N}$, and satisfies the conclusion of Theorem 1.2. It suffices to show that K α-works if $K^{(\alpha)} = \{0\}$.

Indeed, if K is homeomorphic to $[1, \omega^\alpha n]$, then there exist distinct elements $x_1, \ldots, x_n$ of K so that K is the union of n pairwise disjoint clopen subsets $K_1, \ldots, K_n$ with $K_i^{(\alpha)} = \{x_i\}$ for all $i \le n$. We set

$$L = \bigcup_{i=1}^n \left\{ \frac{1}{2}x - \frac{1}{2}x_i : x \in K_i \right\}.$$

Choose $m \in \mathbb{N}$ such that $|x_i|[m, \infty) < \varepsilon/3$ for all $i \le n$. It is easy to check that if $y \in B_{l_1(\omega)}$, $s(y) \subset [m, \infty)$ and y is $\varepsilon/3$-disjoint from L, then it is ε-disjoint from K. But $L^{(\alpha)} = \{0\}$ and thus if L α-works, so does K.

We proceed by transfinite induction on α to show that K α-works if $K^{(\alpha)} = \{0\}$. The argument for the case $\alpha = 1$ is contained in the general inductive step and so we omit it. Assume the assertion holds for all ordinals smaller than α and let (α_n) be a sequence of ordinals such that either $\alpha_n + 1 = \alpha$ for all $n \in \mathbb{N}$, or (α_n) is a strictly increasing sequence of ordinals whose limit is α. We may write

$$K = \bigcup_{n=1}^\infty K_n \cup \{0\}$$

where (K_n) is a sequence of pairwise disjoint clopen subsets of K with K_n homeomorphic to $[1, \omega^{\alpha_n}]$. We fix $m \in \mathbb{N}$. It is enough to exhibit z, an $(\alpha+1)$-average of (y_i) which is ε-disjoint from K and such that $y_m < z$.

We first choose $l \in \mathbb{N}$ such that $2/l < \varepsilon/2$. Inductively we choose $1 = n_0 < n_1 < \ldots$ and (z_i), a convex block subsequence of (y_i) consisting of α-averages of (y_i), so that for all $i \in \mathbb{N}$:

(7) $|x|(s(z_i)) < \varepsilon/l$ for all $x \in K_n$ and all $n \ge n_i$ (since $s(z_i)$ is finite and the K_n's cluster at 0).

(8) z_{i+1} is ε/l-disjoint from $\bigcup_{n<n_i} K_n$ (by the induction hypothesis).

(If $\alpha = 1$, then (8) is obtained from the fact that K consists of a w^*-null sequence and 0.) Note that z_1 can be chosen arbitrarily. We let $z = (z_1 + \ldots + z_l)/l$, which is an $(\alpha+1)$-average of (y_i) with $y_m < z_1$. We claim that z is ε-disjoint from K. To show this, let $x \in K_n$ for some $n \in \mathbb{N}$ and choose $i \in \mathbb{N}$ so that $n_{i-1} \le n < n_i$. If $j < i$, then (7) yields that $|x|(s(z_j)) < \varepsilon/l$. If $j > i$, then (8) implies that z_j is ε/l-disjoint from K_n. In this case choose $E_j \subset s(z_j)$ such that

$$z_j(E_j) > 1 - \varepsilon/l \quad \text{and} \quad |x|(E_j) < \varepsilon/l.$$

Finally, set $E = \bigcup_{j>i} E_j \cup \bigcup_{j<i} s(z_j)$, where j runs in $\{1, \ldots, l\}$. It follows that $|x|(E) < \varepsilon$ and $z(E) > 1 - \varepsilon$. Hence, x and z are ε-disjoint for every $x \in K$. The proof of Theorem 1.2 is now complete. ■

COROLLARY 6.1. *There exists an l_1-predual which is isomorphic to a quotient of $C(\omega^\omega)$ yet isomorphic to a subspace of $C(\alpha)$ for no $\alpha < \omega_1$.*

Proof. Let K be the set of all probability measures on $[1, \omega]$. Let also $X_\infty = X(K)$ be the l_1-predual corresponding to K according to Theorem 5.1. We denote by C the l_1-basis of X_∞^* constructed in the proof of Theorem 5.1. Let $\alpha < \omega_1$ and observe that there exists a subset of C which spans a w^*-closed subspace, Z_α, of X_∞^*, w^*-isometric to X_α^*. It now follows by Corollary 1 of [9] (an alternative proof of this result for the case of l_1-preduals is given in [7]) that Z_α is contractively complemented in X_∞^* via a w^*-continuous projection. Hence, X_α is isometric to a contractively complemented subspace of X_∞. Corollary 6.1 now follows since X_α is isomorphic to a subspace of $C(\omega^{\omega^\alpha})$ for no $\alpha < \omega_1$. ■

7. Final remarks. The proof of Theorem 1.2 actually shows that if K is homeomorphic to $[1, \omega^\omega]$ and $\varepsilon > 0$, then there exists a 2-average of (δ_i) which is ε-disjoint from K. This observation in turn implies, via Theorem 5.1, the existence of a subspace of $C(\omega^{\omega^3})$ isomorphic to a quotient of $C(\omega^\omega)$ yet not isomorphic to a subspace of $C(\omega^\omega)$. We do not know if such a phenomenon occurs in $C(\omega^{\omega^2})$ and therefore we ask:

QUESTION 1. Does there exist a subspace of $C(\omega^{\omega^2})$ isomorphic to a quotient of $C(\omega^\omega)$ yet not isomorphic to a subspace of $C(\omega^\omega)$?

A negative answer to this question yields that every complemented subspace of $C(\omega^{\omega^2})$ whose Szlenk index is equal to ω^2 is isomorphic to a subspace of $C(\omega^\omega)$.

Let us say that the Banach space X λ-*embeds* in the Banach space Y, $\lambda > 0$, if there exists a subspace Z of Y isomorphic to X and such that the Banach–Mazur distance between X and Z is at most λ.

We next let X denote the l_1-predual corresponding to the w^*-closure in $l_1(\omega)$ of the set of the 1-averages of (δ_n). In other words, $X = X(K)$, where $K = \overline{A(1)}^{w^*}$. According to Theorem 5.1, X is the c_0-sum of a sequence of spaces (X_n), each isomorphic to c_0. An argument similar to the one given in the proof of Theorem 1.1 yields the following property of the sequence (X_n): If (α_n) is a sequence in $[1, \omega^\omega)$ and (λ_n) is a sequence of positive reals so that X_n λ_n-embeds in $C(\alpha_n)$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \lambda_n = \infty$. Further, X is isometric to a subspace of $C(\omega^{\omega^2})$. We do not know if X is isomorphic to a subspace of $C(\omega^\omega)$.

QUESTION 2. Let X be a Banach space isomorphic to the c_0-sum of a sequence (X_n) of spaces isomorphic to c_0. Assume X is isomorphic to a subspace of $C(\omega^\omega)$. Do there exist $\lambda > 0$ and a sequence of ordinals (α_n) in $[1, \omega^\omega)$ so that X_n λ-embeds in $C(\alpha_n)$ for all $n \in \mathbb{N}$?

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