Proof of Theorem 2. If $G_F(X)$ acts transitively on $S(X)$, then the only $G_F(X)$-invariant subspaces of X are the trivial ones, hence $H = X$ in Lemma 2 and $(\cdot,\cdot)$ is defined on the whole X. Moreover, $(\cdot,\cdot)$ is $G_F(X)$-invariant and $(x_0, x_0) = \|x_0\|^2 = 1$. Transitivity of $G_F(X)$ now implies that $(x,x) = \|x\|^2 =$ for every $x \in X$, which proves the theorem. ■

3. Concluding remarks and questions. In a sense, the proof of Theorem 2 is algebra. Thus, it is not surprising that Theorem 2 holds if X is assumed to be a quasi-normed space. But, actually, an almost isotropic quasi-normed space having a non-trivial finite-dimensional perturbation of the identity must be locally convex, its quasi-norm being, in fact, a norm [5]. Recall that an *almost isotropic* quasi-normed space is one in which the isometry group acts with dense orbits on the unit sphere. Theorem 2 suggests the following questions:

QUESTION 1. Let X be a normed space for which, given $x, y \in S(X)$ and $\varepsilon > 0$, there exists $T \in G_F(X)$ such that $\|y - Tx\| \le \varepsilon$. Must X be an inner product space?

QUESTION 2. Find operator ideals J (containing F) for which Theorem 2 remains true if $T - \mathrm{Id} \in F$ is replaced by $T - \mathrm{Id} \in J$.

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Weighted inequalities and the shape of approach regions

by

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Abstract. We characterize geometric properties of a family of approach regions by means of analytic properties of the class of weights related to the boundedness of the maximal operator associated with this family.

1. Introduction. In [NS], Nagel and Stein studied under which conditions on a general domain $\Omega \subset \mathbb{R}_+^{n+1}$, the associated maximal operator M_Ω is of weak type $(1,1)$. J. Sueiro [Su] gave an extension of this result for spaces of homogeneous type. Following the ideas of [Su], Pan [Pa] studied weak type weighted norm estimates for M_Ω also in spaces of homogeneous type. Later, Sánchez-Colomer and Soria [SS1] gave strong-type weighted norm estimates for M_Ω in the Euclidean space, and they also studied the relationship between weighted inequalities for this operator and the geometry of Ω (see [SS2]).

In this paper we find another (easier) characterization of the weak-type inequalities for M_Ω, in terms of the classical A_p condition plus an extra property related to being a Carleson measure (see Theorem 2.12). For this, we use some of the techniques given in [AC].

This result allows us to prove that the equivalence of weighted inequalities for M_Ω and the classical Hardy–Littlewood maximal function M completely determines the geometry of the family of approach regions Ω (see Theorem 3.4). We work in the setting of spaces of homogeneous type, extending previous results in $\mathbb{R}^n$ for the special case of regions obtained by translation of a fixed one (see [SS2]).

To this end, we observe that there exists a class of "power" weights (see Corollary 3.2) which are the key to establishing the correspondence between analytic properties (boundedness of maximal operators) and geometric properties of the domains Ω. The main idea behind this technique

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is to find an equivalent metric in the given space which enjoys some extra invariance properties.

2. Definitions and previous results. Let X be a topological space with a nonnegative Borel measure μ. Suppose we have a nonnegative real-valued function δ defined in $X \times X$ with the following properties:

(i) $\delta(x,y) = 0$ if and only if $x = y$.

(ii) There is a constant $D \geq 1$ such that $\delta(x,y) \leq D\delta(y,x)$.

(iii) There is a constant $A \geq 1$ such that $\delta(x,y) \leq A(\delta(x,z) + \delta(y,z))$, for all x, y, z in X.

(iv) The balls $B(x,r) = \{y \in X : \delta(x,y) < r\}$ are measurable sets for all x in X and $r > 0$. Moreover, $\{B(x,r)\}_{r>0}$ is a basis of open neighborhoods for all x in X.

(v) There is a constant $K > 1$ such that $0 < \mu(B(x,2r)) \leq K\mu(B(x,r))$ for all x in X and $r > 0$.

(vi) There is a constant $M > 1$ such that $B(x,Mr) \setminus B(x,r) \neq \emptyset$ for all x in X and $r > 0$.

A measure satisfying (v) is called a *doubling measure*. Although our purpose is to be in the setting of spaces of homogeneous type, we need to assume our function δ nonsymmetric in general (we call it a *quasidistance*). In the sequel, we will write every positive constant as C; it may change from one occurrence to the next.

We give some technical results that are important for later purposes.

LEMMA 2.1 (see [ST]). *Let $a > 0$. If $B(x,r) \cap B(y,r') \neq \emptyset$ and $r \leq ar'$, then $B(x,r) \subset B(y,c_0r')$ with $c_0 = A^2(1+a) + ADa$.*

PROPOSITION 2.2. *There exist $\alpha, \beta > 0$ and $0 < K_1 < 1$ such that*

$$K_1t^\beta\mu(B(x,r)) \leq \mu(B(x,tr)) \leq Kt^\alpha\mu(B(x,r)) \tag{1}$$

for all x in X, $r > 0$ and $t \geq 1$.

Proof. The right inequality is well known, with $\alpha = \log_2(K)$, and condition (vi) is not needed. To see the left inequality, we claim that there exist constants $A_0 > 1$ and $0 < K_1 < 1$ such that $\mu(B(x,r)) \leq K_1\mu(B(x,A_0r))$ for all $x \in X$ and $r > 0$. Granted this, let $n \geq 1$ be so that $A_0^{n-1} \leq t < A_0^n$. Applying the last inequality we obtain

$$\mu(B(x,tr)) \geq \mu(B(x,A_0^{n-1}r)) \geq K_1^{1-n}\mu(B(x,r)) \geq K_1t^\beta\mu(B(x,r)),$$

where $\beta = \log_{A_0}(K_1^{-1})$.

We now prove the claim. Let c_0 be the constant in Lemma 2.1 when $a = 1$. Choose $L > Dc_0$, and take $y \in B(x,MLr) \setminus B(x,Lr)$ which is not empty by (vi). Then $Lr \leq \delta(x,y) < MLr$, and by (ii), we have $Lr/D \leq \delta(y,x) < DMLr$. By the choice of L, we have $c_0r < \delta(y,x) < DMLr$. Now, using Lemma 2.1, we obtain

$$B(x,r) \cap B(y,r) = \emptyset \quad \text{and} \quad B(y,r) \subset B(x,c_0MLr).$$

Let $A_0 = c_0ML$. By construction, we have

$$\mu(B(x,r)) \leq \mu(B(x,A_0r)) - \mu(B(y,r)).$$

The right inequality in (1) gives us the existence of a constant $C > 1$ such that $\mu(B(x,A_0r)) \leq C\mu(B(y,r))$, and hence

$$\mu(B(x,r)) \leq \mu(B(x,A_0r)) - \frac{1}{C}\mu(B(x,A_0r)) = K_1\mu(B(x,A_0r)). \ \blacksquare$$

This proposition says that our measure is *invariant* in some sense. Also, observe that we avoid measures with atoms. The next two results are proved in [ST]:

LEMMA 2.3. *Let $\mathcal{F}$ be a family of balls with bounded radii. Then there is a countable subfamily of pairwise disjoint balls $B(x_k,r_k)$ such that each ball in $\mathcal{F}$ is contained in one of the balls $B(x_k,c_0r_k)$, where c_0 is the constant of the previous lemma in the case $a = 2$.*

Given a locally integrable function f, the *noncentered Hardy–Littlewood maximal function* of f *with respect to the measure* μ is

$$M_\mu f(x) = \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(z)|\,d\mu(z).$$

Analogously, given a nonnegative measure ν, the *Hardy–Littlewood maximal function* of ν *with respect to the measure* μ is

$$M_\mu\nu(x) = \sup_{B\ni x} \nu(B)/\mu(B).$$

We write $M = M_\mu$ if there is no possible confusion.

As a consequence of Lemma 2.3, it is now easy to prove the following weak-type estimates:

THEOREM 2.4. (a) *The Hardy–Littlewood maximal operator is of weak type* $(1,1)$ *and strong type* (p,p) *for* $1 < p \leq \infty$ *on* $L^p(\mu)$.

(b) *There is a constant $C_\mu > 0$ such that*

$$\mu(\{x \in X : M\nu(x) > \lambda\}) \leq \frac{C_\mu}{\lambda}\nu(X)$$

for all nonnegative measures ν of finite total variation.

The proof of the following theorem is given in [CW] for spaces of homogeneous type, and works in our setting with slight modifications.

THEOREM 2.5. *Let μ be a doubling measure. Let $f \in L^1(\mu)$ be a positive function with bounded support. There exists a countable family $\{B(x_k, r_k)\}_k$ of balls such that*:

(a) $O = \{x \in X : Mf(x) > 1/2\} = \bigcup_k B(x_k, r_k)$.
(b) *There is a constant* $C_\mu > 0$ *such that* $\sum_k \mu(B(x_k, r_k)) \le C_\mu \|f\|_1$.
(c) $B(x_k, 3Ar_k) \cap O^c \ne \emptyset$ *for all* k.

The *tent* of an open set O is the set $T(O) = \{(y,t) \in X \times (0,\infty) : B(y,t) \subset O\}$.

DEFINITION 2.6. We say that two measures, ϱ defined on $X \times (0,\infty)$ and ν defined on X, are a *Carleson pair* if there exists a constant $C_{\varrho,\nu} > 0$ such that

$$\varrho(T(B)) \le C_{\varrho,\nu}\, \nu(B) \quad \text{for all balls } B \subset X.$$

In this case, we use the notation $(\varrho,\nu) \in \mathcal{C}(X)$.

PROPOSITION 2.7. *Let $(\varrho,\nu) \in \mathcal{C}(X)$ with ν doubling. Then there exists a constant $C'_{\varrho,\nu} > 0$ such that*

$$\varrho(T(O)) \le C'_{\varrho,\nu}\, \nu(O) \quad \text{for all } O \subset X \text{ open}.$$

Proof. We can assume that $\nu(O) < \infty$. Let $f = \chi_O \in L^1(\nu)$. Then $O \subset \{x \in X : M_\nu f(x) > 1/2\}$. We use Theorem 2.5 to obtain a family $\{B(x_k, r_k)\}_k$ of balls satisfying:

(a) $O \subset \bigcup_k B(x_k, r_k)$.
(b) There is a constant $C_\nu > 0$ such that $\sum_k \nu(B(x_k, r_k)) \le C_\nu\, \nu(O)$.
(c) $B(x_k, 3Ar_k) \cap O^c \ne \emptyset$ for all k.

Take $(x,s) \in T(O)$. By definition, $B(x,s) \subset O$ and (a) implies that there is k_0 so that $x \in B(x_{k_0}, r_{k_0})$. Now, (c) implies there exists $y \in B(x_{k_0}, 3Ar_{k_0}) \setminus B(x,s)$ and hence

$$\begin{aligned} s &\le \delta(x,y) \le A(\delta(x,x_{k_0}) + \delta(y,x_{k_0})) \le AD(\delta(x_{k_0},x) + \delta(x_{k_0},y)) \\ &\le AD(1+3A) r_{k_0}. \end{aligned}$$

By Lemma 2.1, there exists $C > 0$ independent of x, s, x_{k_0} and r_{k_0} such that $B(x,s) \subset B(x_{k_0}, Cr_{k_0})$, that is, $(x,s) \in T(B(x_{k_0}, Cr_{k_0}))$. Therefore $T(O) \subset \bigcup_k T(B(x_k, Cr_k))$. Now, using (b) and the hypothesis on the measures, we have

$$\begin{aligned} \varrho(T(O)) &\le \sum_k \varrho(T(B(x_k, Cr_k))) \le C_{\varrho,\nu} \sum_k \nu(B(x_k, Cr_k)) \\ &\le C_{\varrho,\nu} K_\nu C^{\alpha(\nu)} \sum_k \nu(B(x_k, r_k)) \le C_{\varrho,\nu} K_\nu C^{\alpha(\nu)} C_\nu\, \nu(O) \\ &\doteq C'_{\varrho,\nu}\, \nu(O), \end{aligned}$$

where K_ν and $\alpha(\nu)$ are the constants appearing in Proposition 2.2 for the measure ν. ■

We say that a family $\Omega = \{\Omega_x\}_{x\in X}$ of measurable sets in $X \times (0,\infty)$ is a *family of approach regions* if $(x,0) \in \overline{\Omega}_x$ for all $x \in X$, with respect to the product topology in $X \times (0,\infty)$. A natural example of approach region is a *cone of width* $\theta > 0$, $\Gamma_\theta(x) = \{(y,t) \in X \times (0,\infty) : x \in B(y, \theta t)\}$. We set $\Gamma(x) = \Gamma_1(x)$.

For a family $\Omega = \{\Omega_x\}_{x\in X}$, let us introduce the following definitions:

DEFINITION 2.8. 1. The *section* of Ω_x at height $t > 0$ is the set $\Omega_x(t) = \{y \in X : (y,t) \in \Omega_x\}$.

2. $S(x,t) = \{y \in X : \Omega_y(t) \cap B(x,t) \ne \emptyset\}$.

3. $P_\Omega(x,t) = \{y \in X : (x,t) \in \Omega_y\}$.

4. Given a nonnegative measure σ on X, we define the outer measure on $X \times (0,\infty)$, $\sigma_\Omega(E) = \sigma(\{x \in X : \Omega_x \cap E \ne \emptyset\})$ for $E \subset X \times (0,\infty)$ (see [AC]).

The *maximal operator related to* Ω for a measurable function f is

$$M_\Omega f(x) = \sup_{(y,t)\in\Omega_x} \frac{1}{\mu(B(y,t))} \int_{B(y,t)} |f(z)|\, d\mu(z).$$

We will always assume that $M_\Omega f$ is a measurable function. In the particular case of $\Omega_x = \Gamma_\theta(x)$, it is known that M and M_{Γ_θ} are equivalent operators, that is, there are positive constants C and C' such that

$$CMf(x) \le M_{\Gamma_\theta} f(x) \le C' Mf(x) \tag{2}$$

for all measurable f and $x \in X$. We write $Mf \sim M_{\Gamma_\theta} f$. So, the operator M_{Γ_θ} has the same estimates as M.

PROPOSITION 2.9. *Let ϱ and ν be two nonnegative measures on X so that ν is doubling. If $M_\Omega : L^p(\nu) \to L^{p,\infty}(\varrho)$ is bounded for some $p \ge 1$, then there exists a constant $C > 0$ such that*

$$\varrho(S(x,t)) \le C\nu(B(x,t)) \quad \text{for all } (x,t) \in X \times (0,\infty).$$

Proof. Take $y \in S(x,t)$. Then there exists $z \in B(x,t)$ so that $z \in \Omega_y(t)$. Now, by condition (iii) on δ, we have $B(z,t) \subset B(x, A(D+1)t)$. Let $f = \chi_{B(x,A(D+1)t)}$. Then $M_\Omega f(y) > 1/2$ for all $y \in S(x,t)$, and therefore

$$\begin{aligned} \varrho(S(x,t)) &\le \varrho(\{x \in X : M_\Omega f(x) > 1/2\}) \le C\|f\|^p_{L^p(\nu)} \\ &= C\nu(B(x, A(D+1)t)) \le C\nu(B(x,t)). \end{aligned}$$

Observe that the constant C is controlled by the norm of the operator M_Ω and the doubling constant K_ν of the measure ν:

$$C = C(K_\nu, \|M_\Omega\|_{L^p(\nu)\to L^{p,\infty}(\varrho)}).\ ■$$

It is proved in [Su] that with no loss of generality, we can always assume that Ω is full on vertical directions, that is, $(y,s) \in \Omega_x$ implies $(y,t) \in \Omega_x$ for all $t \geq s$. This is equivalent to the fact that $P_\Omega(y,s) \subset P_\Omega(y,t)$ whenever $s \leq t$. We will also take this condition for granted.

We now characterize the boundedness of $M_\Omega : L^p(\nu) \to L^{p,\infty}(\varrho)$ for some $p \geq 1$.

THEOREM 2.10. *Let ϱ and ν be two nonnegative measures on X. If $M : L^p(\nu) \to L^{p,\infty}(\nu)$ is bounded for some $p \geq 1$, then the following conditions are equivalent:*

(a) *There exists $C > 0$ such that*

$$\varrho(\{x \in X : M_\Omega f(x) > \lambda\}) \leq C\nu(\{x \in X : M_\Gamma f(x) > \lambda\})$$

for all $\lambda > 0$ and measurable f.

(b) *$M_\Omega : L^p(\nu) \to L^{p,\infty}(\varrho)$ is bounded.*

(c) *There exists $C > 0$ such that $\varrho(S(x,t)) \leq C\nu(B(x,t))$ for all $(x,t) \in X \times (0,\infty)$.*

(d) *$(\varrho_\Omega, \nu) \in \mathcal{C}(X)$.*

Proof. It is known that ν is necessarily a doubling measure. (a)$\Rightarrow$(b) follows trivially by using (2). That (b)$\Rightarrow$(c) is Proposition 2.9. Let us see that (c)$\Rightarrow$(d). Take $y \in X$ so that $\Omega_y \cap T(B(x,t)) \neq \emptyset$. There is $(z,s) \in \Omega_y$ with $B(z,s) \subset B(x,t)$. Since Ω_y is full on vertical directions, $(z,t) \in \Omega_y$. Therefore $z \in \Omega_y(t) \cap B(x,t)$, and hence $\{y \in X : \Omega_y \cap T(B(x,t)) \neq \emptyset\} \subset S(x,t)$. So, using the definition of ϱ_Ω, we have

$$\varrho_\Omega(T(B(x,t)) \leq \varrho(S(x,t)) \leq C\nu(B(x,t)).$$

Now, suppose $(\varrho_\Omega, \nu) \in \mathcal{C}(X)$. Observe that

$$\left\{(y,t) : \frac{1}{\mu(B(y,t))} \int_{B(y,t)} |f(z)|\, d\mu(z) > \lambda\right\} \subset T(O),$$

where $O = \{x \in X : M_\Gamma f(x) > \lambda\}$, for all functions f. Then, applying Proposition 2.7, we obtain

$$\begin{aligned}\varrho(\{x \in X : M_\Omega f(x) > \lambda\}) &= \varrho_\Omega\left(\left\{(y,t) : \frac{1}{\mu(B(y,t))} \int_{B(y,t)} |f|\, d\mu > \lambda\right\}\right)\\ &\leq \varrho_\Omega(T(O)) \leq C\nu(O)\\ &= C\nu(\{x \in X : M_\Gamma f(x) > \lambda\}). \ \blacksquare\end{aligned}$$

REMARKS 2.11. 1. The boundedness of M is only needed in the implication (a)$\Rightarrow$(b).

2. The implication (d)$\Rightarrow$(a) says that we can transfer estimates on M to M_Ω if we have a Carleson type condition.

A *weight* u on (X,μ,δ) is a positive and locally integrable function. We set $u(D) = \int_D u(z)\, d\mu(z)$ for $D \subset X$ measurable. We say that u is *doubling* if the measure $u(z)\, d\mu(z)$ is doubling.

A weight u is in the A_p *class*, $1 \leq p < \infty$, if $M : L^p(u) \to L^{p,\infty}(u)$ is bounded. By the A_p *constant* $\|u\|_{A_p}$ of a weight u in A_p we mean the norm of the maximal operator. It is well known that every weight u in A_p is doubling, and it is easy to check that, if we write K_u for the doubling constant of u, then

$$K_u \leq K^p \|u\|_{A_p}, \tag{3}$$

where K is the doubling constant of the ambient measure μ.

Define A_p^Ω to be the class of weights u such that $M_\Omega : L^p(u) \to L^{p,\infty}(u)$ is bounded, $1 \leq p < \infty$, and let the A_p^Ω *constant* $\|u\|_{A_p^\Omega}$ be the norm of M_Ω.

Set

$$W(\Omega) = \{u \in L^1_{\mathrm{loc}}(\mu), u \geq 0 : \exists C > 0,\ u(S(x,t)) \leq Cu(B(x,t)),\ \forall (x,t)\},$$

and define the $W(\Omega)$ *constant* $\|u\|_{W(\Omega)}$ to be the infimum of the constants C appearing in this definition.

THEOREM 2.12. *For $1 \leq p < \infty$, we have $A_p^\Omega = A_p \cap W(\Omega)$, and there exists $C > 0$ such that $\|u\|_{A_p} \leq C\|u\|_{A_p^\Omega}$ for all u in A_p^Ω.*

Proof. The previous theorem says that $A^p \cap W(\Omega) \subset A_p^\Omega$ and $A_p^\Omega \subset W(\Omega)$. Now, it is proved in [SS1] that if $(x,0) \in \overline{\Omega}_x$, then $M_c f(x) \leq M_\Omega f(x)$ for all functions f, where M_c is the centered Hardy–Littlewood maximal operator. Therefore, $A_p^\Omega \subset A_p$, and $\|u\|_{A_p} \leq C\|u\|_{A_p^\Omega}$ with $C > 0$ independent of u. $\blacksquare$

3. The shape of approach regions.

Let (X,μ,d) be a space of homogeneous type satisfying conditions (i) to (vi), but with $D = 1$ in (ii), that is, d is symmetric, and therefore, a quasidistance. Set $B^d(x,t) = \{y \in X : d(x,y) < t\}$. We will introduce a nonnegative nonsymmetric function δ on $X \times X$ also satisfying these conditions, but having the property that the measure of a δ-ball is comparable to its radius (see [ST] for the details).

For a fixed $x \in X$, consider the function

$$r_x(t) = \begin{cases} \exp\left(\dfrac{-1}{1+t}\right) \displaystyle\int_{1/2}^{1} \mu(B(x,st))\, ds & \text{if } t \neq 0,\\ 0 & \text{if } t = 0.\end{cases}$$

This function is strictly increasing, continuous (continuity at 0 is given by Proposition 2.2), $r_x(t) \to \infty$ as $t \to \infty$ when X is not a compact space, and the measure of a ball $B^d(x,t)$ is comparable to $r_x(t)$. In fact, using Proposition 2.2, we can see

$$2r_x(t) \le \mu(B^d(x,t)) \le eK2^{\alpha+1}r_x(t). \tag{4}$$

The function r_x has an inverse r_x^{-1} for all x in X. If X is a compact space, there exists a constant $c_x > 0$ such that r_x^{-1} is defined in $[0, c_x)$.

We define the normalized "quasidistance" $\delta(x,y) = r_x(d(x,y))$. The δ-balls are $B^\delta(x,t) = \{y \in X : \delta(x,y) < t\}$. Observe that $B^d(x,t) = B^\delta(x, r_x(t))$ for all x in X and $t > 0$, and then $\mu(B^\delta(x,t))$ is comparable to t by (4). Now, it is not difficult to see that δ and $B^\delta(x,t)$ satisfy (i) to (vi).

This new quasidistance is not symmetric in general, and this is why we need this more general condition in the previous section. Consider the Hardy–Littlewood maximal operator with respect to δ,

$$M^\delta f(x) = \sup_{B^\delta \ni x} \frac{1}{\mu(B^\delta)} \int_{B^\delta} |f(z)|\, d\mu(z),$$

for a measurable function f. Since $B^\delta(x, r_x(t)) = B^d(x,t)$ and r_x is one-to-one, we have $Mf(x) = M^\delta f(x)$ for all x in X, where M is the maximal operator with respect to d. Consequently, the class of A_p weights is not modified with the change of quasidistance. Given a family of approach regions in (X,μ,d), we define the corresponding family in (X,μ,δ) as follows. If $\Omega = \{\Omega_x\}_{x\in X}$, set

$$\Omega_x^\delta = \{(y,s) : (y, r_y^{-1}(s)) \in \Omega_x\} = \{(y, r_y(t)) : (y,t) \in \Omega_x\}.$$

Observe that $P_\Omega(x,t) = P_{\Omega^\delta}(x, r_x(t))$ for all x in X and $t > 0$, and so Ω is full on vertical directions if and only if Ω^δ is. The maximal operator associated with Ω^δ on (X,μ,δ) is

$$M^\delta_{\Omega^\delta} f(x) = \sup_{(y,s)\in\Omega^\delta} \frac{1}{\mu(B^\delta(y,s))} \int_{B^\delta(y,s)} |f(z)|\, d\mu(z),$$

for a measurable function f. It is easy to check that $M^\delta_{\Omega^\delta} f(x) = M_\Omega f(x)$ for all x in X and all functions f, where M_Ω is the maximal operator related to Ω on (X,μ,d), and then, if we define $A_p^{\Omega^\delta}$ as the class of weights u such that $M^\delta_{\Omega^\delta} : L^p(u) \to L^{p,\infty}(u)$ is bounded, we have $A_p^{\Omega^\delta} = A_p^\Omega$ with the same constant.

In the sequel, we assume that δ is the ambient quasidistance, and we write $B(x,t) = B^\delta(x,t)$ and $\Omega = \Omega^\delta$. The next result is essentially proved in [CR], but we give the proof in our general setting for the sake of completeness.

PROPOSITION 3.1. *Let ν be a Borel measure on X such that $M\nu \not\equiv \infty$. Then $M\nu^\varepsilon \in A_1$ for all $0 \le \varepsilon < 1$, with A_1 constant depending only on ε.*

Proof. We need to see that $M : L^1(M\nu^\varepsilon) \to L^{1,\infty}(M\nu^\varepsilon)$ with norm depending only on ε. It is well known that this is equivalent to proving that there exists a constant $C = C(\varepsilon)$ such that

$$\frac{1}{\mu(B)} \int_B M\nu(x)^\varepsilon\, d\mu(x) \le C \operatorname*{ess\,inf}_{x\in B} M\nu(x)^\varepsilon$$

for all balls $B \subset X$ (see [ST] for a proof of the equivalence in this context).

For a fixed B_0, take $x \in B_0$ and consider $\mathcal{Q}_1 = \{B \ni x : \mu(B) \le \mu(2B_0)\}$ and $\mathcal{Q}_2 = \{B \ni x : \mu(B) > \mu(2B_0)\}$, where $2B_0$ is the ball with the same center as B_0 but with twice its radius. We have

$$M\nu(x) \le \sup_{B\in\mathcal{Q}_1} \frac{\nu(B)}{\mu(B)} + \sup_{B\in\mathcal{Q}_2} \frac{\nu(B)}{\mu(B)} = A(x) + B(x),$$

and then $M\nu(x)^\varepsilon \le A(x)^\varepsilon + B(x)^\varepsilon$.

If $B \in \mathcal{Q}_2$, then $2B_0 \subset c_0 B$ by Lemma 2.1, and

$$\frac{\nu(B)}{\mu(B)} \le C\frac{\nu(c_0B)}{\mu(c_0B)} \le C \operatorname*{ess\,inf}_{y\in c_0B} M\nu(y) \le C \operatorname*{ess\,inf}_{y\in B_0} M\nu(y),$$

hence $B(x) \le C \operatorname{ess\,inf}_{y\in B_0} M\nu(y)$. If $B \in \mathcal{Q}_1$, then $B \subset 2c_0B_0$ by Lemma 2.1. Consider the measure ν_0 so that $d\nu_0(y) = \chi_{2c_0B_0}(y)d\nu(y)$. Then $A(x) \le M\nu_0(x)$, and therefore, it is enough to prove

$$\frac{1}{\mu(B_0)} \int_{B_0} M\nu_0(y)^\varepsilon\, d\mu(y) \le C \operatorname*{ess\,inf}_{y\in B_0} M\nu(y)^\varepsilon,$$

with C depending only on ε. Applying Fubini's theorem, we get

$$\begin{aligned}\frac{1}{\mu(B_0)} \int_{B_0} M\nu_0(y)^\varepsilon\, d\mu(y) &= \frac{1}{\mu(B_0)} \int_0^\infty \varepsilon t^{\varepsilon-1} \mu(\{y \in B_0 : M\nu_0(y) > t\})\, dt \\ &= \frac{\varepsilon}{\mu(B_0)} \Big(\int_0^R + \int_R^\infty\Big) t^{\varepsilon-1}\mu(\{M\nu_0 > t\} \cap B_0)\, dt \\ &= I_1 + I_2,\end{aligned}$$

for $R > 0$ to be chosen later. We obtain $I_1 \le R^\varepsilon$ if we get the distribution function bounded by the total mass of B_0. In the second integral, we use the boundedness of the maximal operator (Theorem 2.4(b)) to obtain

$$I_2 \le \frac{C_\mu}{\mu(B_0)} \int_R^\infty \varepsilon t^{\varepsilon-2} \nu_0(X)\, dt \le \frac{C_\mu}{\mu(B_0)} \nu_0(X) \frac{\varepsilon}{1-\varepsilon} R^{\varepsilon-1}.$$

Since $\nu_0(X) = \nu(2c_0B_0)$, taking $R = \nu(2c_0B_0)/\mu(B_0)$ and using the doubling condition on μ, we finally have

$$I_1 + I_2 \le C(\varepsilon)\left(\frac{\nu(2c_0B_0)}{\mu(B_0)}\right)^{\varepsilon} \le C(\varepsilon)\left(\frac{\nu(2c_0B_0)}{\mu(2c_0B_0)}\right)^{\varepsilon} \le C(\varepsilon)\operatorname*{ess\,inf}_{y\in B_0} M\nu(y)^{\varepsilon}. \ \blacksquare$$

COROLLARY 3.2. *For all $x \in X$, the weight $u(\xi) = \delta(x,\xi)^{-\varepsilon}$ is in A_1 for all $0 \le \varepsilon < 1$, with A_1 constant independent of x.*

Proof. The result follows from the previous theorem on taking $\nu = \delta_x$, the Dirac delta at x, since $M\delta_x(\xi)$ is pointwise equivalent to $\delta(x,\xi)^{-1}$. $\blacksquare$

If u_1 and u_2 are two A_1 weights, Hölder's inequality shows that $u_1u_2^{1-p}$ is an A_p weight for $1 < p < \infty$. The last result now yields that there exists $0 < \gamma = \gamma(p) \le 1$ such that $u(\xi) = \delta(x,\xi)^{\gamma}$ is an A_p weight for all $x \in X$, with A_p constant independent of x.

We need some kind of regularity of the family of approach regions to prove our main result. However, in the case of the existence of a group or pseudo-group structure in X, this additional condition allows us to work with a larger class of regions than those generated by translating a fixed one, as we can see in the examples below.

DEFINITION 3.3. We say that a family Ω of approach regions is *regular* if there is a constant $C > 0$ such that for all $(x,t) \in X \times (0,\infty)$ the following condition is satisfied:

$$\forall y \in \Omega_x(t),\ \exists y^* \in X \text{ with } \delta(y,x) = \delta(y^*,x) \text{ such that } B(y^*,t) \subset S(x,Ct).$$

Some examples of regular approach regions are the following:

1. Assume that (X,μ,δ) is a space of homogeneous type and X is a group with identity e. Let Ω_e be an approach region of e, and for each $x \in X$ set $\Omega_x = \{(yx,t) : (y,t) \in \Omega_e\}$. Then $S(x,t) = [\Omega_e(t)]^{-1}B(x,t)$ (see [Su]). If δ is left-invariant, then $S(x,t) = \bigcup_{z\in\Omega_e(t)} B(z^{-1}x,t)$. Now, $y \in \Omega_x(t)$ if and only if $yx^{-1} \in \Omega_e(t)$. Take $y^* = xy^{-1}x$ which satisfies $\delta(y,x) = \delta(y^*,x)$ by left-invariance. Then $B(xy^{-1}x,t) \subset S(x,t)$, and consequently, Ω is regular with $C = 1$. This is the case of Euclidean spaces.

2. Let (X,μ,δ) be a space of homogeneous type. Consider a family of approach regions given by cones of width bounded by a constant $M \ge 1$, that is, $\Omega_x = \Gamma_{\theta(x)}(x)$ and $0 < \theta(x) \le M$. Then $\Omega_x(t) = B(x,\theta(x)t)$ and $B(x,t) \subset S(x,t)$ for all x in X and $t > 0$. Take $y \in \Omega_x(t) \subset B(x,Mt)$ and use Lemma 2.1 to obtain $B(y,t) \subset B(x,c_0Mt)$ and hence $B(y,t) \subset S(x,c_0Mt)$. The family is regular with $C = c_0M$ and $y^* = y$.

3. Set $X = \mathbb{R}$, and let μ be the Lebesgue measure and δ the usual distance $\delta(x,y) = |x-y|$. The family $\{\Gamma_{\theta(x)}(x)\}_{x\in\mathbb{R}}$ of cones with $\theta(x) = |x|$

is also regular. We can see that regularity holds even when the regions are not translates of a fixed one.

We now prove our main result:

THEOREM 3.4. *If Ω is a regular family of approach regions, then the following conditions are equivalent:*

(a) *There exist $C > 0$ and $\theta > 0$ such that $M_\Omega f(x) \le CM_{\Gamma_\theta} f(x)$ for all x in X and all measurable functions f.*

(b) *$A_p^\Omega = A_p$ for all $1 \le p < \infty$, with equivalent constants.*

(c) *There is $p \ge 1$ such that $A_p = A_p^\Omega$, with equivalent constants.*

(d) *There exists $0 < \gamma \le 1$ such that the family $\{\delta(x,\cdot)^\gamma\}_{x\in X}$ of weights is in $W(\Omega)$ uniformly in x.*

(e) *There exists $\theta > 0$ such that $\Omega_x \subset \Gamma_\theta(x)$ for all x in X.*

Proof. It is obvious that (e)⇒(a) and (b)⇒(c). The implication (a)⇒(b) is easy if we recall Theorem 2.12. Now, suppose $A_p^\Omega = A_p$ for some $p \ge 1$, with equivalent constants. We can assume that $p > 1$ by the extrapolation theorem of Rubio de Francia, as proved in [Ja]. We have seen that there is $0 < \gamma = \gamma(p) \le 1$ such that the family $\{\delta(x,\cdot)^\gamma\}_{x\in X}$ of weights is in A_p, uniformly in x. So, by hypothesis, this family is in A_p^Ω uniformly in x. Proposition 2.9 and (3) show that this family is in $W(\Omega)$ uniformly in x, that is, there exists a constant $C > 0$ such that $\|\delta(x,\cdot)^\gamma\|_{W(\Omega)} \le C$ for all x in X.

Suppose the family of weights is uniformly in $W(\Omega)$. Take $(y,t) \in \Omega_x$ for a fixed $x \in X$. By the regularity of Ω and conditions (ii) and (iii) on δ, we have

$$\begin{aligned}
\delta(y,x)^\gamma = \delta(y^*,x)^\gamma \le D^\gamma \delta(x,y^*)^\gamma &= \frac{D^\gamma}{\mu(B(y^*,t))} \int_{B(y^*,t)} \delta(x,y^*)^\gamma \, d\mu(\xi) \\
&\le \frac{(AD)^\gamma}{\mu(B(y^*,t))} \int_{B(y^*,t)} (\delta(x,\xi) + \delta(y^*,\xi))^\gamma \, d\mu(\xi) \\
&< (AD)^\gamma \left(\frac{1}{\mu(B(y^*,t))} \int_{S(x,Ct)} \delta(x,\xi)^\gamma \, d\mu(\xi) + t^\gamma \right).
\end{aligned}$$

Set $u_x(\xi) = \delta(x,\xi)^\gamma$. Using the hypothesis on the family $\{u_x\}_{x\in X}$, and the fact that $\mu(B(y^*,t))$ is comparable to the radius t, we have

$$\begin{aligned}
\delta(y,x)^\gamma &\le (AD)^\gamma \left(\frac{u_x(S(x,Ct))}{\mu(B(y^*,t))} + t^\gamma \right) \\
&\le C(AD)^\gamma \left(\frac{u_x(B(x,Ct))}{t} + t^\gamma \right).
\end{aligned} \tag{5}$$

Let us now see that $u_x(B(x,t))$ is comparable to $t^{\gamma+1}$ for all $t>0$:

$$\begin{aligned} u_x(B(x,t)) &= \int\limits_{B(x,t)} \delta(x,\xi)^\gamma\, d\mu(\xi) = \sum_{k\ge 0} \int\limits_{2^{-k-1}t\le \delta(x,\xi)<2^{-k}t} \delta(x,\xi)^\gamma\, d\mu(\xi) \\ &\sim t^\gamma \sum_{k\ge 0} 2^{-k\gamma}\mu(\{\xi : 2^{-k-1}t \le \delta(x,\xi) < 2^{-k}t\}) \\ &\sim t^\gamma \sum_{k\ge 0} 2^{-k\gamma}(\mu(B(x,2^{-k}t)) - \mu(B(x,2^{-k-1}t))) \\ &\sim t^{\gamma+1} \sum_{k\ge 0} 2^{-k\gamma}(2^{-k} - 2^{-k-1}) \sim t^{\gamma+1}. \end{aligned}$$

Finally, returning to (5), we get $\delta(y,x) \le C^{1/\gamma}ADt$, that is, $(y,t) \in \Gamma_\theta(x)$ where $\theta = C^{1/\gamma}AD$, and the proof is complete. ∎

REMARKS 3.5. 1. The implication (b)⇒(c) can be viewed as a local result: Fix $x \in X$; if Ω_x satisfies the condition in Definition 3.3 for all $t>0$, then $u_x = \delta(x,\cdot)^\gamma \in W(\Omega)$ implies there exists $\theta_x > 0$ such that $\Omega_x \subset \Gamma_{\theta_x}(x)$.

2. A space (X,μ,d) of homogeneous type is called *regular* if there is $\alpha > 0$ such that $\mu(B(x,r)) \sim r^\alpha$ for all x in X. Our result holds for such spaces and the change of quasidistance is not needed.

3. We observe that in Definition 3.3 we could have assumed that the point y^* satisfies just $\delta(y,x) \approx \delta(y^*,x)$. Also in statement (d) of Theorem 3.4, the restriction $\gamma \le 1$ is not really needed.

COROLLARY 3.6. *Let (X,μ,d) be a space of homogeneous type. Suppose Ω is a family of approach regions in (X,μ,d) such that the related family Ω^δ in (X,μ,δ) is regular. Then the following conditions are equivalent*:

(a) *$A_p = A_p^\Omega$ for all $p \ge 1$, with equivalent constants.*

(b) *There is $p \ge 1$ such that $A_p = A_p^\Omega$ with equivalent constants.*

(c) *There is $\theta > 0$ such that $\Omega_x \subset \Gamma_\theta(x)$ for all x in X.*

Proof. We only need to see that (b) implies (c). We saw before that $A_p^\Omega = A_p^{\Omega^\delta}$ with the same constants. The previous theorem yields that Ω_x^δ is contained in a cone $\Gamma_\theta^\delta(x)$ with respect to δ, and then Ω_x is also contained in a cone. In fact, we will see that there exists $\theta > 0$ such that $\Omega_x \subset \Gamma_\theta(x)$ if and only if there exists $\theta' > 0$ such that $\Omega_x^\delta \subset \Gamma_{\theta'}^\delta(x)$. Using Proposition 2.2 and the fact that

$$\exp\left(\frac{-1}{1+t}\right) \le \exp\left(\frac{-1}{1+\theta t}\right) \le e\exp\left(\frac{-1}{1+t}\right)$$

for all $\theta \ge 1$ and $t > 0$, we have

$$K_1\theta^\beta r_y(t) \le r_y(\theta t) \le eK\theta^\alpha r_y(t). \tag{6}$$

Assume first that $\Omega_x \subset \Gamma_\theta(x)$. We can assume that $\theta \ge 1$. Take $(y,s) \in \Omega_x^\delta$, that is, $(y, r_y^{-1}(s)) \in \Omega_x$. By hypothesis, $d(y,x) < \theta r_y^{-1}(s)$ and by definition $\delta(y,x) < r_y(\theta r_y^{-1}(s))$; using (6) we get $\delta(y,x) < eK\theta^\alpha s$, and so $(y,s) \in \Gamma_{\theta'}^\delta(x)$ with $\theta' = eK\theta^\alpha$.

On the other hand, assume that $\Omega_x^\delta \subset \Gamma_{\theta'}^\delta(x)$ with $\theta' \ge 1$. Take $(y,s) \in \Omega_x$. Then $(y, r_y(s)) \in \Omega_x^\delta$ and therefore $\delta(y,x) < \theta' r_y(s)$. Using the definition of δ and (6), we get $d(y,x) < (\theta'/K_1)^{1/\beta}s$, and so $(y,s) \in \Gamma_\theta(x)$ with $\theta = (\theta'/K_1)^{1/\beta}$. ∎

We now give a version of our result in the case of a group structure in X (see [SS2]).

COROLLARY 3.7. *Let (X,μ,d) be a space of homogeneous type. Suppose that X is a group and that d and μ are left-invariant, that is,*

1. *$yB(x,t) = B(yx,t)$ for all x and y in X and $t > 0$.*
2. *$\mu(xE) = \mu(E)$ for all measurable sets E and $x \in X$.*

Given an approach region Ω_e for the identity element of X, set $\Omega_x = \{(yx,t) : (y,t) \in \Omega_e\}$. Then the following conditions are equivalent:

(a) *$A_p = A_p^\Omega$ for all $p \ge 1$.*

(b) *There is $p \ge 1$ such that $A_p = A_p^\Omega$.*

(c) *There is $\theta > 0$ such that $\Omega_x \subset \Gamma_\theta(x)$ for all x in X.*

Proof. The assumptions on d and μ show that $r_x(t) = r_e(t)$ for all x in X (and therefore δ is symmetric). Then it is easy to see that $\Omega_x^\delta = \{(yx,t) : (y,t) \in \Omega_e^\delta\}$ using the definition of Ω^δ, and we saw before that this kind of family of approach regions is regular. So, we are in the hypothesis of Theorem 3.4, but we do not need the equivalence of the constants because in the case of translated approach regions, if one region is contained in a cone, so are all the rest. ∎

COROLLARY 3.8. *Let (X,μ,d) be a space of homogeneous type as in the previous corollary. There exists a family Ω of approach regions for which A_p^Ω is not A_p, but $u \equiv 1 \in A_p^\Omega$ for $p \ge 1$; i.e., $\emptyset \ne A_p^\Omega \ne A_p$.*

Proof. Sueiro [Su] gives a family $\Omega = \{\Omega_x\}_{x\in X}$ of translated regions which is not nontangential, and so not contained in a cone, for which the operator M_Ω is of weak type (p,p) on $L^p(\mu)$ for $p \ge 1$, that is, $u \equiv 1 \in A_p^\Omega$ for all $p \ge 1$. Using the previous corollary, we get $A_p^\Omega \ne A_p$ for this family. ∎

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Uniqueness of unconditional bases in c_0-products

by

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Abstract. We give counterexamples to a conjecture of Bourgain, Casazza, Lindenstrauss and Tzafriri that if X has a unique unconditional basis (up to permutation) then so does $c_0(X)$. We also give some positive results including a simpler proof that $c_0(\ell_1)$ has a unique unconditional basis and a proof that $c_0(\ell_{p_n}^{N_n})$ has a unique unconditional basis when $p_n \downarrow 1$, $N_{n+1} \geq 2N_n$ and $(p_n - p_{n+1}) \log N_n$ remains bounded.

1. Introduction. A Banach space X is said to have a *unique unconditional basis* (or more precisely, a unique unconditional basis up to permutation) if it has an unconditional basis and if whenever (u_n) and (v_n) are two normalized unconditional bases of X, then there is a permutation π of $\mathbb{N}$ such that (v_n) and $(u_{\pi(n)})$ are equivalent. Since unconditional bases correspond to discrete or atomic order-continuous lattice structures on X, this can be reworded as a statement that such a lattice-structure is essentially unique.

The earliest examples of Banach spaces with unique unconditional bases are c_0, ℓ_1 ([10]) and ℓ_2 ([9]). It was shown by Lindenstrauss and Zippin [12] that amongst spaces with symmetric bases this is the complete list. Later Edelstein and Wojtaszczyk showed that direct sums of these spaces also have unique unconditional bases. All these results can be found in [11]. In [3] the authors attempted a complete classification and showed that the spaces $c_0(\ell_1), c_0(\ell_2), \ell_1(c_0)$ and $\ell_1(\ell_2)$ all have unique unconditional bases while $\ell_2(\ell_1)$ does not. They also found an unexpected additional space, 2-convexified Tsirelson (see [5] for the definition), with a unique unconditional basis. Recently, the authors found a new approach to this type of problem and were able to add some more spaces, including Tsirelson space (see [5]) itself and certain Nakano spaces [4] (as pointed out in [4], some spaces considered by Gowers [8] provide further examples); we also showed

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