6.10. REMARK. In general a necessary and sufficient condition for $\xi(\omega)$ to be cyclic and separating for both M and M_0 is that $LR(\omega) = M$.

Theorem 6.9 shows how far the subset of $R(M, M_0)$ of its elements dominated by ϱ can be from being dense in $R(M, M_0)$, while this is the case for $m(\omega)$ in $(M_*)^+$.

Acknowledgements. This paper has been supported by the Italian M.U.R.S.T. through 60% and 40% grants.

References

[1] L. Accardi and C. Cecchini, *Conditional expectations in von Neumann algebras and a theorem of Takesaki*, J. Funct. Anal. 45 (1982), 245–273.

[2] H. Araki, *Some properties of modular conjugation operator of von Neumann algebras and a noncommutative Radon–Nikodym theorem with a chain rule*, Pacific J. Math. 50 (1974), 309–354.

[3] C. Cecchini, *An abstract characterization of ω-conditional expectations*, Math. Scand. 66 (1990), 155–160.

[4] C. Cecchini and D. Petz, *State extensions and a Radon–Nikodym theorem for conditional expectations on von Neumann algebras*, Pacific J. Math. 138 (1989), 9–23.

[5] —, —, *Classes of conditional expectations over von Neumann algebras*, J. Funct. Anal. 92 (1990), 8–29.

[6] F. Combes et C. Delaroche, *Groupe modulaire d'une espérance conditionnelle dans une algèbre de von Neumann*, Bull. Soc. Math. France 103 (1975), 385–426 (1976).

[7] A. Connes, *Sur le théorème de Radon–Nikodym pour les poids normaux fidèles semifinis*, ibid. 97 (1973), 253–258.

Dipartimento di Matematica e Informatica
Università di Udine
Via delle Scienze 206
33100 Udine, Italy
E-mail: cecchini@dimi.uniud.it

Received December 12, 1996 (3801)

STUDIA MATHEMATICA 135 (1) (1999)

Interpolation of the measure of non-compactness by the real method

by

FERNANDO COBOS (Madrid),
PEDRO FERNÁNDEZ-MARTÍNEZ (Murcia)
and ANTÓN MARTÍNEZ (Vigo)

Dedicated to Professor David E. Edmunds on the occasion of his 65th birthday

Abstract. We investigate the behaviour of the measure of non-compactness of an operator under real interpolation. Our results refer to general Banach couples. An application to the essential spectral radius of interpolated operators is also given.

Introduction. In 1960 Krasnosel'skiĭ [11] proved that compactness of an operator can be interpolated between L_p-spaces. A motivation for this result might have been a remark by S. G. Kreĭn on the interpolation character that certain compactness results for integral operators between L_p-spaces established by Kantorovich in 1956 seemed to have (see [12], p. 118).

At the beginning of the sixties, with the foundation of abstract interpolation theory, Krasnosel'skiĭ's result led to the investigation of interpolation properties of compact operators between abstract Banach spaces. The main contributions during that period are due to J. L. Lions, J. Peetre, E. Gagliardo, A. Calderón, A. Persson, S. G. Kreĭn, Yu. I. Petunin and K. Hayakawa (see [2] and [16] for precise references).

More recently, the paper [4] by Cobos, Edmunds and Potter opened a new research period in this area, and in 1992, culminating the efforts of several authors (see the paper [6] by Cobos and Peetre for references) M. Cwikel [9] proved that compactness of an operator can be interpolated between any Banach couples by the real method.

In the present paper we investigate the behaviour under real interpolation of the measure of non-compactness, a concept that means more than only continuity but not so much as compactness. Previous results on this

1991 *Mathematics Subject Classification*: 46B70, 46M35.
All three authors have been supported in part by DGICYT (PB94-0252).

question have been obtained by Edmunds and Teixeira [15] (see also the monograph by A. Pietsch [14], Prop. 12.1.11 and 12.1.12). These results require the assumption that one of the Banach couples degenerates to a Banach space, i.e. $A_0 = A_1$ or $B_0 = B_1$, or that they are different but the image couple (B_0, B_1) satisfies a certain approximation condition.

We consider here the case of general couples without assuming any approximation hypothesis on them, and we show that the following logarithmically convex inequality holds:

$$\beta(T:(A_0,A_1)_{\theta,q}\to(B_0,B_1)_{\theta,q}) \le c\beta(T:A_0\to B_0)^{1-\theta}\beta(T:A_1\to B_1)^{\theta}. \tag{1}$$

In the special case where one restriction of T is compact, say $\beta(T : A_0 \to B_0) = 0$, we recover Cwikel's compactness theorem. As another application, combining Nussbaum's formula [13] for the essential spectral radius of an operator with inequality (1), we derive an estimate for the essential spectral radius of the interpolated operator that complements previous results of Albrecht [1].

Our techniques are based on the decomposition of the interpolated operator described in [5], Thm. 1.3 (see also [8], Thm. 3.1), and a more refined splitting of the different terms. Those operators that in the compact case were the easiest to deal with require now a very careful study using the construction of the real interpolation method.

We also show the necessity of a constant in inequality (1).

1. Real interpolation and measure of non-compactness. Let A and B be Banach spaces and let $T \in \mathcal{L}(A, B)$ be a bounded linear operator acting from A into B. The *(ball) measure of non-compactness* of T is defined by

$$\beta(T)=\inf\Big\{r>0:\text{there exists a finite number of elements } b_1,\dots,b_n\in B \text{ such that } T(\mathcal{U}_A)\subseteq\bigcup_{j=1}^{n}\{b_j+r\mathcal{U}_B\}\Big\}$$

where $\mathcal{U}_A$ (resp. $\mathcal{U}_B$) stands for the closed unit ball of A (resp. B).

Clearly, $\beta(T) \le \|T\|$, and $\beta(T) = 0$ if and only if T is compact. We refer to [10] and [3] for other properties of this notion.

Let $\overline{A} = (A_0, A_1)$ be a *Banach couple*, that is to say, two Banach spaces A_0 and A_1 which are continuously embedded in a common Hausdorff topological vector space. Let $A_0 + A_1$ be the sum of the spaces and $A_0 \cap A_1$ their intersection. These spaces become Banach spaces when endowed with the

norms

$$\|a\|_{A_0+A_1}=\inf\{\|a_0\|_{A_0}+\|a_1\|_{A_1}:a=a_0+a_1,\ a_j\in A_j\},$$
$$\|a\|_{A_0\cap A_1}=\max\{\|a\|_{A_0},\|a\|_{A_1}\}.$$

Given any positive number t, the K- and the J-*functionals* are defined by

$$\begin{aligned} K(t,a)&=K(t,a;A_0,A_1)\\ &=\inf\{\|a_0\|_{A_0}+t\|a_1\|_{A_1}:a=a_0+a_1,\ a_j\in A_j\},\qquad a\in A_0+A_1,\\ J(t,a)&=J(t,a;A_0,A_1)=\max\{\|a\|_{A_0},t\|a\|_{A_1}\},\qquad a\in A_0\cap A_1.\end{aligned}$$

Note that for each t, $K(t,\cdot)$ (resp. $J(t,\cdot)$) is a norm on A_0+A_1 (resp. $A_0\cap A_1$) equivalent to $\|\cdot\|_{A_0+A_1}$ (resp. $\|\cdot\|_{A_0\cap A_1}$).

Let $1 \le q \le \infty$ and $0 < \theta < 1$. The *real interpolation space* $\overline{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ consists of all $a \in A_0 + A_1$ which have a finite norm

$$\|a\|_{\theta,q}=\begin{cases}\Big(\int_0^\infty (t^{-\theta}K(t,a))^q\,\frac{dt}{t}\Big)^{1/q} & \text{if } 1\le q<\infty,\\ \sup_{t>0}\{t^{-\theta}K(t,a)\} & \text{if } q=\infty\end{cases}$$

(see [2] and [16] for properties of these spaces).

Let (B_0, B_1) be another Banach couple and let T be a linear operator mapping $A_0 + A_1$ into $B_0 + B_1$ such that its restrictions to A_j are continuous mappings from A_j into B_j with norm M_j $(j = 0, 1)$. Then it is well known that T maps $(A_0, A_1)_{\theta,q}$ continuously into $(B_0, B_1)_{\theta,q}$ with norm

$$\|T\|_{\theta,q}\le\|T\|_{A_0,B_0}^{1-\theta}\|T\|_{A_1,B_1}^{\theta}. \tag{2}$$

Let $\beta_0(T), \beta_1(T)$ and $\beta_{\theta,q}(T)$ be the measures of non-compactness of T as a mapping from A_0 to B_0, A_1 to B_1 and $(A_0, A_1)_{\theta,q}$ into $(B_0, B_1)_{\theta,q}$, respectively. It is natural to wonder if a similar formula to (2) holds for the measure of non-compactness, i.e. if

$$\beta_{\theta,q}(T)\le\beta_0(T)^{1-\theta}\beta_1(T)^{\theta}. \tag{2'}$$

The following example shows that (2′) can only be true in general with an additional constant.

EXAMPLE 1.1. We work with the spaces ℓ_p of p-summable sequences. Take $(A_0, A_1) = (\ell_1, \ell_1)$, $(B_0, B_1) = (\ell_1, \ell_\infty)$ and choose $T = I$ the identity operator, $I\xi = \xi$.

It is easy to check that

$$\beta_0(I)=\beta(I:\ell_1\to\ell_1)=1.$$

On the other hand, according to [3], (3.5.17),

$$\beta_1(I)=\beta(I:\ell_1\to\ell_\infty)=1/2.$$

Take $\theta \in (0,1)$ close enough to 1 so that

$$\theta^{1-\theta}2^{\theta} > 1,$$

and put $1/q = 1-\theta$. We compute the interpolation norm on $\ell_1 = (\ell_1, \ell_1)_{\theta,q}$:

$$\begin{aligned}\|\xi\|_{\theta,q} &= \Big(\int_0^\infty (t^{-\theta}K(t,\xi;\ell_1,\ell_1))^q \frac{dt}{t}\Big)^{1/q}\\ &= \Big(\int_0^\infty (t^{-\theta}\min\{1,t\}\|\xi\|_{\ell_1})^q \frac{dt}{t}\Big)^{1/q}\\ &= \Big(\int_0^1 t^{(1-\theta)q}\frac{dt}{t} + \int_1^\infty t^{-\theta q}\frac{dt}{t}\Big)^{1/q}\|\xi\|_{\ell_1} = \theta^{-(1-\theta)}\|\xi\|_{\ell_1}.\end{aligned}$$

Hence, the identity map $I : \ell_1 \to (\ell_1,\ell_1)_{\theta,q}$ has norm $\theta^{-(1-\theta)}$.

Next we estimate the norm of I acting from $\ell_q = (\ell_1, \ell_\infty)_{\theta,q}$ (endowed with the interpolation norm) into ℓ_q. Using the fact that

$$K(t,\xi;\ell_1,\ell_\infty) = \int_0^t \xi^*(s)\,ds$$

where ξ^* is the non-increasing rearrangement of ξ on $(0,\infty)$, we derive that

$$\begin{aligned}\|\xi\|_{\ell_q} &= \Big(\int_0^\infty (\xi^*(t))^q\,dt\Big)^{1/q} = \Big(\int_0^\infty (t^{-\theta}t\xi^*(t))^q \frac{dt}{t}\Big)^{1/q}\\ &\le \Big(\int_0^\infty \Big(t^{-\theta}\int_0^t \xi^*(s)\,ds\Big)^q \frac{dt}{t}\Big)^{1/q}\\ &= \Big(\int_0^\infty (t^{-\theta}K(t,\xi;\ell_1,\ell_\infty))^q \frac{dt}{t}\Big)^{1/q} = \|\xi\|_{\theta,q}.\end{aligned}$$

Since we can factorize $I : \ell_1 \to \ell_q$ by means of the diagram

$$\begin{array}{ccc} (\ell_1,\ell_1)_{\theta,q} & \xrightarrow{\ I\ } & (\ell_1,\ell_\infty)_{\theta,q}\\ I\uparrow & & \downarrow I\\ \ell_1 & & \ell_q \end{array}$$

if $(2')$ were true then we would obtain

$$\begin{aligned}\beta(I:\ell_1\to\ell_q) &\le \|I:\ell_1\to(\ell_1,\ell_1)_{\theta,q}\|\beta_{\theta,q}(I)\|I:(\ell_1,\ell_\infty)_{\theta,q}\to\ell_q\|\\ &\le \theta^{-(1-\theta)}\beta_0(I)^{1-\theta}\beta_1(I)^{\theta} \le \theta^{-(1-\theta)}2^{-\theta} < 1\end{aligned}$$

by our choice of θ. Take $\theta^{-(1-\theta)}2^{-\theta} < s < 1$. Then there would exist finitely many vectors $\xi_r = (\xi_n^r) \in \ell_q$, $r = 1,\dots,m$, such that

$$\mathcal{U}_{\ell_1} = I(\mathcal{U}_{\ell_1}) \subseteq \bigcup_{r=1}^m \{\xi_r + s\mathcal{U}_{\ell_q}\}. \tag{3}$$

Let e_j be the sequence which is zero at all coordinates but the jth where it is 1. It follows from (3) that there is a subsequence (n') of $\mathbb{N}$ and some r, say $r = 1$, such that

$$\|\xi_1 - e_{n'}\|_{\ell_q} \le s \quad \text{for all } n'.$$

Consequently,

$$0 < 1 - s \le \xi^1_{n'} \quad \text{for all } n',$$

which contradicts that $\xi_1 \in \ell_q$.

We next show that indeed

$$\beta_{\theta,q}(T) \le c\beta_0(T)^{1-\theta}\beta_1(T)^{\theta}. \tag{2''}$$

To this end, in what follows we endow $(B_0,B_1)_{\theta,q}$ with the discrete K-norm

$$\|b\|_{\theta,q;K} = \Big(\sum_{m=-\infty}^{\infty}(2^{-\theta m}K(2^m,b;B_0,B_1))^q\Big)^{1/q}$$

(the sum should be replaced by the supremum if $q=\infty$), and we equip $(A_0,A_1)_{\theta,q}$ with the discrete J-norm

$$\|a\|_{\theta,q;J} = \inf\Big\{\Big(\sum_{m=-\infty}^{\infty}(2^{-\theta m}J(2^m,u_m;A_0,A_1))^q\Big)^{1/q} : a = \sum_{m=-\infty}^{\infty}u_m,\ (u_m)\subset A_0\cap A_1\Big\}.$$

It is known (see [2] or [16]) that both norms are equivalent to $\|\cdot\|_{\theta,q}$. Moreover,

$$\|\cdot\|_{\theta,q;K} \le \frac{1}{3-2^{\theta}-2^{1-\theta}}\|\cdot\|_{\theta,q;J}.$$

For these discrete norms the interpolation property (2) still holds but with the additional constant $\delta = 2^{\theta}(3-2^{\theta}-2^{1-\theta})^{-1}$. More precisely,

$$\|T\|_{\theta,q;d} = \|T\|_{(\overline{A}_{\theta,q},\|\cdot\|_{\theta,q;J}),(\overline{B}_{\theta,q},\|\cdot\|_{\theta,q;K})} \le \delta\|T\|^{1-\theta}_{A_0,B_0}\|T\|^{\theta}_{A_1,B_1}.$$

In the sequel, we denote by $\beta_{\theta,q}(T)$ the measure of non-compactness of $T : (A_0,A_1)_{\theta,q} \to (B_0,B_1)_{\theta,q}$ computed by using the discrete norms.

Given any sequence (W_m) of Banach spaces and any sequence (λ_m) of non-negative numbers, we write $\ell_q(\lambda_m W_m)$ for the vector-valued ℓ_q space, that is to say,

$$\ell_q(\lambda_m W_m) = \Big\{w = (w_m) : w_m \in W_m \text{ and } \|w\|_{\ell_q(\lambda_m W_m)} = \Big(\sum_{m=-\infty}^{\infty}(\lambda_m\|w_m\|_{W_m})^q\Big)^{1/q} < \infty\Big\}.$$

Put

$$G_m = (A_0 \cap A_1, J(2^m, \cdot\,; A_0, A_1)), \qquad m \in \mathbb{Z},$$

and

$$F_m = (B_0 + B_1, K(2^m, \cdot\,; B_0, B_1)), \qquad m \in \mathbb{Z}.$$

Vector-valued spaces generated by the sequences (G_m) and (F_m) will be of special interest for us.

THEOREM 1.2. *Let (A_0, A_1) and (B_0, B_1) be Banach couples and let T be a linear operator such that $T : A_0 \to B_0$ and $T : A_1 \to B_1$ are bounded. Then for any $1 \le q \le \infty$ and $0 < \theta < 1$ we have*

$$\beta_{\theta,q}(T) \le 16\delta\beta_0(T)^{1-\theta}\beta_1(T)^{\theta}$$

where $\delta = 2^{\theta}(3 - 2^{\theta} - 2^{1-\theta})^{-1}$.

Proof. Consider the operator $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$. Clearly,

$$\pi : \ell_1(G_m) \to A_0 \quad \text{and} \quad \pi : \ell_1(2^{-m}G_m) \to A_1$$

are bounded with norms ≤ 1. Moreover, since the norm of $(A_0, A_1)_{\theta,q}$ is the discrete J-norm,

$$\pi : \ell_q(2^{-\theta m}G_m) \to (A_0, A_1)_{\theta,q}$$

is a metric surjection.

Let now j be the operator that associates with any $b \in B_0 + B_1$ the constant sequence $j(b) = (\ldots, b, b, b, \ldots)$. The restrictions

$$j : B_0 \to \ell_\infty(F_m) \quad \text{and} \quad j : B_1 \to \ell_\infty(2^{-m}F_m)$$

are bounded with norms ≤ 1. Moreover,

$$j : (B_0, B_1)_{\theta,q} \to \ell_q(2^{-\theta m}F_m)$$

is a metric injection (recall that $(B_0, B_1)_{\theta,q}$ is equipped with the discrete K-norm).

We then have the following diagram of bounded operators:

$$\begin{array}{ccccccc}
\ell_1(G_m) & \xrightarrow{\pi} & A_0 & \xrightarrow{T} & B_0 & \xrightarrow{j} & \ell_\infty(F_m) \\
\ell_1(2^{-m}G_m) & \xrightarrow{\pi} & A_1 & \xrightarrow{T} & B_1 & \xrightarrow{j} & \ell_\infty(2^{-m}F_m) \\
\hline\hline
\ell_q(2^{-\theta m}G_m) & \xrightarrow{\pi} & (A_0, A_1)_{\theta,q} & \xrightarrow{T} & (B_0, B_1)_{\theta,q} & \xrightarrow{j} & \ell_q(2^{-\theta m}F_m)
\end{array}$$

Write $\widehat{T} = jT\pi$. Properties of π and j yield that

$$\begin{aligned}
\beta_0(\widehat{T}) &= \beta(\widehat{T} : \ell_1(G_m) \to \ell_\infty(F_m)) \\
&\le \|\pi : \ell_1(G_m) \to A_0\| \beta(T : A_0 \to B_0) \|j : B_0 \to \ell_\infty(F_m)\| \\
&\le \beta(T : A_0 \to B_0) = \beta_0(T).
\end{aligned}$$

Similarly, $\beta_1(\widehat{T}) \le \beta_1(T)$, while

$$\begin{aligned}
\beta_{\theta,q}(T) &= \beta(T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}) \\
&\le 2\beta(jT : (A_0, A_1)_{\theta,q} \to \ell_q(2^{-\theta m}F_m)) \\
&= 2\beta(\widehat{T} : \ell_q(2^{-\theta m}G_m) \to \ell_q(2^{-\theta m}F_m)).
\end{aligned}$$

Put, for simplicity, $\beta(\widehat{T}) = \beta(\widehat{T} : \ell_q(2^{-\theta m}G_m) \to \ell_q(2^{-\theta m}F_m))$. In order to establish the theorem it suffices to show that

$$\beta(\widehat{T}) \le 8\delta\beta_0(T)^{1-\theta}\beta_1(T)^{\theta}. \tag{4}$$

Given any $n \in \mathbb{N}$, define operators P_n, Q_n^+, Q_n^- on the Banach couple $(\ell_1(G_m), \ell_1(2^{-m}G_m))$ by

$$\begin{aligned}
P_n(u_m) &= (\ldots, 0, 0, u_{-n}, u_{-n+1}, \ldots, u_{n-1}, u_n, 0, 0, \ldots), \\
Q_n^+(u_m) &= (\ldots, 0, 0, u_{n+1}, u_{n+2}, \ldots), \\
Q_n^-(u_m) &= (\ldots, u_{-n-2}, u_{-n-1}, 0, 0 \ldots).
\end{aligned}$$

It is not hard to check that these operators satisfy the following conditions:

(I) The identity operator on $\ell_1(G_m) + \ell_1(2^{-m}G_m)$ can be decomposed as $I = P_n + Q_n^+ + Q_n^-$ for $n = 1, 2, \ldots$

(II) The operators are uniformly bounded, i.e.

$$\|P_n : \ell_1(G_m) \to \ell_1(G_m)\| = \|P_n : \ell_1(2^{-m}G_m) \to \ell_1(2^{-m}G_m)\| = 1$$

and similarly for Q_n^+ and Q_n^-.

(III) We have

$$\|Q_n^+ : \ell_1(G_m) \to \ell_1(2^{-m}G_m)\| = 2^{-(n+1)} = \|Q_n^- : \ell_1(2^{-m}G_m) \to \ell_1(G_m)\|.$$

On the couple $(\ell_\infty(F_m), \ell_\infty(2^{-m}F_m))$ we can define analogous operators that satisfy the corresponding versions of (I), (II) and (III). We call them R_n, S_n^+, S_n^-.

In order to estimate $\beta(\widehat{T})$ we decompose $\widehat{T}$ by using (I) as

$$\begin{aligned}
\widehat{T} &= \widehat{T}(P_n + Q_n^+ + Q_n^-) = \widehat{T}P_n + (R_n + S_n^+ + S_n^-)\widehat{T}(Q_n^+ + Q_n^-) \\
&= \widehat{T}P_n + R_n\widehat{T}(Q_n^+ + Q_n^-) + S_n^+\widehat{T}Q_n^- + S_n^-\widehat{T}Q_n^+ + S_n^+\widehat{T}Q_n^+ + S_n^-\widehat{T}Q_n^-.
\end{aligned}$$

Thus

$$\begin{aligned}
\beta(\widehat{T}) \le\ & \beta(S_n^+\widehat{T}Q_n^-) + \beta(S_n^-\widehat{T}Q_n^+) + \beta(S_n^+\widehat{T}Q_n^+) \\
&+ \beta(S_n^-\widehat{T}Q_n^-) + \beta(\widehat{T}P_n) + \beta(R_n\widehat{T}(Q_n^+ + Q_n^-)).
\end{aligned}$$

Here all operators act from $\ell_q(2^{-\theta m}G_m)$ into $\ell_q(2^{-\theta m}F_m)$.

To proceed with these terms, we recall that the following interpolation formulae hold with equivalence of norms:

$$\begin{aligned}
\ell_q(2^{-\theta m}G_m) &= (\ell_1(G_m), \ell_1(2^{-m}G_m))_{\theta,q}, \\
\ell_q(2^{-\theta m}F_m) &= (\ell_\infty(F_m), \ell_\infty(2^{-m}F_m))_{\theta,q}.
\end{aligned}$$

Furthermore, writing down for the case of couples the arguments given in [7], Thm. 3.1, one can verify that the inclusions

$$\ell_q(2^{-\theta m}G_m) \hookrightarrow ((\ell_1(G_m), \ell_1(2^{-m}G_m))_{\theta,q}, \|\cdot\|_{\theta,q;J})$$

and

$$((\ell_\infty(F_m), \ell_\infty(2^{-m}F_m))_{\theta,q}, \|\cdot\|_{\theta,q;K}) \hookrightarrow \ell_q(2^{-\theta m}F_m)$$

have norm less than or equal to 1. Therefore

$$\begin{aligned}\beta(S_n^+\widehat{T}Q_n^-) &\le \beta_{\theta,q}(S_n^+\widehat{T}Q_n^-) \le \|S_n^+\widehat{T}Q_n^-\|_{\theta,q;d}\\ &\le \delta\|S_n^+\widehat{T}Q_n^-\|_0^{1-\theta}\|S_n^+\widehat{T}Q_n^-\|_1^{\theta}.\end{aligned}$$

Using now the factorization

$$\begin{array}{ccc} \ell_1(G_m) & \xrightarrow{\widehat{T}} & \ell_\infty(F_m) \\ {\scriptstyle Q_n^-}\uparrow & & \downarrow{\scriptstyle S_n^+} \\ \ell_1(2^{-m}G_m) & & \ell_\infty(2^{-m}F_m) \end{array}$$

and properties (II) and (III), we obtain

$$\begin{aligned}\beta(S_n^+\widehat{T}Q_n^-) &\le \delta\|\widehat{T}\|_0^{1-\theta}(2^{-(n+1)}\|\widehat{T}\|_0 2^{-(n+1)})^{\theta}\\ &= \delta 2^{-2(n+1)\theta}\|\widehat{T}\|_0 \to 0 \quad \text{as } n \to 0.\end{aligned}$$

A similar argument shows that also

$$\beta(S_n^-\widehat{T}Q_n^+) \to 0 \quad \text{as } n \to \infty.$$

Consider next the case of $S_n^+\widehat{T}Q_n^+$. We have

$$\begin{aligned}\beta(S_n^+\widehat{T}Q_n^+) &\le \beta_{\theta,q}(S_n^+\widehat{T}Q_n^+) \le \delta\|S_n^+\widehat{T}Q_n^+\|_0^{1-\theta}\|S_n^+\widehat{T}Q_n^+\|_1^{\theta}\\ &\le \delta\|\widehat{T}Q_n^+\|_0^{1-\theta}\|S_n^+\widehat{T}Q_n^+\|_1^{\theta}.\end{aligned}$$

Since

$$\|\widehat{T}Q_1^+\|_0 \ge \|\widehat{T}Q_2^+\|_0 \ge \ldots \ge 0,$$

there exists $\lambda \ge 0$ such that $\|\widehat{T}Q_n^+\|_0 \to \lambda$ as $n \to \infty$. Find vectors $u_n \in \mathcal{U}_0 = \mathcal{U}_{\ell_1(G_m)}$ such that

$$\|\widehat{T}Q_n^+u_n\|_{\ell_\infty(F_m)} \to \lambda \quad \text{as } n \to \infty.$$

On the other hand, given any $\varepsilon > 0$, there are finitely many vectors $b_1, \ldots, b_s$ in B_0 so that

$$T\pi\mathcal{U}_0 \subseteq \bigcup_{r=1}^{s}\{b_r + (\beta_0(T)+\varepsilon)\mathcal{U}_{B_0}\}.$$

Hence, for some subsequence (n') of $\mathbb{N}$ and some r, say $r = 1$, it follows that

$$T\pi Q_{n'}^+u_{n'} \in \{b_1 + (\beta_0(T)+\varepsilon)\mathcal{U}_{B_0}\} \quad \text{for all } n'.$$

According to (III), we then get, for any $m \in \mathbb{Z}$,

$$\begin{aligned}K(2^m, b_1) &\le \|b_1 - T\pi Q_{n'}^+u_{n'}\|_{B_0} + 2^m\|T\pi Q_{n'}^+u_{n'}\|_{B_1}\\ &\le (\beta_0(T)+\varepsilon) + 2^m\|T\|_1\|Q_{n'}^+u_{n'}\|_{\ell_1(2^{-m}G_m)}\\ &\le (\beta_0(T)+\varepsilon) + 2^{m-n'}\|T\|_1 \to \beta_0(T)+\varepsilon \quad \text{as } n' \to \infty.\end{aligned}$$

This implies that

$$\|jb_1\|_{\ell_\infty(F_m)} = \sup_{m\in\mathbb{Z}}\{K(2^m, b_1)\} \le \beta_0(T) + \varepsilon$$

and so

$$\begin{aligned}\lambda &= \lim_{n'\to\infty}\|\widehat{T}Q_{n'}^+u_{n'}\|_{\ell_\infty(F_m)}\\ &\le \sup_{n'}(\|\widehat{T}Q_{n'}^+u_{n'} - jb_1\|_{\ell_\infty(F_m)} + \|jb_1\|_{\ell_\infty(F_m)}) \le 2(\beta_0(T)+\varepsilon).\end{aligned}$$

Consequently, given any $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for $n \ge N_1$,

$$\|\widehat{T}Q_n^+\|_0^{1-\theta} \le (2\beta_0(T))^{1-\theta} + \varepsilon. \tag{5}$$

We now turn to $\|S_n^+\widehat{T}Q_n^+\|_1$. Given any $\varepsilon > 0$, it is easy to see that there exists a finite set $\{v_r\}_{r=1}^s \subseteq \ell_1(2^{-m}G_m)$, formed by sequences having only a finite number of non-zero coordinates, so that for any $v \in \mathcal{U}_1 = \mathcal{U}_{\ell_1(2^{-m}G_m)}$,

$$\min_{1\le r\le s}\{\|\widehat{T}v - \widehat{T}v_r\|_{\ell_\infty(2^{-m}F_m)}\} \le 2\beta_1(T) + \varepsilon. \tag{6}$$

Since $\{\widehat{T}v_r\}_{r=1}^s \subseteq \ell_\infty(F_m) \cap \ell_\infty(2^{-m}F_m)$, we deduce from (III) that there is $N_2 \in \mathbb{N}$ such that if $n \ge N_2$ then

$$\|S_n^+\widehat{T}v_r\|_{\ell_\infty(2^{-m}F_m)} \le \varepsilon \quad \text{for any } r = 1, \ldots, s. \tag{7}$$

Inequalities (6), (7) and the fact that $Q_n^+\mathcal{U}_1 \subseteq \mathcal{U}_1$ yield that for $n \ge N_2$ and any $v \in \mathcal{U}_1$ we may choose $r \in [1, s]$ so that

$$\begin{aligned}\|S_n^+\widehat{T}Q_n^+v\|_{\ell_\infty(2^{-m}F_m)} &\le \|S_n^+\widehat{T}Q_n^+v - S_n^+\widehat{T}v_r\|_{\ell_\infty(2^{-m}F_m)}\\ &\quad + \|S_n^+\widehat{T}v_r\|_{\ell_\infty(2^{-m}F_m)}\\ &\le \|\widehat{T}Q_n^+v - \widehat{T}v_r\|_{\ell_\infty(2^{-m}F_m)} + \varepsilon \le 2\beta_1(T) + 2\varepsilon.\end{aligned}$$

In other words,

$$\|S_n^+\widehat{T}Q_n^+\|_1 \le 2\beta_1(T) + 2\varepsilon \quad \text{if } n \ge N_2. \tag{8}$$

From (5) and (8) it follows that for any $\varepsilon > 0$, choosing n large enough,

$$\beta(S_n^+\widehat{T}Q_n^+) \le 2\delta\beta_0(T)^{1-\theta}\beta_1(T)^{\theta} + \varepsilon.$$

The term $\beta(S_n^-\widehat{T}Q_n^-)$ can be estimated similarly.

For the remaining two terms our arguments are based on the construction of the real interpolation method. We start with $\beta(\widehat{T}P_n)$. Let ℓ_q^{2n+1} be $\mathbb{R}^{2n+1}$

with the q-norm. Since ℓ_q^{2n+1} is finite-dimensional, given any $\varepsilon > 0$, there is a finite set $\{\mu_r\}_{r=1}^s \subseteq \mathcal{U}_{\ell_q^{2n+1}}$ such that for any $\lambda \in \mathcal{U}_{\ell_q^{2n+1}}$,

$$\min_{1\le r\le s}\{\|\lambda-\mu_r\|_{\ell_q^{2n+1}}\}\le\varepsilon.$$

Take any $k_i > \beta_i(T)$ $(i = 0, 1)$, and find $\nu \in \mathbb{Z}$ so that $2^{\nu-1} < k_1/k_0 \le 2^\nu$. Given any $u = (u_m) \in \mathcal{U}_{\ell_q(2^{-\theta m}G_m)}$, since

$$\|(2^{-\theta(-n)}J(2^{-n},u_{-n}),\ldots,2^{-\theta 0}J(2^0,u_0),\ldots,(2^{-\theta n}J(2^n,u_n))\|_{\ell_q^{2n+1}} \le\Big(\sum_{m=-\infty}^{\infty}(2^{-\theta m}J(2^m,u_m))^q\Big)^{1/q}\le 1,$$

we can find $r \in [1, s]$ such that

$$2^{-\theta}2^{-\theta(m-\nu)}\left(\frac{k_1}{k_0}\right)^{-\theta}J\left(2^{m-\nu}\frac{k_1}{k_0},u_m\right) \le 2^{-\theta}2^{-\theta(m-\nu)}2^{-\theta\nu+\theta}J(2^{m-\nu+\nu},u_m)=2^{-\theta m}J(2^m,u_m)\le\mu_m^r+\varepsilon$$

for any $m = -n, \ldots, n$, where $\mu_r = (\mu_{-n}^r, \ldots, \mu_n^r)$. That is to say,

$$\max\left\{2^{-\theta}2^{-\theta(m-\nu)}\left(\frac{k_1}{k_0}\right)^{-\theta}\|u_m\|_{A_0},\,2^{-\theta}2^{(1-\theta)(m-\nu)}\left(\frac{k_1}{k_0}\right)^{1-\theta}\|u_m\|_{A_1}\right\}\le\mu_m^r+\varepsilon$$

for some $r \in [1, s]$ and any $m = -n, \ldots, n$. Then, according to the definition of $\beta_i(T)$, we can pick finite sets $\{b_{m,j}^i\} \subseteq B_i$, $i = 0, 1$, $j = 1, \ldots, h$, where h depends on i, such that

$$\min_{1\le j\le h}\{\|Tu_m-b_{m,j}^i\|_{B_i}\}\le k_i(\mu_m^r+\varepsilon)2^\theta 2^{-(i-\theta)(m-\nu)}\left(\frac{k_1}{k_0}\right)^{\theta-i} =2^\theta 2^{-(i-\theta)(m-\nu)}k_0^{1-\theta}k_1^\theta(\mu_m^r+\varepsilon)$$

with the same r for $m = -n, \ldots, n$. Put

$$\psi_{m,r}^i=2^\theta 2^{-(i-\theta)(m-\nu)}k_0^{1-\theta}k_1^\theta(\mu_m^r+\varepsilon),$$

and for any j, g and r, choose

$$d_{m,p}\in\{b_{m,j}^0+\psi_{m,r}^0\mathcal{U}_{B_0}\}\cap\{b_{m,g}^1+\psi_{m,r}^1\mathcal{U}_{B_1}\}$$

provided the intersection is not void. The number of the $d_{m,p}$'s may change with m, say it is $w = w(m)$, but it is finite. All $d_{m,p}$'s belong to $B_0 \cap B_1$ and, by construction, given any $u = (u_m) \in \mathcal{U}_{\ell_q(2^{-\theta m}G_m)}$ there are some $r \in [1, s]$ and some $\{d_{m,p}\}_{m=-n}^n$ so that

$$J(2^{m-\nu},Tu_m-d_{m,p})=\max\{\|Tu_m-d_{m,p}\|_{B_0},2^{m-\nu}\|Tu_m-d_{m,p}\|_{B_1}\} \le\max\{2\psi_{m,r}^0,2^{m-\nu}2\psi_{m,r}^1\}=2\psi_{m,r}^0$$

where $m = -n, \ldots, n$.

Denote by D the collection of all sums $\sum_{m=-n}^n d_{m,p}$ where $p \in [1, w(m)]$. Then D is a finite subset of $(B_0, B_1)_{\theta,q}$ such that for any $u \in \mathcal{U}_{\ell_q(2^{-\theta m}G_m)}$ there exists some $\sum_{m=-n}^n d_{m,p} \in D$ with

$$\begin{aligned}\Big\|T\pi P_n u-\sum_{m=-n}^n d_{m,p}\Big\|_{\theta,q;K} &=\Big\|\sum_{m=-n}^n(Tu_m-d_{m,p})\Big\|_{\theta,q;K}\\ &\le\frac{\delta}{2^\theta}\Big\|\sum_{m=-n}^n(Tu_m-d_{m,p})\Big\|_{\theta,q;J}\\ &\le\frac{\delta}{2^\theta}\Big(\sum_{m=-n}^n(2^{-\theta(m-\nu)}J(2^{m-\nu},Tu_m-d_{m,p}))^q\Big)^{1/q}\\ &\le\frac{\delta}{2^\theta}2\cdot 2^\theta k_0^{1-\theta}k_1^\theta\Big(\sum_{m=-n}^n(\mu_m^r+\varepsilon)^q\Big)^{1/q}\le 2\delta k_0^{1-\theta}k_1^\theta(1+(2n+1)^{1/q}\varepsilon).\end{aligned}$$

Consequently,

$$\beta(\widehat{T}P_n)\le\beta(T\pi P_n:\ell_q(2^{-\theta m}G_m)\to(B_0,B_1)_{\theta,q})\le 2\delta\beta_0(T)^{1-\theta}\beta_1(T)^\theta.$$

Finally, for $\beta(R_n\widehat{T}(Q_n^+ + Q_n^-))$, we note that

$$\beta(R_n\widehat{T}(Q_n^++Q_n^-))=\beta(R_njT\pi(Q_n^++Q_n^-)) \le\beta(R_njT:(A_0,A_1)_{\theta,q}\to\ell_q(2^{-\theta m}F_m)).$$

Let again $k_i > \beta_i(T)$ $(i = 0, 1)$, take $\nu \in \mathbb{Z}$ satisfying $2^{\nu-1} < k_1/k_0 \le 2^\nu$, and given $\varepsilon > 0$, find $\{\mu_r\}_{r=1}^s \subseteq \mathcal{U}_{\ell_q^{2n+1}}$ so that for any $\lambda \in \mathcal{U}_{\ell_q^{2n+1}}$,

$$\min_{1\le r\le s}\{\|\lambda-\mu_r\|_{\ell_q^{2n+1}}\}\le\varepsilon.$$

For $a \in \mathcal{U}_{(A_0,A_1)_{\theta,q}}$, since

$$\Big(\sum_{m=-n}^n(2^{-\theta(m+\nu)}K(2^{m+\nu},a))^q\Big)^{1/q}\le\|a\|_{\theta,q;K}\le 2^{-\theta}\delta\|a\|_{\theta,q;J}\le 2^{-\theta}\delta,$$

there is some $r \in [1, s]$ with

$$2^{-\theta(m+\nu)}K(2^{m+\nu},a)\le 2^{-\theta}\delta(\mu_m^r+\varepsilon),\qquad m=-n,\ldots,n.$$

Thus

$$K\left(2^m\frac{k_1}{k_0},a\right)\le K(2^{m+\nu},a)\le 2^{\theta(m+\nu)}2^{-\theta}\delta(\mu_m^r+\varepsilon) <\delta\left(2^m\frac{k_1}{k_0}\right)^\theta(\mu_m^r+\varepsilon),\qquad m=-n,\ldots,n.$$

Using now the definition of the K-functional, we can find decompositions

$$a = a_m^0 + a_m^1, \quad a_m^i \in A_i \ (i=0,1), \quad m = -n, \ldots, n.$$

such that

$$\|a_m^0\|_{A_0} + 2^m \frac{k_1}{k_0} \|a_m^1\|_{A_1} \le \delta \Big(2^m \frac{k_1}{k_0} \Big)^{\theta} (\mu_m^r + \varepsilon).$$

Then, by our choice of $k_i > \beta_i(T)$ $(i=0,1)$, there are finite sets $\{b_{m,j}^i\}_{j=1}^h \subseteq B_i$ so that

$$\min_{1\le j\le h} \{\|Ta_m^i - b_{m,j}^i\|_{B_i}\} \le \delta 2^{m(\theta-i)} k_0^{1-\theta} k_1^{\theta} (\mu_m^r + \varepsilon)$$

Write W for the collection of all vector-valued sequences $z_w = (z_m^w)$ defined by

$$z_m^w = \begin{cases} 0 & \text{if } m \notin [-n,n], \\ b_{m,j}^0 + b_{m,g}^1 & \text{if } -n \le m \le n, \end{cases}$$

where j and g run over their domains. Then W is a finite subset of $\ell_q(2^{-\theta m} F_m)$ and given any $a \in \mathcal{U}_{(A_0,A_1)_{\theta,q}}$, if we choose $z_w \in W$ so that $z_m^w = b_{m,j}^0 + b_{m,g}^1$ $(m=-n,\ldots,n)$ with

$$\|Ta_m^0 - b_{m,j}^0\|_{B_0} \le \delta 2^{m\theta} k_0^{1-\theta} k_1^{\theta} (\mu_m^r + \varepsilon),$$
$$\|Ta_m^1 - b_{m,g}^1\|_{B_1} \le \delta 2^{m(\theta-1)} k_0^{1-\theta} k_1^{\theta} (\mu_m^r + \varepsilon),$$

it follows that

$$\begin{aligned} &\|R_n jTa - z_w\|_{\ell_q(2^{-\theta m}F_m)} \\ &= \Big(\sum_{m=-n}^{n} (2^{-\theta m} K(2^m, Ta_m^0 - b_{m,j}^0 + Ta_m^1 - b_{m,g}^1))^q \Big)^{1/q} \\ &\le 2\delta k_0^{1-\theta} k_1^{\theta} \Big(\sum_{m=-n}^{n} (\mu_m^r + \varepsilon)^q \Big)^{1/q} \le 2\delta k_0^{1-\theta} k_1^{\theta} (1 + (2n+1)^{1/q} \varepsilon). \end{aligned}$$

Therefore

$$\beta(R_n jT : (A_0,A_1)_{\theta,q} \to \ell_q(2^{-\theta m} F_m)) \le 2\delta \beta_0(T)^{1-\theta} \beta_1(T)^{\theta},$$

and so

$$\beta(R_n \widehat{T}(Q_n^+ + Q_n^-)) \le 2\delta \beta_0(T)^{1-\theta} \beta_1(T)^{\theta}.$$

In conclusion,

$$\beta(\widehat{T}) \le 8\delta \beta_0(T)^{1-\theta} \beta_1(T)^{\theta}.$$

This proves (4) and completes the proof. ∎

The theorem shows the deviation from compactness that the interpolated operator may have. As a special case, taking $\beta_0(T) = 0$, we recover Cwikel's compactness theorem mentioned in the Introduction.

We close the paper with an application to spectral theory. Let A be a complex Banach space, let $T \in \mathcal{L}(A,A)$ and let $\mathrm{sp_e}(T)$ be the *essential spectrum* of T, that is to say, the collection of all complex numbers λ satisfying at least one of the following conditions:

(i) the range of $T - \lambda I$ is not closed;
(ii) λ is a limit point of the spectrum of T;
(iii) $\bigcup_{r=1}^{\infty} \mathrm{Ker}(T - \lambda I)^r$ is infinite-dimensional.

The *essential spectral radius* is defined by

$$r_{\mathrm e}(T) = \sup\{|\lambda| : \lambda \in \mathrm{sp_e}(T)\}$$

(see [13], [10] and [3] for more details on these notions).

COROLLARY 1.3. *Let (A_0, A_1) be a (complex) Banach couple and let T be a linear operator such that its restrictions $T : A_i \to A_i$ are bounded $(i = 0,1)$. Then for any $0 < \theta < 1$ and $1 \le q \le \infty$,*

$$r_{\mathrm e}(T : (A_0,A_1)_{\theta,q} \to (A_0,A_1)_{\theta,q}) \le r_{\mathrm e}(T : A_0 \to A_0)^{1-\theta} r_{\mathrm e}(T : A_1 \to A_1)^{\theta}.$$

Proof. According to a result of Nussbaum [13], Thm. 1, the essential spectral radius can be expressed in terms of the measure of non-compactness by the formula

$$r_{\mathrm e}(T) = \lim_{n\to\infty} \beta(T^n)^{1/n}.$$

Hence, using Theorem 1.2, we conclude that

$$\begin{aligned} &r_{\mathrm e}(T : (A_0,A_1)_{\theta,q} \to (A_0,A_1)_{\theta,q}) \\ &\quad \le \lim_{n\to\infty} c^{1/n} \beta(T^n : A_0 \to A_0)^{(1-\theta)/n} \beta(T^n : A_1 \to A_1)^{\theta/n} \\ &\quad = r_{\mathrm e}(T : A_0 \to A_0)^{1-\theta} r_{\mathrm e}(T : A_1 \to A_1)^{\theta}. \ \blacksquare \end{aligned}$$

Note that the inequality for the essential spectral radius holds without any additional constant, and it does not depend on the norm chosen in $(A_0, A_1)_{\theta,q}$.

In the case $1 \le q < \infty$, where $A_0 \cap A_1$ is dense in $(A_0,A_1)_{\theta,q}$, Corollary 1.3 was established by Albrecht in [1] using different techniques.

References

[1] E. Albrecht, *Spectral interpolation*, in: Oper. Theory Adv. Appl. 14, Birkhäuser, Basel, 1984.
[2] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin, 1976.
[3] B. Carl and I. Stephani, *Entropy, Compactness and the Approximation of Operators*, Cambridge Univ. Press, Cambridge, 1990.
[4] F. Cobos, D. E. Edmunds and A. J. B. Potter, *Real interpolation and compact linear operators*, J. Funct. Anal. 88 (1990), 351–365.

[5] F. Cobos, T. Kühn and T. Schonbek, *One-sided compactness results for Aronszajn–Gagliardo functors*, ibid. 106 (1992), 274–313.

[6] F. Cobos and J. Peetre, *Interpolation of compactness using Aronszajn–Gagliardo functors*, Israel J. Math. 68 (1989), 220–240.

[7] —, —, *Interpolation of compact operators: The multidimensional case*, Proc. London Math. Soc. 63 (1991), 371–400.

[8] F. Cobos and L. E. Persson, *Real interpolation of compact operators between quasi-Banach spaces*, Math. Scand. 82 (1998), 138–160.

[9] M. Cwikel, *Real and complex interpolation and extrapolation of compact operators*, Duke Math. J. 65 (1992), 333–343.

[10] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1987.

[11] M. A. Krasnosel'skiĭ, *On a theorem of M. Riesz*, Soviet Math. Dokl. 1 (1960), 229–231.

[12] M. A. Krasnosel'skiĭ, P. P. Zabreĭko, E. I. Pustyl'nik and P. E. Sobolevskiĭ, *Integral Operators in Spaces of Summable Functions*, Noordhoff, Leiden, 1976.

[13] R. D. Nussbaum, *The radius of the essential spectrum*, Duke Math. J. 37 (1970), 473–479.

[14] A. Pietsch, *Operator Ideals*, North-Holland, Amsterdam, 1980.

[15] M. F. Teixeira and D. E. Edmunds, *Interpolation theory and measures of non-compactness*, Math. Nachr. 104 (1981), 129–135.

[16] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense de Madrid
28040 Madrid, Spain
E-mail: cobos@eucmax.sim.ucm.es

Departamento de Matemática Aplicada
Universidad de Murcia
Campus de Espinardo
30071 Espinardo (Murcia), Spain
E-mail: pedrofdz@fcu.um.es

Departamento de Matemática Aplicada
Escuela Técnica Superior de Ingenieros Industriales
Universidad de Vigo
Lagoas-Marcosende
36200 Vigo, Spain
E-mail: antonmar@uvigo.es

Received July 14, 1997
Revised version November 23, 1998 (3924)

STUDIA MATHEMATICA 135 (1) (1999)

Tail and moment estimates for some types of chaos

by

RAFAŁ LATAŁA (Warszawa)

Abstract. Let X_i be a sequence of independent symmetric real random variables with logarithmically concave tails. We consider a variable $X = \sum_{i\neq j} a_{i,j} X_i X_j$, where $a_{i,j}$ are real numbers. We derive approximate formulas for the tails and moments of X and of its decoupled version, which are exact up to some universal constants.

Definitions and notation. Let X_i, X'_j be two independent sequences of independent symmetric random variables with logarithmically concave tails, i.e. the functions $N_i, N'_j : [0,\infty) \to [0,\infty]$ defined by

$$N_i(t) = -\ln P(|X_i| \ge t) \quad \text{and} \quad N'_j(t) = -\ln P(|X'_j| \ge t)$$

are convex. Since it is only a matter of normalization we may and will assume that for all i and j,

$$\inf\{t : N_i(t) \ge 1\} = \inf\{t : N'_j(t) \ge 1\} = 1. \tag{1}$$

Define the functions $\widehat{N}_i$ by

$$\widehat{N}_i(t) = \begin{cases} t^2 & \text{for } |t| \le 1, \\ N_i(|t|) & \text{for } |t| > 1. \end{cases}$$

For sequences (a_i) of real numbers and $p > 0$ we put

$$\|(a_i)\|_{\mathcal{N},p} = \sup\Big\{\sum a_i b_i : \sum \widehat{N}_i(b_i) \le p\Big\}$$

and

$$\|(a_i)\|_p = \Big(\sum a_i^p\Big)^{1/p}.$$

In a similar way we define $\widehat{N}'_j$ and $\|(a_j)\|_{\mathcal{N}',p}$.

For matrices $(a_{i,j})$ and $p > 0$ we define

$$\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} = \sup\Big\{\sum a_{i,j} b_i c_j : \sum \widehat{N}_i(b_i) \le p,\ \sum \widehat{N}'_j(c_j) \le p\Big\}.$$

1991 *Mathematics Subject Classification*: Primary 60E15.
Research partially supported by KBN Grant 2 P301 022 07.