[5] F. Cobos, T. Kühn and T. Schonbek, *One-sided compactness results for Aronszajn–Gagliardo functors*, ibid. 106 (1992), 274–313.
[6] F. Cobos and J. Peetre, *Interpolation of compactness using Aronszajn–Gagliardo functors*, Israel J. Math. 68 (1989), 220–240.
[7] —, —, *Interpolation of compact operators: The multidimensional case*, Proc. London Math. Soc. 63 (1991), 371–400.
[8] F. Cobos and L. E. Persson, *Real interpolation of compact operators between quasi-Banach spaces*, Math. Scand. 82 (1998), 138–160.
[9] M. Cwikel, *Real and complex interpolation and extrapolation of compact operators*, Duke Math. J. 65 (1992), 333–343.
[10] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1987.
[11] M. A. Krasnosel'skiĭ, *On a theorem of M. Riesz*, Soviet Math. Dokl. 1 (1960), 229–231.
[12] M. A. Krasnosel'skiĭ, P. P. Zabreĭko, E. I. Pustyl'nik and P. E. Sobolevskiĭ, *Integral Operators in Spaces of Summable Functions*, Noordhoff, Leiden, 1976.
[13] R. D. Nussbaum, *The radius of the essential spectrum*, Duke Math. J. 37 (1970), 473–479.
[14] A. Pietsch, *Operator Ideals*, North-Holland, Amsterdam, 1980.
[15] M. F. Teixeira and D. E. Edmunds, *Interpolation theory and measures of non-compactness*, Math. Nachr. 104 (1981), 129–135.
[16] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

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Tail and moment estimates for some types of chaos

by

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Abstract. Let X_i be a sequence of independent symmetric real random variables with logarithmically concave tails. We consider a variable $X=\sum_{i\neq j} a_{i,j}X_iX_j$, where $a_{i,j}$ are real numbers. We derive approximate formulas for the tails and moments of X and of its decoupled version, which are exact up to some universal constants.

Definitions and notation. Let X_i, X'_j be two independent sequences of independent symmetric random variables with logarithmically concave tails, i.e. the functions $N_i, N'_j : [0,\infty) \to [0,\infty]$ defined by

$$N_i(t) = -\ln P(|X_i| \ge t) \quad \text{and} \quad N'_j(t) = -\ln P(|X'_j| \ge t)$$

are convex. Since it is only a matter of normalization we may and will assume that for all i and j,

$$\inf\{t : N_i(t) \ge 1\} = \inf\{t : N'_j(t) \ge 1\} = 1. \tag{1}$$

Define the functions $\widehat{N}_i$ by

$$\widehat{N}_i(t) = \begin{cases} t^2 & \text{for } |t| \le 1, \\ N_i(|t|) & \text{for } |t| > 1. \end{cases}$$

For sequences (a_i) of real numbers and $p > 0$ we put

$$\|(a_i)\|_{\mathcal{N},p} = \sup\Big\{\sum a_i b_i : \sum \widehat{N}_i(b_i) \le p\Big\}$$

and

$$\|(a_i)\|_p = \Big(\sum a_i^p\Big)^{1/p}.$$

In a similar way we define $\widehat{N}'_j$ and $\|(a_j)\|_{\mathcal{N}',p}$.

For matrices $(a_{i,j})$ and $p > 0$ we define

$$\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} = \sup\Big\{\sum a_{i,j} b_i c_j : \sum \widehat{N}_i(b_i) \le p,\ \sum \widehat{N}'_j(c_j) \le p\Big\}.$$

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We denote by (ε_i) the Bernoulli sequence, i.e. a sequence of i.i.d. symmetric r.v.'s taking values ± 1. A sequence of independent standard $\mathcal{N}(0,1)$ Gaussian random variables is denoted by (g_i) and the canonical Gaussian measure on $\mathbb{R}^n$ by γ_n.

For a random variable X and $p>0$ we write

$$\|X\|_p = (E|X|^p)^{1/p}.$$

We also use the notation $a \sim_C b$ to mean that $C^{-1}a \le b \le Ca$.

In this paper we prove the following theorem:

THEOREM 1. *Let $(a_{i,j})$ be a square summable matrix and $X = \sum a_{i,j} X_i X_j'$. Then for each $p \ge 1$,*

$$\|X\|_p \sim_C \|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} + \|(A_i)\|_{\mathcal{N},p} + \|(B_j)\|_{\mathcal{N}',p},$$

where $A_i = (\sum_j a_{i,j}^2)^{1/2}$, $B_j = (\sum_i a_{i,j}^2)^{1/2}$ and C is a universal constant.

We postpone the proof till the end of this article and now present some corollaries and examples.

COROLLARY 1. *Let $(a_{i,j})$ be a square summable matrix such that $a_{i,i}=0$ and $a_{i,j}=a_{j,i}$ for all i,j. Then for each $p \ge 1$,*

$$\Big\|\sum a_{i,j} X_i X_j\Big\|_p \sim_{\widetilde{C}} \|(a_{i,j})\|_{\mathcal{N},\mathcal{N},p} + \|(A_i)\|_{\mathcal{N},p},$$

where $A_i = (\sum_j a_{i,j}^2)^{1/2}$ and $\widetilde{C}$ is a universal constant.

Proof. Let X_i' be an independent copy of X_i. Then by the result of de la Peña and Montgomery-Smith (cf. [6]) about decoupling chaos we have, for $p \ge 1$,

$$\Big\|\sum a_{i,j} X_i X_j\Big\|_p \sim_K \Big\|\sum a_{i,j} X_i X_j'\Big\|_p$$

with some universal constant K. Hence Corollary 1 is an immediate consequence of Theorem 1 if we notice that $A_i = B_i$ by the symmetry of the matrix $(a_{i,j})$.

COROLLARY 2. *There exist universal constants $0 < c < C < \infty$ such that under the assumptions of Corollary 1, for each $t \ge 1$,*

$$P\Big(\Big|\sum a_{i,j} X_i X_j\Big| \ge C(\|(a_{i,j})\|_{\mathcal{N},\mathcal{N},t} + \|(A_i)\|_{\mathcal{N},t})\Big) \le e^{-t}$$

and

$$P\Big(\Big|\sum a_{i,j} X_i X_j\Big| \ge c(\|(a_{i,j})\|_{\mathcal{N},\mathcal{N},t} + \|(A_i)\|_{\mathcal{N},t})\Big) \ge \min(c, e^{-t}).$$

Proof. The first inequality follows from Corollary 1 and Chebyshev's inequality. To get the second inequality we first use Corollary 1 and Proposition 1 below to get

$$\Big\|\sum a_{i,j} X_i X_j\Big\|_{2p} \le 4\widetilde{C}^2 \Big\|\sum a_{i,j} X_i X_j\Big\|_p \quad \text{for } p \ge 1.$$

The inequality may now be obtained by Corollary 1, Proposition 2 and the Paley–Zygmund inequality as in [2].

By simple calculations we may easily derive from Corollary 1 the following two examples of interest.

EXAMPLE 1. If a matrix $(a_{i,j})$ satisfies the assumptions of Corollary 1 then for some universal constant K and any $p \ge 1$ we have

$$\Big\|\sum a_{i,j} g_i g_j\Big\|_p \sim_K p\|(a_{i,j})\|_{l_2 \to l_2} + \sqrt{p}\,\|(a_{i,j})\|_{\mathrm{HS}},$$

where

$$\|(a_{i,j})\|_{l_2\to l_2} = \sup\Big\{\sum a_{i,j} b_i c_j : \|(b_i)\|_2, \|(c_j)\|_2 \le 1\Big\}$$

and

$$\|(a_{i,j})\|_{\mathrm{HS}} = \Big(\sum a_{i,j}^2\Big)^{1/2}.$$

EXAMPLE 2. Under the assumptions of Corollary 1 we have

$$\Big\|\sum a_{i,j}\varepsilon_i\varepsilon_j\Big\|_p \sim_K \sup\Big\{\sum a_{i,j} b_i c_j : \|(b_i)\|_2, \|(c_j)\|_2 \le \sqrt{p},\ |b_i|, |c_j| \le 1\Big\} + \sum_{i \le p} A_i^* + \sqrt{p}\Big(\sum_{i>p} (A_i^*)^2\Big)^{1/2},$$

where A_i^* denotes the nondecreasing rearrangement of the sequence A_i and K is a universal constant.

REMARK. Example 1 may also be derived in a simpler way. Using the invariance of Gaussian r.v.'s under orthogonal transformations, it is enough to prove that for any sequence (d_i) of real numbers we have

$$\Big\|\sum d_i g_i g_i'\Big\|_p \sim_{K_1} p\|(d_i)\|_\infty + \sqrt{p}\,\|(d_i)\|_2.$$

This easily follows from the results of [2] (see Theorem 2 below).

The following theorem was established in a slightly less general setting by Gluskin and Kwapień in [2] and in full generality in [4].

THEOREM 2. *There exists a universal constant $C_1 < \infty$ such that for any square summable sequence (a_i) and $p \ge 1$ we have*

$$\Big\|\sum a_i X_i\Big\|_p \sim_{C_1} \|(a_i)\|_{\mathcal{N},p}. \tag{2}$$

In particular, for any $p,q \geq 1$ there exists a constant $C_{p,q}$, which depends only on p and q, such that

$$\Big\|\sum a_i X_i\Big\|_p \leq C_{p,q}\Big\|\sum a_i X_i\Big\|_q. \tag{3}$$

REMARK. The inequality (3) may also be obtained by hypercontractive methods or direct calculations.

We will also use the following theorem of M. Talagrand (see [8] and [5] for a simpler proof with better constants).

THEOREM 3. *Let λ be the measure on $\mathbb{R}$ with density $\frac{1}{2}e^{-|x|}$ and let λ^n be the product measure $\bigotimes_{i=1}^n \lambda$ on $\mathbb{R}^n$. Then for any Borel subset A of $\mathbb{R}^n$ with $\lambda^n(A) > 0$ and any $s > 0$ we have*

$$\lambda^n(A + V_s) \geq 1 - \lambda^n(A)^{-1}e^{-s},$$

where

$$V_s = \Big\{x \in \mathbb{R}^n : \sum \min(|x_i|, x_i^2) \leq 36s\Big\}.$$

In the next part of the paper we need some additional definitions. We say that a measure μ on $\mathbb{R}$ is *symmetric unimodal* if it has a density with respect to the Lebesgue measure which is symmetric and nonincreasing on $[0,\infty)$. A nonnegative Borel measure μ on $\mathbb{R}^n$ will be called *logconcave* if

$$\mu(tA + (1-t)B) \geq \mu^t(A)\mu^{1-t}(B)$$

for any nonempty Borel sets A, B in $\mathbb{R}^n$ and $t \in (0,1)$. A real random variable is called *symmetric unimodal* (resp. *logconcave*) if its distribution is symmetric unimodal (resp. logconcave).

By results of Borell [1] products of logconcave measures are logconcave and nondegenerate measures on $\mathbb{R}$ are logconcave if and only if they have logconcave densities with respect to the Lebesgue measure. In particular, any symmetric nondegenerate logconcave real r.v. has logconcave tails and is symmetric unimodal.

PROPOSITION 1. *The following inequalities are satisfied:*

$$\|(a_i)\|_{\mathcal{N},\lambda p} \leq \lambda\|(a_i)\|_{\mathcal{N},p} \quad \text{for } \lambda \geq 1,\ p > 0, \tag{4}$$

$$\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',\lambda p} \leq \lambda^2\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} \quad \text{for } \lambda \geq 1,\ p > 0, \tag{5}$$

and

$$\sqrt{p}\Big(\sum_{i>p}(a_i^*)^2\Big)^{1/2} \leq \|(a_i)\|_{\mathcal{N},p} \leq pa_1^* + \sqrt{p}\Big(\sum_i (a_i^*)^2\Big)^{1/2}, \tag{6}$$

where (a_i^*) is a nonincreasing rearrangement of the sequence $(|a_i|)$.

Proof. Inequalities (4) and (5) follow easily from the observation that $\widehat{N}_i(tx) \leq t\widehat{N}_i(x)$ for any $t \in [0,1]$ and real number x. To prove (6) fix a sequence (b_i) such that $\sum_i \widehat{N}_i(b_i) \leq p$ and let $J = \{i : b_i \geq 1\}$. Then since $\widehat{N}_i(x) \geq x$ for $x \geq 1$ we have $\sum_{i\in J} a_i b_i \leq pa_1^*$, and since $\widehat{N}_i(x) = x^2$ for $|x| \leq 1$ we get $\sum_{i\notin J} a_i b_i \leq \sqrt{p}(\sum_i (a_i^*)^2)^{1/2}$.

To prove the other inequality in (6) let $k = \lfloor p\rfloor + 1$, $A = (k(a_k^*)^2 + \sum_{i>k}(a_i^*)^2)^{1/2}$, $b_i = \operatorname{sgn}(a_i)\sqrt{p}\,a_k^*/A$ for $|a_i| \geq a_k^*$ and $b_i = \sqrt{p}\,a_i/A$ for $|a_i| \leq a_k^*$. Then $|b_i| \leq 1$, $\sum \widehat{N}_i(b_i) = \sum b_i^2 = p$ and

$$\sum a_i b_i \geq \sqrt{p}A \geq \sqrt{p}\Big(\sum_{i>p}(a_i^*)^2\Big)^{1/2}.$$

PROPOSITION 2. *For any random variable X_i with logconcave tails normalized as in (1) we have*

$$1/2 < 1 - e^{-1} \leq E|X_i| \leq 1 \tag{7}$$

and

$$1/2 < 2 - 4e^{-1} \leq E|X_i|^2 \leq 2. \tag{8}$$

Proof. By our normalization property (1) and the convexity of N_i we get $0 \leq N_i(t) \leq t$ for $t \in [0,1]$ and $N_i(t) \geq \max(0, k(t-1)+1)$ for some $k \geq 1$ and all $t \geq 0$. The assertion easily follows by integration by parts.

LEMMA 1. *Let $\mu_1,\ldots,\mu_n$ and $\nu_1,\ldots,\nu_n$ be symmetric logconcave probability measures on $\mathbb{R}$ such that*

$$\forall_i\ \forall_{t>0}\ \mu_i([-t,t]) \leq \nu_i([-t,t]), \tag{9}$$

$\mu = \mu_1 \otimes \ldots \otimes \mu_n$ and $\nu = \nu_1 \otimes \ldots \otimes \nu_n$. Then for any convex symmetric Borel set K in $\mathbb{R}^n$ we have

$$\mu(K) \leq \nu(K).$$

Proof. It is enough to prove that for any symmetric logconcave measure on $\mathbb{R}^{n-1}$ and convex symmetric set K we have

$$\mu_1 \otimes \mu(K) \leq \nu_1 \otimes \mu(K).$$

Let $K_t = \{x \in \mathbb{R}^{n-1} : (t,x) \in K\}$ and $f(t) = \mu(K_t)$ for $t \in \mathbb{R}$. By the convexity of K we have, for any $\lambda \in (0,1)$ and $s,t \in \mathbb{R}$ such that $K_t, K_s \neq \emptyset$,

$$\lambda K_t + (1-\lambda)K_s \subset K_{\lambda t + (1-\lambda)s}.$$

Therefore f is logconcave on $\mathbb{R}$ and since it is also symmetric, it is nonincreasing on $[0,\infty)$. Hence approximating f by $\sum a_j I_{[-t_j,t_j]}$ and using (9) we obtain

$$\mu_1 \otimes \mu(K) = \int_{\mathbb{R}} f(t)\,d\mu_1(t) \leq \int_{\mathbb{R}} f(t)\,d\nu_1(t) = \nu_1 \otimes \mu(K).$$

LEMMA 2. *For all $t>0$ we have*

$$\gamma_1([-t,t]) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-x^2/2}\,dx \ge e^{-2/t^2}.$$

Proof. Since for any $x>0$,

$$e^{-x} + e^{-1/x} \le \frac{1}{1+x} + \frac{1}{1+x^{-1}} = 1,$$

the conclusion follows from the well-known (and easy to check) estimate $\gamma_1([-t,t]) \ge 1 - e^{-t^2/2}$.

LEMMA 3. *For any matrix $(a_{i,j})$ and $C \ge 2\sum_{i,j} a_{i,j}^2$ we have*

$$\gamma_n\Big(\max_j \Big|\sum_{i=1}^n a_{i,j}x_i\Big| \le 1, \sum_j \Big|\sum_{i=1}^n a_{i,j}x_i\Big|^2 \le C\Big) \ge \frac{1}{2}\exp\Big(-2\sum_{i,j} a_{i,j}^2\Big).$$

Proof. From a result of Khatri [3] and Sidak [7] we have

$$(10)\quad \gamma_n\Big(\max_j \Big|\sum_{i=1}^n a_{i,j}x_i\Big| \le 1, \sum_j \Big|\sum_{i=1}^n a_{i,j}x_i\Big|^2 \le C\Big)$$
$$\ge \Big(\prod_j \gamma_n\Big(\Big|\sum_{i=1}^n a_{i,j}x_i\Big| \le 1\Big)\Big)\gamma_n\Big(\sum_j \Big|\sum_{i=1}^n a_{i,j}x_i\Big|^2 \le C\Big).$$

By Lemma 2 we have

$$(11)\quad \gamma_n\Big(\Big|\sum_{i=1}^n a_{i,j}x_i\Big| \le 1\Big) = \gamma_1\Big(\Big[-\Big(\sum_{i=1}^n a_{i,j}^2\Big)^{-1/2}, \Big(\sum_{i=1}^n a_{i,j}^2\Big)^{-1/2}\Big]\Big)$$
$$\ge \exp\Big(-2\sum_{i=1}^n a_{i,j}^2\Big).$$

Since $E\sum_j |\sum_{i=1}^n a_{i,j}g_i|^2 = \sum_{i,j} a_{i,j}^2$, from Chebyshev's inequality we obtain

$$(12)\quad \gamma_n\Big(\sum_j \Big|\sum_{i=1}^n a_{i,j}x_i\Big|^2 \le C\Big) = 1 - P\Big(\sum_j \Big|\sum_{i=1}^n a_{i,j}g_i\Big|^2 > C\Big) \ge \frac{1}{2}.$$

The conclusion follows from (10)–(12).

LEMMA 4. *Let $Y_1,\ldots,Y_n$ be symmetric unimodal real r.v.'s and $d_i = EY_i^2$. Then for any matrix $(b_{i,j})$ we have*

$$P\Big(\max_j \Big|\sum_{i=1}^n b_{i,j}Y_i\Big| \le 1, \sum_j \Big|\sum_{i=1}^n b_{i,j}Y_i\Big|^2 \le 1 + 4\sum d_i b_{i,j}^2\Big)$$
$$\ge \frac{1}{4}\exp\Big(-4\sum_{i,j} d_i b_{i,j}^2\Big).$$

Proof. Let Y_i have the distribution μ_i with density f_i and $\mu = \bigotimes_{i=1}^n \mu_i$. Since f_i are nonnegative, symmetric, nonincreasing on $[0,\infty)$ and μ_i are probability measures, there exist probability measures $m_1,\ldots,m_n$ on $\mathbb{R}$ such that for each i,

$$f_i(x) = \int_0^\infty \frac{1}{2t} I_{[-t,t]}(x)\,dm_i(t).$$

We also have

$$\int_0^\infty t^2\,dm_i(t) = 3\int_0^\infty \int_{\mathbb{R}} x^2 \frac{1}{2t} I_{[-t,t]}(x)\,dx\,dm_i(t) = 3\int_{\mathbb{R}} x^2 f_i(x)\,dx = 3d_i.$$

For any Borel set A in $\mathbb{R}^n$ we have

$$(13)\qquad \mu(A) = \int_{\mathbb{R}^n} \nu^n_{t_1,\ldots,t_n}(A)\,dm_1(t_1)\ldots dm_n(t_n),$$

where $\nu^n_{t_1,\ldots,t_n}$ denotes the uniform probability measure on $[-t_1,t_1]\times\ldots\times[-t_n,t_n]$. We will also write ν^n_t instead of $\nu^n_{t,\ldots,t}$.

From Lemma 1 it immediately follows that for any convex symmetric set K in $\mathbb{R}^n$, $\nu^n_{\sqrt{\pi/2}}(K) \ge \gamma_n(K)$. Hence by Lemma 3,

$$(14)\quad \nu^n_{t_1,\ldots,t_n}\Big(x\in\mathbb{R}^n : \max_j \Big|\sum_{i=1}^n b_{i,j}x_i\Big| \le 1,\ \sum_j \Big|\sum_{i=1}^n b_{i,j}x_i\Big|^2 \le C\Big)$$
$$= \nu^n_{\sqrt{\pi/2}}\Big(x\in\mathbb{R}^n : \max_j \Big|\sum_{i=1}^n b_{i,j}t_i\sqrt{\frac{2}{\pi}}x_i\Big| \le 1, \sum_j \Big|\sum_{i=1}^n b_{i,j}t_i\sqrt{\frac{2}{\pi}}x_i\Big|^2 \le C\Big)$$
$$\ge \frac{1}{2}\exp\Big(-\frac{4}{\pi}\sum_{i,j} t_i^2 b_{i,j}^2\Big) I_{\{(4/\pi)\sum_{i,j} t_i^2 b_{i,j}^2 \le C\}}.$$

Since the function e^{-x} is convex we obtain

$$\int_{\mathbb{R}^n} \exp\Big(-\frac{4}{\pi}\sum_{i,j} t_i^2 b_{i,j}^2\Big)\,dm_1(t_1)\ldots dm_n(t_n)$$
$$\ge \exp\Big(-\int_{\mathbb{R}^n} \frac{4}{\pi}\sum_{i,j} t_i^2 b_{i,j}^2\,dm_1(t_1)\ldots dm_n(t_n)\Big) = \exp\Big(-\frac{12}{\pi}\sum_{i,j} d_i b_{i,j}^2\Big).$$

Using the above and the obvious estimate

$$\int_{\mathbb{R}^n} \exp\Big(-\frac{4}{\pi}\sum_{i,j} t_i^2 b_{i,j}^2\Big) I_{\{(4/\pi)\sum_{i,j} t_i^2 b_{i,j}^2 > C\}}\,dm_1(t_1)\ldots dm_n(t_n) \le e^{-C}$$

we obtain, by (13) and (14),

$$\mu\Big(x\in\mathbb{R}^n : \max_j \Big|\sum_{i=1}^n b_{i,j}x_i\Big| \le 1, \sum_j \Big|\sum_{i=1}^n b_{i,j}x_i\Big|^2 \le 1+4\sum d_i b_{i,j}^2\Big)$$
$$\ge \frac12 \exp\Big(-4\sum_{i,j} d_i b_{i,j}^2\Big) - \frac12 \exp\Big(-1-4\sum_{i,j} d_i b_{i,j}^2\Big) \ge \frac14 \exp\Big(-4\sum_{i,j} d_i b_{i,j}^2\Big).$$

LEMMA 5. *Let $Y_1,\ldots,Y_n$ be symmetric unimodal r.v.'s such that $EY_i^2 \le 4$. Then for any $p>0$,*

$$P\Big(\Big\|\Big(\sum_{i=1}^n a_{i,j}Y_i\Big)\Big\|_{\mathcal{N}',p} \le 10\|(B_j)\|_{\mathcal{N}',p}\Big) \ge \frac14 e^{-8p}.$$

Proof. For $p\le 1$ we have $\|(a_j)\|_{\mathcal{N}',p} = \sqrt{p}\,\|(a_j)\|_2$ and the assertion follows easily from Chebyshev's inequality. So we assume that $p>1$. Without loss of generality we may also assume that $B_1 \ge B_2 \ge \ldots$ and $\|(B_j)\|_{\mathcal{N}',p} = p$. Let

$$b_{i,j} = \begin{cases} a_{i,j}/B_j & \text{for } j\le p,\\ a_{i,j} & \text{for } j>p\end{cases}$$

and $d_i = EY_i^2/4$. Then by (6) we get

$$\sum_{i,j} d_i b_{i,j}^2 \le \sum_{i,j} b_{i,j}^2 = \lfloor p\rfloor + \sum_{j>p} B_j^2 \le p + p^{-1}\|(B_j)\|_{\mathcal{N}',p}^2 \le 2p.$$

Moreover, if $\max_j |\sum_i b_{i,j}y_i| \le 1$ and $\sum_j |\sum_i b_{i,j}y_i|^2 \le 1+4\sum_{i,j} d_i b_{i,j}^2$ then by (6),

$$\Big\|\Big(\sum_{i=1}^n a_{i,j}y_i\Big)\Big\|_{\mathcal{N}',p} \le \Big\|\Big(\sum_{i=1}^n a_{i,j}y_i\Big)_{j\le p}\Big\|_{\mathcal{N}',p}$$
$$+ p\max_{j>p}\Big|\sum_i a_{i,j}y_i\Big| + \sqrt{p}\Big(\sum_{j>p}\Big|\sum_i a_{i,j}y_i\Big|^2\Big)^{1/2}$$
$$\le \|(B_j)\|_{\mathcal{N}',p} + p + \sqrt{p(8p+1)} \le 5p.$$

Hence by Lemma 4,

$$P\Big(\Big\|\Big(\sum_{i=1}^n \frac{a_{i,j}Y_i}{2}\Big)\Big\|_{\mathcal{N}',p} \le 5\|(B_j)\|_{\mathcal{N}',p}\Big) \ge \frac14 \exp\Big(-4\sum_{i,j} d_i b_{i,j}^2\Big) \ge \frac14 e^{-8p}.$$

COROLLARY 3. *There exists $C_3<\infty$ such that for any matrix $(a_{i,j})$ and $p>0$,*

$$E\Big\|\Big(\sum_{i=1}^n a_{i,j}X_i\Big)\Big\|_{\mathcal{N}',p} \le C_3(\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} + \|(B_j)\|_{\mathcal{N}',p}).$$

Proof. For $p<1$ the statement follows easily by (8), so we asssume that $p\ge 1$. Suppose first that all X_i's are also unimodal. Then $P(|X_i|<t) \ge tP(|X_i|\le 1) = t(1-e^{-1})$ for $t\in[0,1]$. So for all $t>0$,

$$N_i(t) = -\ln P(|X_i|\ge t) \ge (1-e^{-1})t \ge t/2. \tag{15}$$

Let F_i be an odd function whose restriction to $\mathbb{R}^+$ is the inverse of N_i. Then X_i has distribution $F_i(\lambda)$, where λ is the same symmetric exponential measure as in Theorem 3. Let

$$A = \Big\{x\in\mathbb{R}^n : \Big\|\Big(\sum_{i=1}^n a_{i,j}F_i(x_i)\Big)\Big\|_{\mathcal{N}',p} \le 10\|(B_j)\|_{\mathcal{N}',p}\Big\}.$$

Then by Lemma 5 and (8),

$$\lambda^n(A) = P\Big(\Big\|\Big(\sum_{i=1}^n a_{i,j}X_i\Big)\Big\|_{\mathcal{N}',p} \le 10\|(B_j)\|_{\mathcal{N}',p}\Big) \ge \frac14 e^{-8p}.$$

Let $s>0$, $x=y+z$ with $y\in A$ and $z\in V_s$. Let $\Delta_i = F_i(x_i)-F_i(y_i)$; then by (15) and since N_i is convex we get $|\Delta_i| \le 2\min(F_i(|z_i|),|z_i|)$. So for $36s\ge p$ we have

$$\Big\|\Big(\sum_{i=1}^n a_{i,j}\Delta_i\Big)\Big\|_{\mathcal{N}',p} \le 2\sup\Big\{\Big\|\Big(\sum_{i=1}^n a_{i,j}b_i\Big)\Big\|_{\mathcal{N}',p} : \sum \widehat{N}_i(b_i) \le 36s\Big\}$$
$$\le \frac{72s}{p}\sup\Big\{\Big\|\Big(\sum_{i=1}^n a_{i,j}b_i\Big)\Big\|_{\mathcal{N}',p} : \sum \widehat{N}_i(b_i) \le p\Big\}$$
$$= \frac{72s}{p}\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p}.$$

Hence for $36s\ge p$,

$$\Big\|\Big(\sum_{i=1}^n a_{i,j}F_i(x_i)\Big)\Big\|_{\mathcal{N}',p} \le 10\|(B_j)\|_{\mathcal{N}',p} + \frac{72s}{p}\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p}.$$

So by Theorem 3,

$$P\Big(\Big\|\Big(\sum_{i=1}^n a_{i,j}X_i\Big)\Big\|_{\mathcal{N}',p} > 10\|(B_j)\|_{\mathcal{N}',p} + \frac{72s}{p}\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p}\Big)$$
$$\le 1-\lambda^n(A+V_s) \le 4e^{8p-s}.$$

Integrating by parts we therefore get, for any $s_0 \ge p/36$,

$$E\Big\|\Big(\sum_{i=1}^n a_{i,j}X_i\Big)\Big\|_{\mathcal{N}',p}$$
$$\le 10\|(B_j)\|_{\mathcal{N}',p} + \|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p}\Big(\frac{72s_0}{p} + \frac{288}{p}\int_{s_0}^{\infty} e^{8p-x}\,dx\Big).$$

Choosing $s_0 = 9p$ we get

$$E\Big\|\Big(\sum_{i=1}^n a_{i,j}X_i\Big)\Big\|_{\mathcal{N}',p} \le \widetilde{C}_3(\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} + \|(B_j)\|_{\mathcal{N}',p}).$$

Now let X_i be arbitrary r.v.'s with logconcave tails and the normalization property (1). Let $\widetilde{X}_i$ be a r.v. with density $c_i e^{-N_i(|x|)}$, where $c_i = (\int_{\mathbb{R}} e^{-N_i(|x|)}\,dx)^{-1}$. Let

$$\alpha_i = \inf\{t>0 : P(|t\widetilde{X}_i| \ge 1) \ge e^{-1}\}, \qquad Y_i = \alpha_i X_i,$$
$$\widetilde{N}_i(t) = -\ln P(|Y_i| \ge t) = -\ln P(|\alpha_i \widetilde{X}_i| \ge t)$$

and

$$M_i(t) = \begin{cases} \widetilde{N}_i(|t|) & \text{for } |t|>1,\\ t^2 & \text{for } |t| \le 1.\end{cases}$$

The functions $\widetilde{N}_i$ are convex and have the normalization property (1), and the variables Y_i are unimodal, so by the first part of this proof,

$$E\Big\|\Big(\sum_{i=1}^n a_{i,j}Y_i\Big)\Big\|_{\mathcal{N}',p} \le \widetilde{C}_3(\|(a_{i,j})\|_{\widetilde{\mathcal{N}},\mathcal{N}',p} + \|(B_j)\|_{\mathcal{N}',p}), \tag{16}$$

where

$$\|(a_{i,j})\|_{\widetilde{\mathcal{N}},\mathcal{N}',p} = \sup\Big\{\sum a_{i,j}b_ic_j : \sum M_i(b_i) \le p,\ \sum \widehat{N}_j'(c_j) \le p\Big\}.$$

Notice that

$$c_i^{-1} \le 2\Big(1 + \int_1^\infty e^{-x}\,dx\Big) < 3 \quad\text{and}\quad c_i^{-1} \ge 2\int_0^1 e^{-x}\,dx > 1.$$

Hence

$$P(|\widetilde{X}_i| \ge t) = 2c_i\int_t^\infty e^{-N_i(x)}\,dx \le 2\int_t^\infty e^{-xN_i(t)/t}\,dx = \frac{2t}{N_i(t)}e^{-N_i(t)},$$

so for $t \ge 2$ we obtain

$$P(|\widetilde{X}_i| \ge t) \le 2e^{-N_i(t)} \le 2e^{-2N_i(t/2)} \le e^{-N_i(t/2)}. \tag{17}$$

Also,

$$P(|\widetilde{X}_i| \ge t) \ge \frac{2}{3}\int_t^\infty e^{-N_i(x)}\,dx \ge te^{-N_i(5t/2)}. \tag{18}$$

From (17) and the normalization property (1) we get $\alpha_i \ge 1/2$. Since $P(|\widetilde{X}_i| \le t) \le 2c_it \le 2t$ we obtain $\alpha_i \le 4$. Therefore, by (18), for $t \ge 5/2$ we get

$$P(|X_i| \ge t) \le P(|5\widetilde{X}_i/2| \ge t) \le P(|5\alpha_i\widetilde{X}_i| \ge t) = P(|5Y_i| \ge t).$$

So by the contraction principle,

$$E\Big\|\Big(\sum_{i=1}^n a_{i,j}X_iI_{\{|X_i|\ge 5/2\}}\Big)\Big\|_{\mathcal{N}',p} \le 5E\Big\|\Big(\sum_{i=1}^n a_{i,j}Y_i\Big)\Big\|_{\mathcal{N}',p}.$$

Also by the contraction principle, since $E|Y_i| \ge 1/2$ by (7), we have

$$\begin{aligned}E\Big\|\Big(\sum_{i=1}^n a_{i,j}X_iI_{\{|X_i|\le 5/2\}}\Big)\Big\|_{\mathcal{N}',p} &\le \frac{5}{2}E\Big\|\Big(\sum_{i=1}^n a_{i,j}\varepsilon_i\Big)\Big\|_{\mathcal{N}',p}\\ &\le 5E\Big\|\Big(\sum_{i=1}^n a_{i,j}Y_i\Big)\Big\|_{\mathcal{N}',p}.\end{aligned}$$

Therefore

$$E\Big\|\Big(\sum_{i=1}^n a_{i,j}X_i\Big)\Big\|_{\mathcal{N}',p} \le 10E\Big\|\Big(\sum_{i=1}^n a_{i,j}Y_i\Big)\Big\|_{\mathcal{N}',p}. \tag{19}$$

Let $t \ge 8$; then by (17) we have

$$P(|Y_i| \ge t) = P(|\alpha_i\widetilde{X}_i| \ge t) \le P(|\widetilde{X}_i| \ge t/4) \le e^{-N_i(t/8)}.$$

Hence $\widetilde{N}_i(t) \ge N_i(t/8)$ for $t \ge 8$, so $M_i(t) \ge \widehat{N}_i(t/8)$ for all t and

$$\|(a_{i,j})\|_{\widetilde{\mathcal{N}},\mathcal{N}',p} \le 8\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p}. \tag{20}$$

Inequalities (16), (19) and (20) complete the proof.

Proof of Theorem 1. First we estimate $\|X\|_p$ from below. By the Jensen inequality and symmetry of X_i we have

$$\|X\|_p \ge \Big\|\sum_i X_iE\Big|\sum_j a_{i,j}X_j'\Big|\Big\|_p.$$

But by (3) and (8) we get

$$E\Big|\sum_j a_{i,j}X_j'\Big| \ge C_{2,1}^{-1}\Big\|\sum_j a_{i,j}X_j'\Big\|_2 \ge C_{2,1}^{-1}\Big(\sum_j \frac{1}{2}a_{i,j}^2\Big)^{1/2} = \widetilde{C}^{-1}A_i.$$

Hence by Theorem 2,

$$\|X\|_p \ge \widetilde{C}^{-1}\Big\|\sum_i A_iX_i\Big\|_p \ge (\widetilde{C}C_1)^{-1}\|(A_i)\|_{\mathcal{N},p}. \tag{21}$$

In a similar way we prove that

$$\|X\|_p \ge (\widetilde{C}C_1)^{-1}\|(B_j)\|_{\mathcal{N}',p}. \tag{22}$$

By Theorem 2 we also find that for any (c_j) with $\sum_j \widehat{N}_j'(c_j) \le p$ we have

$$\|X\|_p \ge C_1^{-1}\Big(E\Big|\sum_{i,j} a_{i,j}c_jX_i\Big|^p\Big)^{1/p} \ge C_1^{-2}\Big\|\Big(\sum_j a_{i,j}c_j\Big)\Big\|_{\mathcal{N},p}.$$

Taking the supremum over all such sequences (c_j) we get

$$\|X\|_p \geq C_1^{-2}\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p}. \tag{23}$$

Now, (21)–(23) complete the proof of this part of Theorem 1.

To prove the estimate from above we first notice that $X_i = Y_i + Z_i$ for some symmetric random variables Y_i and Z_i such that

$$P(|Y_i| \geq t) = e^{-\widetilde{N}_i(t)}, \quad \text{where} \quad \widetilde{N}_i(t) = \begin{cases} t & \text{for } t \leq 1, \\ N_i(t) & \text{for } t > 1, \end{cases}$$

and $|Z_i| \leq 1$ a.e.; we also assume that the Y_i are independent and so are the Z_i. In the same way we split $X_i' = Y_i' + Z_i'$. By the contraction principle and since $E|Y_i|, E|Y_i'| \geq 1/2$ by (7), we have

$$\Big\|\sum a_{i,j}Z_iY_j'\Big\|_p \leq \Big\|\sum a_{i,j}\varepsilon_iY_j'\Big\|_p \leq 2\Big\|\sum a_{i,j}Y_iY_j'\Big\|_p$$

and

$$\Big\|\sum a_{i,j}Z_iZ_j'\Big\|_p \leq \Big\|\sum a_{i,j}\varepsilon_iZ_j'\Big\|_p \leq 2\Big\|\sum a_{i,j}Y_iZ_j'\Big\|_p \leq 4\Big\|\sum a_{i,j}Y_iY_j'\Big\|_p.$$

Hence

$$\begin{aligned}\|X\|_p &\leq \Big\|\sum a_{i,j}Z_iZ_j'\Big\|_p + \Big\|\sum a_{i,j}Z_iY_j'\Big\|_p \\ &\quad + \Big\|\sum a_{i,j}Y_iZ_j'\Big\|_p + \Big\|\sum a_{i,j}Y_iY_j'\Big\|_p \\ &\leq 9\Big\|\sum a_{i,j}Y_iY_j'\Big\|_p.\end{aligned}$$

So it is enough to prove that for $p \geq 1$,

$$\Big\|\sum a_{i,j}Y_iY_j'\Big\|_p \leq C(\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} + \|(A_i)\|_{\mathcal{N},p} + \|(B_j)\|_{\mathcal{N}',p}).$$

To simplify the notation let

$$m_p = \|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} + \|(A_i)\|_{\mathcal{N},p} + \|(B_j)\|_{\mathcal{N}',p}.$$

Then $m_p \geq \|(A_i)\|_{\mathcal{N},1} + \|(B_j)\|_{\mathcal{N}',1} = 2(\sum_{i,j} a_{i,j}^2)^{1/2}$. Since $EY_i^2, E(Y_j')^2 \leq 2$ by (8), we get

$$\begin{aligned}P\Big(\Big|\sum a_{i,j}Y_iY_j'\Big| \geq 2m_p\Big) \\ \leq P\Big(\Big|\sum a_{i,j}Y_iY_j'\Big|^2 \geq 4E\Big|\sum a_{i,j}Y_iY_j'\Big|^2\Big) \leq \frac{1}{4}.\end{aligned} \tag{24}$$

From Corollary 3 we have

$$P\Big(\Big\|\Big(\sum_{i=1}^n a_{i,j}Y_i\Big)\Big\|_{\mathcal{N}',p} \geq 4C_3m_p\Big) \leq \frac{1}{4} \tag{25}$$

and

$$P\Big(\Big\|\Big(\sum_{j=1}^n a_{i,j}Y_j'\Big)\Big\|_{\mathcal{N},p} \geq 4C_3m_p\Big) \leq \frac{1}{4}. \tag{26}$$

Let $F_i, F_j' : \mathbb{R} \to \mathbb{R}$ be odd functions whose restrictions to $\mathbb{R}^+$ are the inverses of $\widetilde{N}_i$, $\widetilde{N}_j'$ respectively. Then Y_i, Y_j' have distributions $F_i(\lambda)$ and $F_j'(\lambda)$, where λ is the same symmetric exponential measure as in Theorem 3. Let

$$\begin{aligned}A = \Big\{(x,x') \in \mathbb{R}^{2n} : \Big|\sum a_{i,j}F_i(x_i)F_j'(x_j')\Big| \leq 2m_p, \\ \Big\|\Big(\sum_{i=1}^n a_{i,j}F_i(x_i)\Big)\Big\|_{\mathcal{N}',p}, \Big\|\Big(\sum_{j=1}^n a_{i,j}F_j'(x_j')\Big)\Big\|_{\mathcal{N},p} \leq 4C_3m_p\Big\};\end{aligned}$$

then by (24)–(26),

$$\lambda^{2n}(A) \geq 1/4.$$

Hence by Theorem 3, for $s > 0$,

$$\lambda^{2n}(A + V_s) \geq 1 - 4e^{-s}.$$

Let $(x,x') = (y+z, y'+z')$ with $(y,y') \in A$, $(z,z') \in V_s$. Let $\Delta_i = F_i(x_i) - F_i(y_i)$ and $\Delta_j' = F_j'(x_j') - F_j'(y_j')$. By the convexity of $\widetilde{N}_i$ we have $|\Delta_i| \leq 2F_i(|x_i - y_i|)$, therefore

$$\sum_i \widehat{N}_i(\Delta_i/2) \leq \sum_i \widehat{N}_i(F_i(|z_i|)) = \sum_i \min(|z_i|, z_i^2) \leq 36s.$$

For a similar reason we have

$$\sum_j \widehat{N}_j'(\Delta_j'/2) \leq 36s.$$

Hence

$$\begin{aligned}\Big|\sum_{i,j} a_{i,j}\Delta_iF_j'(y_j')\Big| &\leq 2\sup\Big\{\sum_i\Big(\sum_j a_{i,j}F_j'(y_j')\Big)b_i : \sum_i \widehat{N}_i(b_i) \leq 36s\Big\} \\ &\leq 2\Big\|\Big(\sum_j a_{i,j}F_j'(y_j')\Big)\Big\|_{\mathcal{N},36s}.\end{aligned}$$

Therefore by (4) we have

$$\begin{aligned}\Big|\sum_{i,j} a_{i,j}\Delta_iF_j'(y_j')\Big| &\leq \frac{72s}{p}\Big\|\Big(\sum_j a_{i,j}F_j'(y_j')\Big)\Big\|_{\mathcal{N},p} \\ &\leq \frac{288s}{p}C_3m_p \quad \text{for } 36s \geq p.\end{aligned} \tag{27}$$

In a similar way we prove

$$\Big|\sum_{i,j} a_{i,j}F_i(y_i)\Delta_j'\Big| \leq \frac{288s}{p}C_3m_p \quad \text{for } 36s \geq p. \tag{28}$$

We also have

$$\Big|\sum_{i,j} a_{i,j}\Delta_i\Delta_j'\Big| \le 4\sup\Big\{\sum_{i,j} a_{i,j}b_ic_j : \sum_i \widehat{N}_i(b_i),\ \sum_j \widehat{N}_j'(c_j) \le 36s\Big\} = 4\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',36s}.$$

So by (5) we get, for $36s \ge p$,

$$\Big|\sum_{i,j} a_{i,j}\Delta_i\Delta_j'\Big| \le 4\Big(\frac{36s}{p}\Big)^2\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} \le 4\Big(\frac{36s}{p}\Big)^2 m_p. \tag{29}$$

By (27)–(29) and the definition of the set A we get

$$\Big|\sum_{i,j} a_{i,j}F_i(x_i)F_j'(x_j')\Big| \le \Big(2+2\frac{288s}{p}C_3+4\Big(\frac{36s}{p}\Big)^2\Big)m_p \le C_4\Big(\frac{s}{p}\Big)^2 m_p \quad \text{for } s \ge p.$$

Therefore for $s \ge p$,

$$P\Big(\Big|\sum_{i,j} a_{i,j}Y_iY_j'\Big| > C_4\Big(\frac{s}{p}\Big)^2 m_p\Big) = \lambda^{2n}\Big((x,x') \in \mathbb{R}^{2n} : \Big|\sum_{i,j} a_{i,j}F_i(x_i)F_j'(x_j')\Big| > C_4\Big(\frac{s}{p}\Big)^2 m_p\Big) \le 1-\lambda^{2n}(A+V_s) \le 4e^{-s}.$$

Hence integration by parts gives

$$E\Big|\sum_{i,j} a_{i,j}Y_iY_j'\Big|^p \le C_4^p m_p^p\Big(1+4\int_1^\infty x^{p-1}e^{-p\sqrt{x}}\,dx\Big) \le C^p m_p^p.$$

Theorem 1 is proved.

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References

[1] C. Borell, *Convex measures on locally convex spaces*, Ark. Mat. 12 (1974), 239–252.
[2] E. D. Gluskin and S. Kwapień, *Tail and moment estimates for sums of independent random variables with logarithmically concave tails*, Studia Math. 114 (1995), 303–309.
[3] C. G. Khatri, *On certain inequalities for normal distributions and their applications to simultaneous confidence bounds*, Ann. Math. Statist. 38 (1967), 1853–1867.

[4] R. Latała, *Tail and moment estimates for sums of independent random vectors with logarithmically concave tails*, Studia Math. 118 (1996), 301–304.
[5] B. Maurey, *Some deviation inequalities*, Geom. Funct. Anal. 1 (1991), 188–197.
[6] V. H. de la Peña and S. J. Montgomery-Smith, *Bounds on the tail probability of U-statistics and quadratic forms*, Bull. Amer. Math. Soc. 31 (1994), 223–227.
[7] Z. Sidak, *Rectangular confidence regions for the means of multivariate normal distributions*, J. Amer. Statist. Assoc. 62 (1967), 626–633.
[8] M. Talagrand, *A new isoperimetric inequality and the concentration of measure phenomenon*, in: Geometric Aspects of Functional Analysis (1989–90), Lecture Notes in Math. 1469, Springer, 1991, 94–124.

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