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A quasi-nilpotent operator with reflexive commutant, II

by

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Abstract. A new example of a non-zero quasi-nilpotent operator T with reflexive commutant is presented. The norms $\|T^n\|$ converge to zero arbitrarily fast.

Let H be a complex separable Hilbert space and let $\mathcal{B}(H)$ denote the algebra of all continuous linear operators on H. If $T \in \mathcal{B}(H)$ then $\{T\}' = \{A \in \mathcal{B}(H) : AT = TA\}$ is called the *commutant* of T. By a *subspace* we always mean a closed linear subspace. If $\mathcal{A} \subset \mathcal{B}(H)$ then $\operatorname{Alg}\mathcal{A}$ denotes the smallest weakly closed subalgebra of $\mathcal{B}(H)$ containing the identity I and $\mathcal{A}$, and $\operatorname{Lat}\mathcal{A}$ denotes the set of all subspaces invariant for each $A \in \mathcal{A}$. If $\mathcal{L}$ is a set of subspaces of H, then $\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{B}(H) : \mathcal{L} \subset \operatorname{Lat}\{T\}\}$. T is said to be *hyperreflexive* if $\{T\}' = \operatorname{Alg}\operatorname{Lat}\{T\}'$, i.e., if the algebra $\{T\}'$ is reflexive.

It can be shown (see [1]) that if T is a nilpotent hyperreflexive operator on a separable Hilbert space then $T = 0$. This is not true for quasinilpotent operators. An example of a non-zero quasinilpotent hyperreflexive operator was given in [5] using a modification of an idea of Wogen [4]. The powers in the example converged to zero slowly; more precisely, the following inequality was true for all positive integers:

$$\|T^n\|^{1/n} \geq 1/\log n.$$

In [6] it was shown that the convergence of the powers of T to zero can be faster, namely for each $p > 0$ there exists a non-zero hyperreflexive operator T for which

$$\|T^n\|^{1/n} \leq 1/n^p.$$

The aim of this note is to show that the convergence $\|T^n\|^{1/n} \to 0$ can be arbitrarily fast:

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THEOREM 1. *Let $(\beta_n)_{n\geq 1}$ be a sequence of positive numbers. Then there exists a non-zero hyperreflexive operator T on a separable Hilbert space H such that $\|T^n\|^{1/n}\leq\beta_n$ for all $n\geq 1$.*

Proof. The set of all non-negative integers will be denoted by $\mathbb{N}$. Set formally $\beta_0=1$. Without loss of generality we can assume that $1=\beta_0\geq\beta_1\geq\beta_2\geq\ldots$ (if necessary, we can replace β_n by $\min\{\beta_j: 0\leq j\leq n\}$).

For $k=0,1,\ldots$ set $m_k=3k(k+1)$. For $n\in\mathbb{N}$ let $f(n)=\min\{k: m_k>n\}$. Thus $f(n)=k$ if and only if $m_{k-1}\leq n<m_k$.

Finally, set $s_0=1$ and, for $k,j\in\mathbb{N}$ with $j^2<k\leq(j+1)^2$, set

$$s_k=\min\left\{\frac{1}{f(n)}\frac{\beta_{n+f(n)}^{n+f(n)}}{\beta_n^n}: 0\leq n\leq m_{(j+1)^2}\right\}.$$

Clearly, $1=s_0\geq s_1\geq s_2\geq\ldots$ Further, $s_{j^2+1}=s_{j^2+2}=\ldots=s_{(j+1)^2}$ so that the sequence (s_n) contains constant subsequences of arbitrary length.

If $n\in\mathbb{N}$, $f(n)=k$ and $j^2<k\leq(j+1)^2$ then $m_{k-1}\leq n<m_k\leq m_{(j+1)^2}$ so that

$$s_{f(n)}\leq\frac{1}{f(n)}\frac{\beta_{n+f(n)}^{n+f(n)}}{\beta_n^n}\qquad(n\in\mathbb{N}).$$

Now let $\mathcal{R}$ be a complex Hilbert space with $\dim\mathcal{R}=2$. Let $\{a,b\}$ be its orthonormal basis and let $c=\frac{1}{\sqrt{2}}(a+b)$, $d=\frac{1}{\sqrt{2}}(a-b)$. Note that $\{c,d\}$ is also an orthonormal basis of $\mathcal{R}$.

For $x\in\mathcal{R}$, $x\neq 0$, we denote by P_x the orthogonal projection in $\mathcal{B}(\mathcal{R})$ onto the one-dimensional space spanned by $\{x\}$. For any integer $n\geq 0$ write

$$\begin{aligned}A_n&=(I-P_a)+s_0s_1\ldots s_nP_a=P_b+s_0s_1\ldots s_nP_a,\\ B_n&=(I-P_b)+s_0s_1\ldots s_nP_b=P_a+s_0s_1\ldots s_nP_b,\\ C_n&=(I-P_c)+s_0s_1\ldots s_nP_c=P_d+s_0s_1\ldots s_nP_c.\end{aligned}$$

Note that $A_0=B_0=C_0=I$. Define the sequence $\{R_n\}_{n\geq0}$ of operators in $\mathcal{B}(\mathcal{R})$ as follows:

$$I,A_1,\ I,B_1,\ I,C_1,\ I,A_1,A_2,A_1,\ I,B_1,B_2,B_1,\ I,C_1,C_2,C_1,\ I,A_1,A_2,A_3,A_2,\ldots$$

More precisely, if $i,k\in\mathbb{N}$ then

$$R_n=\begin{cases}A_i & \text{if } n=m_k+i,\ 0\leq i\leq k+1,\\ A_i & \text{if } n=m_k+2(k+1)-i,\ 1\leq i\leq k,\\ B_i & \text{if } n=m_k+2(k+1)+i,\ 0\leq i\leq k+1,\\ B_i & \text{if } n=m_k+4(k+1)-i,\ 1\leq i\leq k,\\ C_i & \text{if } n=m_k+4(k+1)+i,\ 0\leq i\leq k+1,\\ C_i & \text{if } n=m_{k+1}-i,\ 1\leq i\leq k.\end{cases}$$

For $n\in\mathbb{N}$ set $g(n)=i$ if and only if $R_n\in\{A_i,B_i,C_i\}$. By the definition of $f(n)$ we have $g(n)\leq f(n)$ for all $n\geq 0$.

Note that R_n is invertible, $\|R_n\|=1$ and

$$\|R_{n+1}R_n^{-1}\|=\max\left\{1,\frac{s_0s_1\ldots s_{g(n+1)}}{s_0s_1\ldots s_{g(n)}}\right\}$$

where $|g(n+1)-g(n)|=1$. If $g(n+1)>g(n)$ then $\|R_{n+1}R_n^{-1}\|\leq 1$. If $g(n+1)<g(n)$ then $\|R_{n+1}R_n^{-1}\|=1/s_{g(n)}\leq 1/s_{f(n)}$. Thus $\|R_{n+1}R_n^{-1}\|\leq 1/s_{f(n)}$ $(n\in\mathbb{N})$. For $0\leq i<j$ we have

$$\begin{aligned}\|R_jR_i^{-1}\|&\leq\|R_jR_{j-1}^{-1}\|\,\|R_{j-1}R_{j-2}^{-1}\|\ldots\|R_{i+1}R_i^{-1}\|\\ &\leq\frac{1}{s_{f(j-1)}s_{f(j-2)}\ldots s_{f(i)}}.\end{aligned}$$

Let H be the orthogonal sum of infinitely many copies of $\mathcal{R}$:

$$H=\mathcal{R}\oplus\mathcal{R}\oplus\ldots\tag{1}$$

For $n\geq 0$ set

$$\alpha_n=s_{f(n)}\beta_{n+1}^{n+1}/\beta_n^n\quad\text{and}\quad T_n=\alpha_nR_{n+1}R_n^{-1}.$$

Let $T\in\mathcal{B}(H)$ be the weighted shift with weights T_n,

$$T(x_0\oplus x_1\oplus\ldots)=0\oplus T_0x_0\oplus T_1x_1\oplus\ldots$$

We show that T satisfies the required conditions.

Let $n\geq 1$. Then

$$T^n\Big(\bigoplus_{i=0}^{\infty}x_i\Big)=\underbrace{0\oplus\ldots\oplus0}_{n}\oplus\bigoplus_{i=0}^{\infty}\alpha_i\alpha_{i+1}\ldots\alpha_{i+n-1}R_{n+i}R_i^{-1}x_i.$$

Thus

$$\begin{aligned}\|T^n\|&=\sup_i\alpha_i\alpha_{i+1}\ldots\alpha_{i+n-1}\|R_{n+i}R_i^{-1}\|\\ &\leq\sup_i\frac{s_{f(i)}s_{f(i+1)}\ldots s_{f(i+n-1)}}{s_{f(i+n-1)}\ldots s_{f(i)}}\frac{\beta_{i+1}^{i+1}}{\beta_i^i}\frac{\beta_{i+2}^{i+2}}{\beta_{i+1}^{i+1}}\ldots\frac{\beta_{i+n}^{i+n}}{\beta_{i+n-1}^{i+n-1}}\\ &\leq\sup_i\frac{\beta_{i+n}^{i+n}}{\beta_i^i}\leq\sup_i\frac{\beta_{i+n}^{i+n}}{\beta_{i+n}^i}=\sup_i\beta_{i+n}^n\leq\beta_n^n.\end{aligned}$$

Hence

$$\|T^n\|^{1/n}\leq\beta_n\qquad(n\geq1).$$

The above-defined operator-weighted shift T is reflexive since it has injective weights of dimension 2 [2, Corollary 3.5]. We shall show that $\{T\}'=\operatorname{Alg}T$ and then T is also hyperreflexive. Similarly to [5, p. 281] let $(U_{ij})_{i,j\ge 0}$ be the matrix of an operator $U\in\{T\}'$ in the decomposition (1). Then

$$0=TU-UT=\begin{pmatrix} -U_{01}T_0 & -U_{02}T_1 & -U_{03}T_2 & \dots\\ T_0U_{00}-U_{11}T_0 & T_0U_{01}-U_{12}T_1 & T_0U_{02}-U_{13}T_2 & \dots\\ T_1U_{10}-U_{21}T_0 & T_1U_{11}-U_{22}T_1 & T_1U_{12}-U_{23}T_2 & \dots\\ T_2U_{20}-U_{31}T_0 & T_2U_{21}-U_{32}T_1 & T_2U_{22}-U_{33}T_2 & \dots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

Since T_n's are invertible we obtain from the first row $U_{0i}=0$ for all $i\ge 1$. Similarly we obtain by induction $U_{ij}=0$ if $i<j$, i.e., the matrix U is lower triangular.

Further, for $i\ge j\ge 1$, we have $T_{i-1}U_{i-1,j-1}-U_{ij}T_{j-1}=0$ so that

$$U_{ij}=T_{i-1}U_{i-1,j-1}T_{j-1}^{-1}.$$

Thus for $i,n\ge 0$ we have by induction

$$\begin{aligned} U_{n+i,n}&=T_{n+i-1}T_{n+i-2}\dots T_iU_{i0}T_0^{-1}\dots T_{n-1}^{-1}\\ &=(T_{n+i-1}T_{n+i-2}\dots T_0)S_i(T_{n-1}T_{n-2}\dots T_0)^{-1}\\ &=\alpha_n\alpha_{n+1}\dots\alpha_{n+i-1}R_{n+i}S_iR_n^{-1},\end{aligned}$$

where $S_i=(T_{i-1}T_{i-2}\dots T_0)^{-1}U_{i0}$.

We are now going to show that each S_i is a scalar multiple of identity. Fix $i\ge 0$. Suppose that $S_ia=\lambda_ia+\mu_ib$. To show that $\mu_i=0$ find $k\in\mathbb{N}$, $k>i$, such that $s_k=s_{k-1}=\dots=s_{k-i}$. Let $n=m_{k-1}+k$. Then $R_n=A_k$, $R_{n+i}=A_{k-i}$, $f(n)=f(n+1)=\dots=f(n+i)=k$ and we have

$$\begin{aligned}\|U\|\ge\|U_{n+i,n}\|&\ge\|U_{n+i,n}a\|=\alpha_n\alpha_{n+1}\dots\alpha_{n+i-1}\|R_{n+i}S_iR_n^{-1}a\|\\ &=\frac{\alpha_n\alpha_{n+1}\dots\alpha_{n+i-1}}{s_0s_1\dots s_k}\|A_{k-i}(\lambda_ia+\mu_ib)\|\\ &=\frac{\alpha_n\alpha_{n+1}\dots\alpha_{n+i-1}}{s_0s_1\dots s_k}\|s_0s_1\dots s_{k-i}\lambda_ia+\mu_ib\|\\ &\ge|\mu_i|\frac{\alpha_n\alpha_{n+1}\dots\alpha_{n+i-1}}{s_0s_1\dots s_k}=|\mu_i|\frac{s_k^i}{s_0\dots s_k}\frac{\beta_{n+i}^{n+i}}{\beta_n^n}\\ &\ge|\mu_i|\frac{s_k^i}{s_{k-i}\dots s_k}\frac{\beta_{n+i}^{n+i}}{\beta_n^n}=|\mu_i|\frac{1}{s_k}\frac{\beta_{n+i}^{n+i}}{\beta_n^n}\\ &\ge|\mu_i|k\frac{\beta_n^n}{\beta_{n+k}^{n+k}}\frac{\beta_{n+i}^{n+i}}{\beta_n^n}=|\mu_i|k\left(\frac{\beta_{n+i}}{\beta_{n+k}}\right)^{n+i}\frac{1}{\beta_{n+k}^{k-i}}\ge|\mu_i|k.\end{aligned}$$

Since k could have been chosen arbitrarily large, we conclude that $\mu_i=0$. Thus $S_ia=\lambda_ia$. Similarly (for $n=m_{k-1}+3k$ and $n=m_{k-1}+5k$, respectively) we can prove that $S_ib=\lambda_i'b$ and that $S_ic=\lambda_i''c$ for some complex numbers λ_i',λ_i''. Thus

$$\frac{1}{\sqrt2}\lambda_i''(a+b)=\lambda_i''c=S_ic=S_i\left(\frac{1}{\sqrt2}(a+b)\right)=\frac{1}{\sqrt2}\lambda_ia+\frac{1}{\sqrt2}\lambda_i'b.$$

Thus $\lambda_i=\lambda_i''=\lambda_i'$, i.e., $S_i=\lambda_iI$. Hence $U_{n+i,n}=\lambda_iT_{n+i-1}T_{n+i-2}\dots T_n$ for all $i,n\ge 0$.

Observe that the only non-zero entries of the matrix of the operator T^i are $(T^i)_{n+i,n}=T_{n+i-1}T_{n+i-2}\dots T_n$ for $n=0,1,2,\dots$ and so formally $U=\sum\lambda_iT^i$.

The rest of the proof is exactly the same as that of Lemma 2.3 in [3]. The operator U can be written as a formal power series $\sum\lambda_iT^i$. The series need not converge but its Cesàro means converge to U strongly. So the commutant of T coincides with $\operatorname{Alg}T$ and therefore it is reflexive. This finishes the proof of Theorem 1.

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