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On the representation of functions by orthogonal series in weighted L^p spaces

by

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Abstract. It is proved that if $\{\varphi_n\}$ is a complete orthonormal system of bounded functions and $\varepsilon > 0$, then there exists a measurable set $E \subset [0,1]$ with measure $|E| > 1-\varepsilon$, a measurable function $\mu(x)$, $0 < \mu(x) \le 1$, $\mu(x) \equiv 1$ on E, and a series of the form $\sum_{k=1}^{\infty} c_k\varphi_k(x)$, where $\{c_k\} \in l_q$ for all $q > 2$, with the following properties:

1. For any $p \in [1,2)$ and $f \in L^p_\mu[0,1] = \{f : \int_0^1 |f(x)|^p \mu(x)\,dx < \infty\}$ there are numbers ε_k, $k = 1,2,\ldots$, $\varepsilon_k = 1$ or 0, such that

$$\lim_{n\to\infty} \int_0^1 \Big| \sum_{k=1}^{n} \varepsilon_k c_k \varphi_k(x) - f(x)\Big|^p \mu(x)\,dx = 0.$$

2. For every $p \in [1,2)$ and $f \in L^p_\mu[0,1]$ there are a function $g \in L^1[0,1]$ with $g(x) = f(x)$ on E and numbers δ_k, $k = 1,2,\ldots$, $\delta_k = 1$ or 0, such that

$$\lim_{n\to\infty} \int_0^1 \Big| \sum_{k=1}^{n} \delta_k c_k \varphi_k(x) - g(x)\Big|^p \mu(x)\,dx = 0, \quad \text{where} \quad \delta_k c_k = \int_0^1 g(t)\varphi_k(t)\,dt.$$

In 1932 M. Riesz proved the existence of a function $f_0 \in L^1[0,2\pi]$ whose trigonometric Fourier series diverges in the $L^1[0,2\pi]$ metric (see [1], pp. 599–602). From this result it follows that it is impossible to find for every integrable function on $[0,2\pi]$ a trigonometric series that converges to the function in the L^1 metric.

In this paper we prove the following result.

THEOREM 1. *For any $\varepsilon > 0$ there exists a measurable function $\mu(x)$, $0 < \mu(x) \le 1$, $|\{x \in [0,2\pi] : \mu(x) = 1\}| > 2\pi - \varepsilon$, such that for any function $f \in L^1_\mu[0,2\pi] \equiv \{f : \int_0^{2\pi} |f(x)|\mu(x)\,dx < \infty\}$ there is a trigonometric series*

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{1}$$

with the properties:

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(a) *the series* (1) *converges to* $f(x)$ *in* $L^1_\mu[0,2\pi]$, *i.e.*

$$\lim_{n\to\infty}\int_0^{2\pi}\Big|\sum_{|k|\le n} c_k e^{ikx} - f(x)\Big|\mu(x)\,dx = 0,$$

(b) $\sum_{k=-\infty}^{\infty}|c_k|^q < \infty$ *for all* $q > 2$.

Theorem 1 follows from the more general Theorem 2, whose statement is as follows.

THEOREM 2. *For any complete orthonormal system* $\{\varphi_n\}$ *of bounded functions in* $L^2[0,1]$, *and* $\varepsilon > 0$, *there exists a measurable set* $E \subset [0,1]$ *with measure* $|E| > 1-\varepsilon$, *a measurable function* $\mu(x)$, $0 < \mu(x) \le 1$, $\mu(x) \equiv 1$ *on* E, *and a series of the form* $\sum_{k=1}^{\infty} c_k\varphi_k(x)$, *where* $\{c_k\} \in l_q$ *for all* $q > 2$, *with the following properties*:

1. *For any* $p \in [1,2)$ *and* $f \in L^p_\mu[0,1] = \{f : \int_0^1 |f(x)|^p\mu(x)\,dx < \infty\}$ *there are numbers* ε_k, $k = 1,2,\ldots$, $\varepsilon_k = 1$ *or* 0, *such that*

$$\lim_{n\to\infty}\int_0^1\Big|\sum_{k=1}^{n}\varepsilon_k c_k\varphi_k(x) - f(x)\Big|^p\mu(x)\,dx = 0.$$

2. *For every* $p \in [1,2)$ *and* $f \in L^p_\mu[0,1]$ *there are a function* $g \in L^1[0,1]$ *with* $g(x) = f(x)$ *on* E *and numbers* δ_k, $k = 1,2,\ldots$, $\delta_k = 1$ *or* 0, *such that*

$$\lim_{n\to\infty}\int_0^1\Big|\sum_{k=1}^{n}\delta_k c_k\varphi_k(x) - g(x)\Big|^p\mu(x)\,dx = 0, \quad \text{where} \quad \delta_k c_k = \int_0^1 g(t)\varphi_k(t)\,dt.$$

It should be noted in connection with the second part of this theorem that in 1912 N. N. Luzin [5] proved that for any measurable function f that is finite almost everywhere on $[0,1]$ and any $\varepsilon > 0$ there exist a measurable set E with $|E| > 1-\varepsilon$ and a function g that is continuous on $[0,1]$ and coincides with f on E. This idea of Luzin's of correcting a function with a view to improving its properties was subsequently developed very strongly (see [2]–[4], [6]).

The following lemma is the basic tool in the proof of Theorem 2.

LEMMA. *Let* $\{\varphi_n\}$ *be a complete orthonormal system in* $L^2[0,1]$ *consisting of bounded functions. Then for any* $\varepsilon > 0$, $f \in L^2[0,1]$, *and* $N_0 > 1$ *there exists a measurable set* $E \subset [0,1]$, *a function* $g \in L^2[0,1]$, *and a polynomial* $P(x) = \sum_{n=N_0}^{N} a_n\varphi_n(x)$ *satisfying the following conditions*:

1) $g(x) = f(x)$, $\quad x \in E$, $\quad |E| > 1-\varepsilon$,

2) $\int_0^1 |g(x)|\,dx \le 4\int_0^1 |f(x)|\,dx$, $\quad |g(x)| \ge |f(x)|$, $\quad x \in [0,1]$,

3) $\int_0^1 |P(x) - g(x)|^2\,dx < \varepsilon^2$,

4) $\sum_{k=N_0}^{N} |a_k|^{2+\varepsilon} < \varepsilon$,

5) $\max_{1\le m\le N}\Big(\int_0^1\Big|\sum_{k=N_0}^{m} a_k\varphi_k(x)\Big|\,dx\Big) \le 4\int_0^1 |f(x)|\,dx$,

6) $\max_{1\le m<N}\Big(\int_e\Big|\sum_{k=N_0}^{m} a_k\varphi_k(x)\Big|^p\,dx\Big)^{1/p} \le \varepsilon + 3\Big(\int_e |f(x)|^p\,dx\Big)^{1/p}$,

for all $p \in [1,2)$ *and every measurable subset* e *of* E.

Proof. This is proved analogously to Lemma 2 of [4] (see pp. 80–84).

Proof of Theorem 2. Let $F = \{f_n\}_{n=1}^{\infty}$ be a countable dense subset of $L^2[0,1]$ and $\varepsilon > 0$. By successive application of the Lemma we can define sequences of functions $\overline{g}_n$, sets $\overline{E}_n$, and polynomials

$$\overline{P}_n(x) = \sum_{k=N_{n-1}}^{N_n-1} c_k^{(n)}\varphi_k(x), \qquad N_0 = 1, \tag{2}$$

satisfying the conditions

(3) $\overline{g}_s(x) = f_s(x)$, $\quad x \in \overline{E}_s \subset [0,1]$,

(4) $|\overline{E}_s| > 1 - \varepsilon\cdot 2^{-2s-1}$,

(5) $\int_0^1 |\overline{g}_s(x)|\,dx \le 4\int_0^1 |f_s(x)|\,dx$, $\quad |\overline{g}_s(x)| \ge |f_s(x)|$, $\quad x \in [0,1]$,

(6) $\int_0^1 |\overline{P}_s(x) - \overline{g}_s(x)|^2\,dx \le 2^{-4(s+1)}$,

(7) $\max_{N_{s-1}\le m\le N_s}\int_0^1\Big|\sum_{k=N_{s-1}}^{m} c_k^{(s)}\varphi_k(x)\Big|\,dx \le 4\int_0^1 |f_s(x)|\,dx$,

(8) $\sum_{k=N_{s-1}}^{N_s-1} |c_k^{(s)}|^{2+2^{-2s}} < \varepsilon\cdot 2^{-2s}$,

(9) $\max_{N_{s-1}\le m<N_s}\Big(\int_e\Big|\sum_{k=N_{s-1}}^{m} c_k^{(s)}\varphi_k(x)\Big|^p dx\Big)^{1/p} \le 2^{-2(s-1)} + \Big(\int_e |f_s(x)|^p\,dx\Big)^{1/p}$,

for all $p \in [1,2)$ and every measurable subset e of $\overline{E}_s$. We set

(10) $\Omega_n = \bigcap_{k=n}^{\infty}\overline{E}_k$, $\quad B = \bigcup_{n=1}^{\infty}\Omega_n = \Omega_1 \cup \Big(\bigcup_{n=2}^{\infty}(\Omega_n\setminus\Omega_{n-1})\Big)$, $\quad \Omega_0 = \emptyset$.

It is obvious (cf. (4)) that $|B| = 1$.

We now define the set E, the function μ and the series $\sum_{k=1}^{\infty} c_k\varphi_k(x)$ as follows:

$$E = \Omega_1 = \bigcap_{k=1}^{\infty} \overline{E}_k, \tag{11}$$

$$\mu(x) = \begin{cases} \mu_1 = 1 & \text{for } x \in E \cup ([0,1] \setminus B), \\ \mu_n & \text{for } x \in \Omega_n \setminus \Omega_{n-1}, n \ge 2, \end{cases} \tag{12}$$

where

$$\begin{gathered} \mu_n = \Big(2^{2n} \prod_{k=1}^{n} h_k\Big)^{-1}, \\ h_k = \sup_{1\le p\le 2}\Big(\int_0^1 |\overline{g}_k(x)|^p\,dx + \max_{N_{k-1}\le m<N_k} \int_0^1 \Big|\sum_{i=N_{k-1}}^{m} c_i^{(k)}\varphi_i(x)\Big|^p dx\Big) + 1, \end{gathered} \tag{13}$$

$$\sum_{k=1}^{\infty} c_k\varphi_k(x) = \sum_{s=1}^{\infty} \overline{P}_s = \sum_{s=1}^{\infty}\Big(\sum_{k=N_{s-1}}^{N_s-1} c_k^{(s)}\varphi_k(x)\Big), \tag{14}$$

with $c_k = c_k^{(s)}$ for $N_{s-1} \le k < N_s$, $s = 1, 2, \ldots$

By (4), (8), (11)–(14) we have $|E| > 1 - \varepsilon$, $0 < \mu(x) \le 1$, $\mu(x)$ is measurable, and $\sum |c_k|^{q_0} < \infty$ for all $q_0 > 2$.

From (10), (12), (13) we have for all $k \ge 1$ and all $p \in [1,2)$,

$$\begin{aligned} \int_{[0,1]\setminus\Omega_k} |\overline{g}_k(x)|^p\mu(x)\,dx &= \sum_{n=k+1}^{\infty}\Big(\int_{\Omega_n\setminus\Omega_{n-1}} |\overline{g}_k(x)|^p\mu_n\,dx\Big) \\ &\le \sum_{n=k+1}^{\infty} 2^{-2n}\Big(\int_0^1 |\overline{g}_k(x)|^p\,dx\Big)h_k^{-1} < \frac{1}{3}\cdot 2^{-2k}. \end{aligned} \tag{15}$$

Analogously for all $k \ge 1$ and all $p \in [1,2)$,

$$\max_{N_{k-1}\le m<N_k} \int_{[0,1]\setminus\Omega_k} \Big|\sum_{i=N_{k-1}}^{m} c_i^{(k)}\varphi_i(x)\Big|^p \mu(x)\,dx \le \frac{1}{3}\cdot 2^{-2k}. \tag{16}$$

It follows from conditions (3), (5), (6), (10) and (15) that

$$\begin{aligned} \int_0^1 |\overline{g}_s(x)|^p\mu(x)\,dx &= \int_{\Omega_s} |f_s(x)|^p\mu(x)\,dx + \int_{[0,1]\setminus\Omega_s} |\overline{g}_s(x)|^p\mu(x)\,dx \\ &\le \int_0^1 |f_s(x)|^p\mu(x)\,dx + 2^{-2s}, \end{aligned} \tag{17}$$

$$\begin{aligned} &\Big(\int_0^1 |\overline{P}_s(x) - f_s(x)|^p\mu(x)\,dx\Big)^{1/p} \\ &\le \Big(\int_0^1 |\overline{P}_s(x) - \overline{g}_s(x)|^p\mu(x)\,dx\Big)^{1/p} + \Big(\int_0^1 |\overline{g}_s(x) - f_s(x)|^p\mu(x)\,dx\Big)^{1/p} \\ &\le 2^{-2(s+1)} + 2\Big(\int_{\Omega_s} |\overline{g}_s(x)|^p\mu(x)\,dx\Big)^{1/p} \le 2^{-2(s+1)} + \frac{1}{3}\cdot 2^{-2s} \le 2^{-2s}. \end{aligned} \tag{18}$$

Taking account of (9), (13) and (16), we have for all $m \in [N_{k-1}, N_k)$, $k = 1, 2, \ldots$, and all $p \in [1, 2)$,

$$\begin{aligned} &\int_0^1 \Big|\sum_{i=N_{k-1}}^{m} c_i^{(k)}\varphi_i(x)\Big|^p \mu(x)\,dx \\ &= \Big(\int_{\Omega_k} + \int_{[0,1]\setminus\Omega_k}\Big)\Big|\sum_{i=N_{k-1}}^{m} c_i^{(k)}\varphi_i(x)\Big|^p\mu(x)\,dx \\ &\le \frac{1}{3}\cdot 2^{-2k} + \sum_{n=1}^{k}\Big[\int_{\Omega_n\setminus\Omega_{n-1}} \Big|\sum_{i=1}^{m} c_i^{(k)}\varphi_i(x)\Big|^p dx\Big]\mu_n \\ &\le \frac{1}{3}\cdot 2^{-2k} + \sum_{n=1}^{k}\Big[2^{-2(k+1)} + \Big(\int_{\Omega_n\setminus\Omega_{n-1}} |f_k(x)|^p\,dx\Big)^{1/p}\Big]^p\mu_n \\ &= \frac{1}{3}\cdot 2^{-2k} + \sum_{n=1}^{k}\Big[\frac{\mu_n^{1/p}}{2^{2k}} + \Big(\int_{\Omega_n\setminus\Omega_{n-1}} |f_k(x)|^p\mu_n\,dx\Big)^{1/p}\Big]^p \\ &= \frac{1}{3}\cdot 2^{-2k} + \sum_{n=1}^{k}\Big[\frac{\mu_n^{1/p}}{2^{2(k+1)}} + \Big(\int_{\Omega_n\setminus\Omega_{n-1}} |f_k(x)|^p\mu_n\,dx\Big)^{1/p}\Big]^p \\ &\le \frac{1}{3}\cdot 2^{-2k} + \sum_{n=1}^{k} 2^p\Big[\frac{\mu_n}{2^{2p(k+1)}} + \int_{\Omega_n\setminus\Omega_{n-1}} |f_k(x)|^p\mu_n\,dx\Big] \\ &= \frac{1}{3}\cdot 2^{-2k-1} + 2^p 2^{-2p(k+1)}\sum_{n=1}^{k}\mu_n + 2^p\sum_{n=1}^{k}\int_{\Omega_k} |f_k(x)|^p\mu(x)\,dx \\ &\le 4\int_0^1 |f_k(x)|^p\mu(x)\,dx + 2^{-2k}. \end{aligned} \tag{19}$$

Let $p \in [1,2)$ and let $f \in L^p_\mu[0,1]$, i.e. $\int_0^1 |f(x)|^p\mu(x)\,dx < \infty$. Now assume that polynomials $\overline{P}_{v_n}(x) = \sum_{i=N_{v_{n-1}}}^{N_{v_n}-1} a_i^{(v_n)}\varphi_i(x)$, $v_1 < \ldots$

$\ldots < v_{k-1}$, have been defined satisfying the conditions

$$(20)\quad \begin{gathered} \int_0^1 \Big| f(x) - \sum_{n=1}^{k'} \overline{P}_{v_n}(x) \Big|^p \mu(x)\,dx < 2^{-2k}, \qquad 1 \le k' \le k-1, \\ \max_{N_{v_n-1} \le m < N_{v_n}} \int_0^1 \Big| \sum_{i=N_{v_n-1}}^{m} a_i^{(v_n)} \varphi_i(x) \Big|^p \mu(x)\,dx < 2^{-2n}. \end{gathered}$$

Choose the function $f_{v_k} \in F = \{f_n\}_{n=1}^{\infty}$ such that

$$(21)\qquad \Big(\int_0^1 \Big| f_{v_k}(x) - \Big[f(x) - \sum_{n=1}^{k-1} \overline{P}_{v_n}(x) \Big] \Big|^p \mu(x)\,dx \Big)^{1/p} < 2^{-2(k+2)}.$$

It follows from (20) and (21) that

$$(22)\qquad \Big(\int_0^1 |f_{v_k}(x)|^p \mu(x)\,dx \Big)^{1/p} < 2^{-2(k-1)} + 2^{-2k}.$$

Taking account of (2), (18), (19) and (21), (22) we have

$$(23)\quad \begin{aligned} &\Big(\int_0^1 \Big| f(x) - \sum_{n=1}^{k} \overline{P}_{v_n}(x) \Big|^p \mu(x)\,dx \Big)^{1/p} \\ &\quad \le \Big(\int_0^1 |\overline{P}_{v_k}(x) - f_{v_k}(x)|^p \mu(x)\,dx \Big)^{1/p} \\ &\qquad + \Big(\int_0^1 \Big| f_{v_k}(x) - \Big[f(x) - \sum_{n=1}^{k-1} \overline{P}_{v_n}(x) \Big] \Big|^p \mu(x)\,dx \Big)^{1/p} \\ &\quad \le 2^{-2v_k} + 2^{-2(k+2)} < 2^{-2k}, \end{aligned}$$

$$(24)\qquad \max_{N_{v_k-1} \le m < N_{v_k}} \int_0^1 \Big| \sum_{i=N_{v_k-1}}^{m} c_i^{(v_k)} \varphi_i(x) \Big|^p \mu(x)\,dx < 2^{-k}.$$

It is clear that we can define by induction polynomials

$$\overline{P}_{v_k}(x) = \sum_{i=N_{v_k-1}}^{N_{v_k}-1} c_i^{(v_k)} \varphi_i(x), \qquad k = 1, 2, \ldots,$$

satisfying conditions (23), (24) for all $k \ge 1$. We set

$$(25)\qquad \varepsilon_i = \begin{cases} 1 & \text{for } N_{v_k-1} \le i < N_{v_k},\ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

By (14), (23)–(25) we have

$$\lim_{n\to\infty} \int_0^1 \Big| \sum_{i=1}^{n} \varepsilon_i c_i \varphi_i(x) - f(x) \Big|^p \mu(x)\,dx = 0,$$

i.e., the first part of Theorem 2 is proved.

We now prove the second part. Let $f \in L^p_\mu[0,1]$, $p \in [1,2)$, and

$$(26)\qquad \bar{f}(x) = \begin{cases} f(x) & \text{for } x \in E, \\ 0 & \text{for } x \notin E. \end{cases}$$

It is easy to see that one can choose a subsequence $\{f_{k_n}\}$ of $F = \{f_n\}$ such that

$$(27)\quad \lim_{N\to\infty} \int_0^1 \Big| \sum_{n=1}^{N} f_{k_n}(x) - \bar{f}(x) \Big|^p dx = 0, \qquad \int_0^1 |f_{k_n}(x)|^p\,dx < 2^{-8pn}, \quad n \ge 2.$$

Assume that functions $\{f_{v_n}\}_{n=1}^{q-1}$ and $\{g_n\}$ and polynomials

$$(28)\qquad \overline{P}_{v_n}(x) = \sum_{i=N_{v_n-1}}^{N_{v_n}-1} c_i^{(v_n)} \varphi_i(x), \qquad n = 1, \ldots, q-1,$$

have been defined satisfying the conditions

$$(29)\quad g_n(x) = f_{k_n}(x), \qquad x \in E,$$

$$(30)\quad \int_0^1 |g_n(x)|\,dx < 2^{-(n+1)}, \qquad \Big(\int_0^1 |g_n(x)|^p \mu(x)\,dx \Big)^{1/p} < 2^{-n-1},$$

$$(31)\quad \begin{gathered} \int_0^1 \Big| \sum_{k=1}^{n} [\overline{P}_{v_k}(x) - g_k(x)] \Big|^p dx < 2^{-2pn}, \\ \max_{N_{v_n-1} \le m < N_{v_n}} \int_0^1 \Big| \sum_{i=N_{v_n-1}}^{m} c_i^{(v_n)} \varphi_i(x) \Big|^p \mu(x)\,dx < 2^{-pn}, \qquad 1 \le n < q. \end{gathered}$$

We now choose $f_{v_q} \in F$ such that

$$(32)\qquad \int_0^1 \Big| f_{v_q}(x) - \Big\{ f_{k_q}(x) - \sum_{i=1}^{q-1} [\overline{P}_{v_i}(x) - g_i(x)] \Big\} \Big|^p dx < 2^{-8qp}.$$

Since (cf. (27) and (31)) $\int_0^1 |f_{k_q}(x) - \sum_{i=1}^{q-1} [\overline{P}_{v_i}(x) - g_i(x)]|^p\,dx < 2^{-3qp}$ it follows from (32) that

$$(33)\qquad \int_0^1 |f_{v_q}(x)|^p dx < 2^{-2qp}.$$

We set

$$g_q(x) = f_{k_q}(x) + [\overline{g}_{v_q}(x) - f_{v_q}(x)]. \tag{34}$$

Taking account of (2)–(6), (17), (19), (27), (28), (31)–(34) we have

$$g_q(x) = f_{k_q}(x) \quad \text{for } x \in E, \tag{35}$$

$$\begin{aligned} \int_0^1 |g_q(x)|\,dx &\le \int_0^1 |\overline{g}_{v_q}(x)|\,dx + \int_0^1 \Big| f_{v_q}(x) - \Big\{ f_{k_q}(x) - \sum_{i=1}^{q-1} [\overline{P}_{v_i}(x) - g_i(x)] \Big\} \Big|\,dx \\ &\quad + \int_0^1 \Big| \sum_{i=1}^{q-1} [\overline{P}_{v_i}(x) - g_i(x)] \Big|\,dx \\ &< 2^{-q-1}, \end{aligned} \tag{36}$$

$$\begin{aligned} \Big(\int_0^1 |g_q(x)|^p \mu(x)\,dx \Big)^{1/p} &\le \Big(\int_0^1 |\overline{g}_{v_q}(x)|^p \mu(x)\,dx \Big)^{1/p} + \Big(\int_0^1 \Big| \sum_{i=1}^{q-1} [\overline{P}_{v_i}(x) - g_i(x)] \Big|^p dx \Big)^{1/p} \\ &\quad + \Big(\int_0^1 \Big| f_{v_q}(x) - \Big\{ f_{k_q}(x) - \sum_{i=1}^{q-1} [\overline{P}_{v_i}(x) - g_i(x)] \Big\} \Big|^p dx \Big)^{1/p} \\ &< 2^{-q-1}, \end{aligned} \tag{37}$$

$$\begin{aligned} \Big(\int_0^1 \Big| \sum_{i=1}^{q} [\overline{P}_{v_i}(x) - g_i(x)] \Big|^p dx \Big)^{1/p} &\le \Big(\int_0^1 |\overline{P}_{v_q}(x) - \overline{g}_{v_q}(x)|^p dx \Big)^{1/p} \\ &\quad + \Big(\int_0^1 \Big| f_{v_q}(x) - \Big\{ f_{k_q}(x) - \sum_{i=1}^{q-1} [\overline{P}_{v_i}(x) - g_i(x)] \Big\} \Big|^p dx \Big)^{1/p} \\ &\le 2^{-2q}, \end{aligned} \tag{38}$$

$$\begin{aligned} \max_{N_{v_q-1} \le m < N_{v_q}} \Big(\int_0^1 \Big| \sum_{i=N_{v_q-1}}^{m} c_i^{(v_q)} \varphi_i(x) \Big|^p \mu(x)\,dx \Big)^{1/p} &\le 2^{-q} + \Big(\int_0^1 |f_{v_q}(x)|^p \mu(x)\,dx \Big)^{1/p} < 2^{-q}. \end{aligned} \tag{39}$$

It is clear that we can define by induction sequences $\{g_q\}$ of functions and $\{\overline{P}_{v_q}\}$ of polynomials satisfying conditions (35)–(39) for all $q \ge 1$. It follows from (36) that $\int_0^1 |\sum_{q=1}^{\infty} g_q(x)|\,dx < \infty$.

We define the function g and the numbers δ_i as follows:

$$g(x) = \sum_{q=1}^{\infty} g_q(x), \tag{40}$$

$$\delta_i = \begin{cases} 1 & \text{for } N_{v_q-1} \le i < N_{v_q},\ q = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{41}$$

By (29), (30) and (40) we obtain $g \in L^1[0,1]$, $g(x) = f(x) = \overline{f}(x)$ on E.

Let n be an arbitrary natural number. Then for some natural q we have

$$N_{v_q-1} \le n < N_{v_{q+1}-1}. \tag{42}$$

It follows from (14), (26), (28), (35)–(42) that

$$\begin{aligned} \Big(\int_0^1 \Big| \sum_{i=1}^{n} \delta_i c_i \varphi_i(x) - g(x) \Big|^p \mu(x) dx \Big)^{1/p} &\le \Big(\int_0^1 \Big| \sum_{i=1}^{q-1} [\overline{P}_{v_i}(x) - g_i(x)] \Big|^p dx \Big)^{1/p} + \sum_{i=q}^{\infty} \Big(\int_0^1 |g_i(x)|^p \mu(x) dx \Big)^{1/p} \\ &\quad + \max_{N_{v_q-1} \le m < N_{v_q}} \Big(\int_0^1 \Big| \sum_{i=N_{v_q-1}}^{m} c_i^{(v_q)} \varphi_i(x) \Big|^p \mu(x) dx \Big)^{1/p} \le 2^{-q+2}. \end{aligned}$$

It can be proved similarly that

$$\int_0^1 \Big| \sum_{i=1}^{n} \delta_i c_i \varphi_i(x) - g(x) \Big|\,dx < 2^{-q}.$$

It follows that

$$\delta_i c_i = \int_0^1 g(t) \varphi_i(x)\,dx, \qquad i = 1, 2, \ldots$$

Theorem 2 is now proved.

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The Conley index in Hilbert spaces and its applications

by

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Abstract. We present a generalization of the classical Conley index defined for flows on locally compact spaces to flows on an infinite-dimensional real Hilbert space H generated by vector fields of the form $f : H \to H$, $f(x) = Lx + K(x)$, where $L : H \to H$ is a bounded linear operator satisfying some technical assumptions and K is a completely continuous perturbation. Simple examples are presented to show how this new invariant can be applied in searching critical points of strongly indefinite functionals having asymptotically linear gradient.

1. Introduction. The purpose of this paper is to present a new generalization of the classical Conley index theory. The standard reference for that theory, developed by Charles Conley in the 70's, is his monograph *Isolated Invariant Sets and the Morse Index* [5]. Referring for all technical details to [5] and the recent paper of Mischaikow [12], we recall that in Conley index theory, with any compact isolated invariant set S of a flow $\eta : \mathbb{R}\times Z \to Z$ on a locally compact metric space Z one can associate an index $h(S)$, which is the homotopy type of a compact pointed space. Instead of isolated invariant sets one can equivalently consider (compact) isolating neighbourhoods for flows as pointed out by Mischaikow [12]. Since our construction of Conley index is analogous to the construction of the Leray–Schauder degree it seems more convenient to work with isolating neighbourhoods than directly with isolated invariant sets.

The aim of this paper is to extend the ideas of Conley to the case where Z is replaced by an infinite-dimensional Hilbert space. To be more precise, we assume that we are given an infinite-dimensional real Hilbert space H together with a bounded linear operator $L : H \to H$ such that H and L satisfy conditions (H.1), (H.2) and (H.3) below. We will be concerned with local flows on H generated by $\mathcal{LS}$-vector fields, i.e. maps $f : H \to H$ which can be written in the form $f(x) = Lx + K(x)$, where $K : H \to H$ is a suffi-

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