STUDIA MATHEMATICA 134 (2) (1999)

Symmetric subspaces of l_1 with large projection constants

by

BRUCE L. CHALMERS (Riverside, Calif.) and
GRZEGORZ LEWICKI (Kraków)

Abstract. We construct k-dimensional ($k \geq 3$) subspaces V^k of l_1, with a very simple structure and with projection constant satisfying $\lambda(V^k) \geq \lambda(V^k, l_1) > \lambda(l_2^{(k)})$.

I. Introduction. Let X be a normed space and let V be a linear subspace of X. Denote by $\mathcal{P}(X, V)$ the set of all projections from X onto V, i.e., the set of all continuous extensions of id : $V \to V$ onto X. Let

$$\lambda(V, X) = \inf\{\|P\| : P \in \mathcal{P}(X, V)\} \tag{1.1}$$

and

$$\lambda(V) = \sup\{\lambda(V, X) : V \subset X\}. \tag{1.2}$$

We call $\lambda(V, X)$ the *relative projection constant* of V in X and $\lambda(V)$ the *absolute projection constant* of V. A projection $P \in \mathcal{P}(X, V)$ is called *minimal* if $\|P\| = \lambda(V, X)$. In [HK] the constant

$$\lambda_k = \sup\{\lambda(X) : X \text{ is a real symmetric space of dimension } k\} \tag{1.3}$$

has been estimated in a very precise manner. It is known that $\lambda_2 = \lambda(l_2^{(2)}) = 4/\pi$. For the proof see [CHFG] or [HK]. The similar result for $k \geq 3$ is not true. In fact, in [HK, Prop. 2] the existence of k-dimensional, $k \geq 3$, real, symmetric spaces X_k with

$$\lambda(X_k) > \lambda(l_2^{(k)}) \tag{1.4}$$

and

$$\lim_k \lambda(X_k)/\sqrt{k} \geq (2 - \sqrt{2/\pi})^{-1} \tag{1.5}$$

has been proved. Observe that

$$\lim_k \lambda(l_2^{(k)})/\sqrt{k} = \sqrt{2/\pi}. \tag{1.6}$$

1991 *Mathematics Subject Classification*: 41A35, 41A65, 47B99.

Research supported by KBN Grant 2 P03A 03610.

Also in [PS] Marcinkiewicz spaces satisfying (1.4) and (1.5) (with equality) have been constructed.

In this paper, for $k \geq 3$, we construct symmetric subspaces V^k of l_1 (see Th. 2.10), having a very simple structure, with $\lambda(V^k, l_1)$ satisfying (1.4) and (1.5). Our method of proof is very simple and quite different from that of [HK] and [PS]. The main tool will be Theorem 3 of [CHM1].

Now we introduce some notation which will be of use later. We denote by S_V the unit sphere in a normed space V. The symbol $\text{ext}(S_V)$ stands for the set of all extreme points of S_V. Note that if V is a k-dimensional subspace of $l_1^{(n)}$ then each $P \in \mathcal{P}(l_1^{(n)}, V)$ has the form

$$Px = \sum_{i=1}^{k} u^i(x)v^i, \tag{1.7}$$

where $v^1, \ldots, v^k$ is a fixed basis of V and $u^1, \ldots, u^k \in l_\infty^{(n)}$ satisfy

$$u^j(v^i) = \sum_{l=1}^{n} u_l^j v_l^i = \delta_{ij}. \tag{1.8}$$

A point $x \in X$ is called a *norming* point for $f \in X^*$ if

$$x \in S_X \quad \text{and} \quad f(x) = \|f\|. \tag{1.9}$$

DEFINITION 1.1. Let V be a finite-dimensional Banach space. V is *symmetric* if there exists a basis $v^1, \ldots, v^k$ in V such that

$$\Big\|\sum_{i=1}^{k} |\alpha_i| v^i\Big\| = \Big\|\sum_{i=1}^{k} \alpha_{\pi(i)} v^i\Big\| \tag{1.10}$$

for any $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and any permutation π of indices.

Now let $P = \sum_{i=1}^k u^i(\cdot)v^i \in \mathcal{P}(l_1^{(n)}, V)$. Define

$$\text{crit}(P) = \{j \in \{1, \ldots, n\} : \|Pe_j\| = \|P\|\}, \tag{1.11}$$

where e_j is the jth unit vector from $\mathbb{R}^k$ and for $j = 1, \ldots, n$,

$$V_j = (v_j^1, \ldots, v_j^k), \quad U_j = (u_j^1, \ldots, u_j^k). \tag{1.12}$$

THEOREM 1.2 [CHM1, Th. 3, p. 294]. *Let $P = \sum_{i=1}^k u^i(\cdot)v^i \in \mathcal{P}(l_1^{(n)}, V)$. Then P is minimal if and only if there exists a nonzero $k \times k$ matrix M such that for every $j \in \text{crit}(P)$,*

$$U_j = (u_j^1, \ldots, u_j^k) = \|P\| a^j, \tag{1.13}$$

where a^j is a norming point for the functional on V associated with MV_j, i.e.,

$$(MV_j)(x) = \sum_{i=1}^{k} (MV_j)_i x_i. \tag{1.14}$$

Here $x = \sum_{i=1}^k x_i v^i$.

REMARK 1.3 (see e.g. [NT]). If V is a symmetric space then M is the identity matrix.

REMARK 1.4. By [CHM2, Th. 1] it is easy to see that if M is invertible and $V_j \neq 0$ for $j = 1, \ldots, n$, then $\text{crit}(P) = \{1, \ldots, n\}$ for any minimal projection P.

II. The results. We start with

LEMMA 2.1. *Let $V = \text{Span}[v^1, \ldots, v^k]$ be a k-dimensional subspace of $l_1^{(n)}$. Then $x = \sum_{i=1}^k x_i v^i \in \text{ext}(S_V)$ if and only if the matrix W consisting of all vectors V_j (see (1.12)) orthogonal to x has rank $k-1$ and $\|x\| = 1$. We understand that V_j is orthogonal to x if*

$$V_j(x) = \sum_{i=1}^{k} (V_j)_i x_i = \sum_{i=1}^{k} v_j^i x_i = 0. \tag{2.1}$$

Proof. If $k = 1$, the result is obvious. So suppose that $k \geq 2$. Let $x \in \text{ext}(S_V)$. Note that there is $j \in \{1, \ldots, n\}$ such that x is orthogonal to V_j, i.e., the jth coordinate of x with respect to the canonical basis of $\mathbb{R}^n$ is 0: if not, modifying slightly $x_1, \ldots, x_k$, we can construct $y, z \in S_V$ different from x such that $x = (y+z)/2$.

Now suppose that $\text{rank}(W) < k-1$ and $k > 2$. Put

$$S = \{j \in \{1, \ldots, n\} : x \text{ is orthogonal to } V_j\}$$

and let $l = \text{card}(S)$. Set $Z = V \cap \bigcap_{j \in S} \ker(V_j)$ (we can consider Z as a subspace of $l_1^{(n-l)}$). Since $\text{rank}(W) < k-1$, $\dim(Z) \geq 2$. Since $x \in \text{ext}(S_V)$ and $x \in Z$, $x \in \text{ext}(S_Z)$. But by the previous part of the proof, $V_j(x) = 0$ for some $j \notin S$, a contradiction with the definition of W.

Now take $x \in S_V$ and suppose that $\text{rank}(W) = k-1$. If $x \notin \text{ext}(S_V)$, then

$$x = (x^1 + x^2)/2 \tag{2.2}$$

for some $x^1, x^2 \in S_V$ different from x. Fix $0 < c < 1$ and define a norm $\|\cdot\|_c$ on V by

$$\|y\|_c = c\sum_{j \in S} |V_j(y)| + \sum_{j \notin S} |V_j(y)| \tag{2.3}$$

(see (2.1)). Since $\operatorname{rank}(W)=k-1$, x^1 and x^2 are not orthogonal to all V_j for $j\in S$. Hence $\|x\|_c=1$ and $\|x^i\|_c<1$ for $i=1,2$, a contradiction with (2.2).

LEMMA 2.2. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^k$ satisfying*

$$\|(x_1,\dots,x_k)\|=\|(x_1,\dots,-x_k)\|. \tag{2.4}$$

Let $u=(u_1,\dots,u_{k-1},0)\in(\mathbb{R}^k,\|\cdot\|)^*$. If $x=(x_1,\dots,x_k)$ is a norming point for u then so is $(x_1,\dots,x_{k-1},0)$.

Proof. Let $x\in S_V$ be a norming point for u. By (2.4), $y=(x_1,\dots,-x_k)$ is also a norming point for u. Hence $u((x+y)/2)=\|u\|$. Since x is a norming point,

$$\|(x_1,\dots,x_{k-1},0)\|=\|(x+y)/2\|=1,$$

as required.

LEMMA 2.3. *The following equalities are true for natural numbers $k\geq 2$:*

$$\frac{\sum_{l=0}^{(k-1)/2}\binom{k}{l}(k-2l)}{2^{k-1}}=\frac{\prod_{l=0}^{(k-1)/2}(k-l)}{2^{k-1}((k-1)/2)!}=\lambda(l_2^{(k)}) \tag{2.5}$$

for k odd;

$$\frac{\sum_{l=0}^{k/2}\binom{k}{l}(k-2l)}{2^{k-1}}=\frac{\prod_{l=0}^{k/2}(k-l)}{2^{k-1}(k/2)!}=\lambda(l_2^{(k-1)}) \tag{2.6}$$

for k even.

Proof. For any $k\in\mathbb{N}\setminus\{0,1\}$ let $[k/2]$ denote the largest natural number less than or equal to $k/2$. Observe that

$$\begin{aligned}\sum_{l=0}^{[k/2]}\binom{k}{l}(k-2l)&=k+\sum_{l=1}^{[k/2]}\frac{\prod_{j=0}^{l-1}(k-j)}{l!}(k-2l)\\&=k\bigg(k-1+\sum_{l=2}^{[k/2]}\frac{\prod_{j=1}^{l-1}(k-j)}{2\cdot3\cdot\ldots\cdot l}(k-2l)\bigg)\\&=\frac{k(k-1)}{2}\bigg(k-2+\sum_{l=3}^{[k/2]}\frac{\prod_{j=2}^{l-1}(k-j)}{3\cdot4\cdot\ldots\cdot l}(k-2l)\bigg).\end{aligned}$$

Repeating this procedure, throwing outside the sum the factor $(k-l+1)/l$ for $l=2,\dots,(k-1)/2$ in the case of k odd and for $l=2,\dots,(k-2)/2$ in the case of k even, we get the proof of the first equalities in (2.5) and (2.6).

Now we prove the second equality in (2.5) by an induction argument. Note that by Rutovitz [RU], $\lambda(l_2^{(k)})=k\Gamma(k/2)/(\sqrt{\pi}\Gamma((k+1)/2))$. For $k=3$,

$$k(k-1)/4=3/2=\lambda(l_2^{(3)}).$$

Observe that for any odd k,

$$\begin{aligned}\frac{\prod_{l=0}^{(k-1)/2}(k-l)}{2^{k-1}((k-1)/2)!}&=\frac{k}{2}\cdot\frac{\prod_{l=2}^{(k-1)/2}(k-l)}{2^{k-3}((k-3)/2)!}\\&=\frac{k(\prod_{l=0}^{(k-3)/2}(k-2-l))}{(k-1)2^{k-3}((k-3)/2)!}\\&=\frac{k}{k-1}\lambda(l_2^{(k-2)})\quad\text{(by the induction hypothesis)}\\&=\frac{(k(k-2)/2)\Gamma((k-2)/2)}{\sqrt{\pi}((k-1)/2)\Gamma((k-1)/2)}=\frac{k\Gamma(k/2)}{\sqrt{\pi}\Gamma((k+1)/2)}=\lambda(l_2^k),\end{aligned}$$

since $\Gamma(x+1)=x\Gamma(x)$ and $\Gamma(k)=(k-1)!$ for $k\in\mathbb{N}$.

To prove the second equality in (2.6), note that for any even k,

$$\frac{\prod_{l=0}^{k/2}(k-l)}{2^{k-1}(k/2)!}=\frac{\prod_{l=0}^{(k-1-1)/2}(k-1-l)}{2^{k-1-1}((k-1-1)/2)!}=\lambda(l_2^{(k-1)}).$$

LEMMA 2.4. *Let k be an even number. Put*

$$\mathcal{A}_k=\{A\subset\{1,\dots,k\}:\operatorname{card}(A)=k/2\}. \tag{2.7}$$

For any $A\in\mathcal{A}_k$ let $x^A\in\mathbb{R}^k$ have $x_i^A=1$ if $i\in A$ and $x_i^A=-1$ in the opposite case. If $y=(y_1,\dots,y_k)\in\mathbb{R}^k$ satisfies $\sum_{i=1}^k y_ix_i^A=0$ for any $A\in\mathcal{A}_k$ then $y_1=\ldots=y_k$.

Proof. Fix $i,j\in\{1,\dots,n\}$, $i\neq j$. Take $A\in\mathcal{A}_k$ such that $i\in A$ and $j\notin A$. Put $A_1=(A\setminus\{i\})\cup\{j\}$. Note that

$$\sum_{k\in A}y_k-\sum_{k\notin A}y_k=0 \tag{2.8}$$

and

$$\sum_{k\in A_1}y_k-\sum_{k\notin A_1}y_k=0. \tag{2.9}$$

Subtracting (2.9) from (2.8) we get $2y_i-2y_j=0$, which proves the result.

Reasoning in the same manner as in Lemma 2.4, we can prove

LEMMA 2.5. *Let k be an odd number. Put*

$$\mathcal{B}_k=\{B\subset\{1,\dots,k\}:\operatorname{card}(B)=2\}.$$

For any $i_1<i_2$ and $B=\{i_1,i_2\}\in\mathcal{B}_k$, let $x^B\in\mathbb{R}^k$ have $x_{i_1}^B=-1$, $x_{i_2}^B=1$ and $x_i^B=0$ for $i\notin B$. If $y=(y_1,\dots,y_k)\in\mathbb{R}^k$ satisfies $\sum_{i=1}^k y_ix_i^B=0$ for any $B\in\mathcal{B}_k$ then $y_1=\ldots=y_k$.

DEFINITION 2.6a. Let $k\in\mathbb{N}$ be odd and let $a\in[0,1]$. Let $\mathcal{A}$ denote the family of all subsets of $\{1,\dots,k\}$ of cardinality $\leq(k-1)/2$. With each

$A \in \mathcal{A}$ we associate a vector $v^A \in \mathbb{R}^k$ such that $v_i^A = -1$ if $i \in A$ and $v_i^A = 1$ in the opposite case. Let D be the $k \times (2^{k-1}+k)$ matrix having as columns the previously defined vectors x^A for $A \in \mathcal{A}$ and the vectors ae_i. Then we denote by V_a^k the k-dimensional subspace of $l_1^{(2^{k-1}+k)}$ spanned by the rows of D.

If k is even let $\mathcal{B}$ denote the family of all subsets of $\{1,\ldots,k\}$ of cardinality $\le (k-2)/2$ plus all the subsets of $\{2,\ldots,k\}$ of cardinality $k/2$. With each $B \in \mathcal{B}$ we can associate the vector x^B such that $x_i^B = -1$ if $i \in B$ and $x_i^B = 1$ in the opposite case. Let D be a $k \times (2^{k-1}+k)$ matrix having as columns the vectors x^B for $B \in \mathcal{B}$ and the vectors ae_i. Then we denote by V_a^k the k-dimensional subspace of $l_1^{(2^{k-1}+k)}$ spanned by the rows of D.

If we denote by v^i the rows of D, it is easy to see that for any $k \in \mathbb{N}$ and $a \in [0,1]$, V_a^k is a symmetric subspace of $l_1^{(2^{k-1}+k)}$ with respect to the basis v^i, $i = 1,\ldots,k$.

DEFINITION 2.6b. Since the symmetric space V_a^k in $\mathbb{R}^k$ is completely determined by specifying the columns of D in the region $[x_1 \ge \ldots \ge x_k \ge 0]$, we will say that V_a^k is *generated* by the two (column) vectors $(1,1,\ldots,1)$ and ae_1.

EXAMPLE 2.7. If $k=3$, then

$$v^1 = (1,-1,1,1,a,0,0), \quad v^2 = (1,1,-1,1,0,a,0), \quad v^3 = (1,1,1,-1,0,0,a).$$

THEOREM 2.8. *Let V_a^k be as in Definition 2.6. Then*

$$\lambda(V_a^k, l_1) = \left(\frac{2^{k-1}}{C_k + ak} + \frac{a}{2^{k-1}+a}\right)^{-1}, \tag{2.10}$$

where

$$C_k = \sum_{l=0}^{(k-1)/2} \binom{k}{l}(k-2l) \tag{2.11}$$

for k odd, and

$$C_k = \sum_{l=0}^{k/2-1} \binom{k}{l}(k-2l) \tag{2.12}$$

for k even.

Proof. First we consider the case of k even. Let $P_a = \sum_{i=1}^k u^i(\cdot)v^i$ be a minimal projection onto V_a^k. Since V_a^k is finite-dimensional, such a projection exists. To find P_a effectively, by Theorem 1.2, Remarks 1.3 and 1.4, we should find norming points for each functional V_j. By the symmetry of V_a^k it is only necessary to find norming points for $(1,0,\ldots,0)$ and $(1,1,\ldots,1)$.

For $x = (x_1,\ldots,x_k) \in \mathbb{R}^k$ put

$$\|x\| = \Big\|\sum_{i=1}^k x_i v^i\Big\|_1.$$

By Lemma 2.2, $(1,0,\ldots,0)/\|(1,0,\ldots,0)\|$ is a norming point for $(1,0,\ldots,0)$; it is unique by the definition of V_a^k and Lemma 2.1. By the symmetry of V_a^k, $(1,1,\ldots,1)/\|(1,1,\ldots,1)\|$ is a norming point for $(1,1,\ldots,1)$; it is unique by the definition of V_a^k, Lemma 2.1 and Lemma 2.4. By Definition 2.6, $\|v^1\|_1 = 2^{k-1}+a$ and $\|\sum_{i=1}^k v^i\|_1 = C_k + ak$. By Theorem 1.2 and (1.8) applied to $i=j=1$ (without loss we can choose $i=j=1$ by symmetry), we have

$$\|P_a\| = \left(\frac{2^{k-1}}{C_k+ak} + \frac{a}{2^{k-1}+a}\right)^{-1},$$

as required.

Now we consider the case of k odd. Since by Lemma 2.1, there is no extreme point of $S_{V_a^k}$ on the line through $(1,\ldots,1)$ and $(0,\ldots,0)$, the proof will go in a slightly different manner. To apply the reasoning from the previous case, for any $\varepsilon > 0$ we replace V_a^k by a suitable space W_a^ε. To define W_a^ε we add to the matrix D from Definition 2.6 (the case of k odd) as columns the vectors εx^B, where x^B have been defined in Lemma 2.5, and $\varepsilon\chi_B$ where $B \in \mathcal{B}_k$ and χ_B denotes the characteristic function of B. Denote by $v^{i,\varepsilon}$ the ith row of the above constructed matrix and let

$$W_a^\varepsilon = \operatorname{span}[v^{1,\varepsilon},\ldots,v^{k,\varepsilon}]. \tag{2.13}$$

If $k=3$, then

$$\begin{array}{rcl} v^{1,\varepsilon} &=& (1,\ -1,\ 1,\ 1,\ a,\ 0,\ 0,\ -\varepsilon,\ \varepsilon,\ 0,\ 0,\ -\varepsilon,\ \varepsilon),\\ v^{2,\varepsilon} &=& (1,\ 1,\ -1,\ 1,\ 0,\ a,\ 0,\ \varepsilon,\ \varepsilon,\ -\varepsilon,\ \varepsilon,\ 0,\ 0),\\ v^{3,\varepsilon} &=& (1,\ 1,\ 1,\ -1,\ 0,\ 0,\ a,\ 0,\ 0,\ \varepsilon,\ \varepsilon,\ \varepsilon,\ \varepsilon). \end{array}$$

Observe that W_a^ε is a k-dimensional, symmetric subspace of $l_1^{(2^{k-1}+k^2)}$. Now we calculate $\lambda(W_a^\varepsilon, l_1)$. As in the previous case, to find a minimal projection onto W_a^ε, we have to find norming points for $(1,0,\ldots,0)$, $(1,1,0,\ldots,0)$ and $(1,1,\ldots,1)$. Define

$$\|(x_1,\ldots,x_k)\|^\varepsilon = \Big\|\sum_{i=1}^k x_i v^{i,\varepsilon}\Big\|_1. \tag{2.14}$$

By Lemmas 2.1 and 2.2, $(1,0,\ldots,0)/\|(1,0,\ldots,0)\|^\varepsilon$ is the only norming point for $(1,0,\ldots,0)$. Analogously, $(1,1,0,\ldots,0)/\|(1,1,0,\ldots,0)\|^\varepsilon$ is the only norming point for $(1,1,0,\ldots,0)$. By Lemmas 2.1 and 2.5, we have $(1,1,\ldots,1)/\|(1,1,\ldots,1)\|^\varepsilon \in \operatorname{ext}(S_{W_a^\varepsilon})$. Hence, by symmetry, it follows that $(1,1,\ldots,1)/\|(1,1,\ldots,1)\|^\varepsilon$ is the only norming point for $(1,1,\ldots,1)$. After elementary but tedious calculations one gets

$$\|(1,0,\ldots,0)\|^{\varepsilon} = 2^{k-1} + a + 2\varepsilon(k-1), \tag{2.15}$$

$$\|(1,1,\ldots,1)\|^{\varepsilon} = C_k + ka + \varepsilon k(k-1), \tag{2.16}$$

$$\|(1,1,0,\ldots,0)\|^{\varepsilon} = 2((1+2\varepsilon)(k-1) + a + D_k), \tag{2.17}$$

where $D_k = \sum_{l=2}^{(k-1)/2}(\binom{k-2}{l} + \binom{k-2}{l-2})$. By Theorem 1.2, (1.8) applied to $i = j = 1$ and (2.15)–(2.17), we have

$$(\lambda(W_a^{\varepsilon}, l_1))^{-1} = \frac{2^{k-1}}{C_k + ak + \varepsilon k(k-1)} + \frac{2\varepsilon(k-1)}{2((1+2\varepsilon)(k-1) + a + D_k)} + \frac{a}{2^{k-1} + a + 2\varepsilon(k-1)}. \tag{2.18}$$

Taking the limit on both sides of (2.18) as $\varepsilon \to 0$ we get

$$\lambda(V_a^k, l_1) = \left(\frac{2^{k-1}}{C_k + ak} + \frac{a}{2^{k-1} + a}\right)^{-1}.$$

REMARK 2.9. By Theorem 2.8 and Lemma 2.3, $\lambda(V_0^k, l_1) = \lambda(l_2^{(k)})$ for k odd and $\lambda(V_0^k, l_1) = \lambda(l_2^{(k-1)})$ for k even.

THEOREM 2.10. *For $k \ge 2$ put*

$$a_k = \frac{\sqrt{k}2^{k-1} - C_k}{k - \sqrt{k}} \quad \textit{and} \quad V^k = V_{a_k}^k.$$

Then for $k \ge 3$, $k \ne 4$, the relative projection constants $\lambda(V^k, l_1)$ satisfy (1.4) *and* (1.5).

Proof. Define $g : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g(a) = \frac{2^{k-1}}{C_k + ak} + \frac{a}{2^{k-1} + a}. \tag{2.19}$$

Observe that

$$g'(a) = 2^{k-1}\left(\frac{1}{(2^{k-1} + a)^2} - \frac{k}{(C_k + ka)^2}\right). \tag{2.20}$$

Note that a_k is the only positive root of g'. By Lemma 2.3 and the Kadets–Snobar theorem [KS], $g'(0) < 0$ and by an easy calculation $g'(2a_k) > 0$. Hence g achieves a global minimum at a_k. Consequently, by Remark 2.9, $\lambda(V^k, l_1) > \lambda(l_2^{(k)})$ for k odd ≥ 3.

Now we show (1.4) for even $k > 4$. By Theorem 2.8, Lemma 2.3 and easy calculations

$$\lambda(V^k) = \frac{k2^{k-1} - C_k}{2^{k-1}(2\sqrt{k} - 1) - C_k} = \frac{k - \lambda(l_2^{(k-1)})}{2\sqrt{k} - 1 - \lambda(l_2^{(k-1)})}. \tag{2.21}$$

Hence we need to prove that

$$\frac{k - \lambda(l_2^{(k-1)})}{2\sqrt{k} - 1 - \lambda(l_2^{(k-1)})} > \lambda(l_2^{(k)}) \tag{2.22}$$

for $k > 4$. By the properties of the function Γ, $\lambda(l_2^{(k-1)})\lambda(l_2^{(k)}) = 2k/\pi$. Hence after an elementary calculation (2.22) is equivalent to

$$k(1 + 2/\pi) > \lambda(l_2^{(k-1)}) + (2\sqrt{k} - 1)\lambda(l_2^{(k)}). \tag{2.23}$$

Note that to prove (2.23) it is enough to show that

$$\sqrt{k}(1 + 2/\pi) > 2\lambda(l_2^{(k)}). \tag{2.24}$$

Now we show that if (2.24) holds for k then it holds for $k+2$. First, note that by elementary calculations

$$\frac{\lambda(l_2^{(k+2)})}{\lambda(l_2^{(k)})} = \frac{k+2}{k+1}. \tag{2.25}$$

Multiplying (2.24) by $(k+2)/(k+1)$, by (2.25), we get

$$\frac{k+2}{k+1}\sqrt{k}(1 + 2/\pi) \ge 2\lambda(l_2^{(k+2)}). \tag{2.26}$$

Hence, to show that (2.24) holds true for $k+2$ it is enough to verify that

$$\frac{k+2}{k+1}\sqrt{k}(1 + 2/\pi) < \sqrt{k+2}(1 + 2/\pi).$$

But the last inequality is equivalent to

$$\sqrt{(k+2)k} \le k + 1,$$

which is evidently true.

To end the proof we list below the necessary numerical results:

(2.27)

k	$k(1+2/\pi)$	$(2\sqrt{k}-1)\lambda(l_2^{(k)}) + \lambda(l_2^{(k-1)})$	$2\sqrt{k}\lambda(l_2^{(k)})$
4	6.54648	6.59296	6.79062
6	9.81972	9.81794	9.98013
8	13.093	13.0296	13.1704
10	16.3662	16.235	16.361

By (2.27), (2.23) is not satisfied for $k = 4$ and it is satisfied for $k = 6$ and $k = 8$. Note that (2.24) is satisfied for $k = 10$. Consequently, $\lambda(V^k, l_1) > \lambda(l_2^{(k)})$ for all $k \ge 3$, $k \ne 4$.

To prove (1.5) (we prove the existence of the limit), note that by (1.6),

$$\lim_k \lambda(V^k, l_1)/\sqrt{k} = \frac{k - \lambda(l_2^{(k-1)})}{\sqrt{k}(2\sqrt{k} - 1 - \lambda(l_2^{(k-1)}))} = \frac{1}{2 - \sqrt{2/\pi}}.$$

The proof of Theorem 2.10 is complete.

Since, by (2.27), $\lambda(V^4, l_1) < \lambda_2^{(4)}$, we must consider the case $k = 4$ separately. To do this, we need

DEFINITION 2.11. For $1 \geq b \geq 0$, let D^b be the 4×48 matrix consisting of the following blocks:

$$(2.28)\quad \begin{bmatrix} 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & b & -b & b & b & b & b & -b & -b \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & b & b & -b & b & b & b & b & b \\ b & b & b & -b & b & -b & b & -b & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ b & b & b & b & -b & -b & -b & b & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

$$(2.29)\quad \begin{bmatrix} 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & b & -b & b & b & b & b & -b & -b \\ b & b & b & -b & b & -b & b & -b & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & b & b & -b & b & b & b & b & b \\ b & b & b & b & -b & -b & -b & b & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

and

$$(2.30)\quad \begin{bmatrix} 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & b & -b & b & b & b & b & -b & -b \\ b & b & b & -b & b & -b & b & -b & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ b & b & b & b & -b & -b & -b & b & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & b & b & -b & b & b & b & b & b \end{bmatrix}.$$

Then we denote by Z^b the space spanned by the rows of D^b. Observe that Z^b is a 4-dimensional, symmetric subspace of $l_1^{(48)}$.

THEOREM 2.12. *Let $b_0 = \sqrt{10} - 3$ and let $V^4 = Z^{b_0}$. Then*

$$(2.31)\qquad 1.70724 = \lambda(V^4, l_1) > \lambda(l_2^{(4)}) = 16/(3\pi) = 1.6977.$$

Proof. Fix $1 \geq b \geq 0$. Let v^i, $i = 1, 2, 3, 4$, denote the ith row of D^b. For any $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ let

$$(2.32)\qquad \|x\| = \Big\| \sum_{i=1}^{4} x_i v^i \Big\|_1.$$

Note that by elementary but tedious calculations for $1 \geq b \geq d \geq 0$,

$$(2.33)\qquad \|(1, 1, d, d)\| = 4(9 + 2bd + 3b + 2d + |b + d + bd - 1|).$$

Analogously, if $1 \geq d > b \geq 0$, then

$$(2.34)\qquad \|(1, 1, d, d)\| = 4(9 + 2bd + 2b + 3d + |b + d + bd - 1|).$$

Indeed, one can check that for any $b, d \in [0, 1]$, the part of the l_1-norm associated with (2.28) is equal to

$$2(4 + 2bd + 3b + 3d + |b - d|).$$

Analogously, the parts of the l_1-norm associated with (2.29) and (2.30) are each equal to

$$2(7 + b + d + bd + |d + b + bd - 1|).$$

Let $P^b \in \mathcal{P}(l_1^{(4)}, V^4)$ be the projection defined by

$$(2.35)\qquad U_j = cV_j/\|V_j\|$$

(see (1.12)), where $c > 0$ is so chosen that the orthogonality conditions (1.8) are satisfied. By Theorem 1.2, (1.8) applied to $i = j = 1$, and (2.33) we have

$$(2.36)\qquad \|P^b\| = \frac{9 + 5b + 2b^2 + |b^2 + 2b - 1|}{6(1 + b^2)}.$$

Now consider the function

$$f(b) = \frac{b^2 + 3b + 10}{6(1 + b^2)}.$$

It is easy to see that $f'(b) = 0$ if and only if $-b^2 - 6b + 1 = 0$. The last equation has the only positive root $b_0 = \sqrt{10} - 3$. By elementary considerations, f has a global maximum at b_0 and $b_0^2 + 2b_0 - 1 = -4(\sqrt{10} - 3) < 0$. Hence

$$(2.37)\qquad \|P^{b_0}\| = f(b_0) = \frac{20 - 3\sqrt{10}}{12(10 - 3\sqrt{10})} = 1.70724$$
$$> \lambda(l_2^{(4)}) = 16/(3\pi) = 1.6977.$$

To finish the proof, we have to show that P^{b_0} is a minimal projection onto V^4. To do this, by Theorem 1.2, Remarks 1.3, 1.4 and symmetry considerations, it suffices to show that $a_0 = (1, 1, b_0, b_0)/\|(1, 1, b_0, b_0)\|$ is the only norming point for $V_1 = (1, 1, b_0, b_0)$. By the symmetry of V^4, there exists a norming point a_d for $(1, 1, b_0, b_0)$ of the form $a_d = (1, 1, d, d)/\|(1, 1, d, d)\|$. Note that, for $d > b_0$ close to b_0, $V_1(a_0) > V_1(a_d)$ if and only if

$$(2.38)\qquad \frac{1 + b_0 d}{10 + b_0 d + 2d + b_0} < \frac{1 + b_0^2}{10 + 3b_0 + b_0^2}$$

and $V_1(a_0) > V_1(a_d)$ for $d < b_0$ close to b_0 if and only if

$$(2.39)\qquad \frac{1 + b_0 d}{10 + b_0 d + d + 2b_0} < \frac{1 + b_0^2}{10 + 3b_0 + b_0^2}.$$

By elementary calculations, (2.38) is equivalent to

$$(2.40)\qquad b_0^2 + 9b_0 - 2 < 0$$

and (2.39) to

$$(2.41)\qquad 1 - 9b_0 - b_0^2 < 0.$$

Now, (2.40) and (2.41) are equivalent to

$$(2.42)\qquad (\sqrt{89} - 9)/4 < b_0 < (\sqrt{89} - 9)/2,$$

which is true since $9.4 < \sqrt{89} < 9.5$. So we have proved that $V_1(a_d) < V_1(a_0)$ for d close to b_0. If $V_1(a_d) \geq V_1(b_0)$ for some $d \neq b_0$, then $V_1(\alpha a_0 + (1 - \alpha)a_d) \geq V_1(a_0)$, which for α close to 1 leads to a contradiction with (2.38) or (2.39). Note that for any $b \in [0, 1]$, $(1, 1, b, b)$ is orthogonal (see (2.1)) to $(-b, -b, 1, 1)$, $(b, -b, 1, -1)$ and $(1, -1, b, -b)$. It is easy to see that the rank of the matrix formed by the above vectors is equal to 3. Hence, by

Lemma 2.1, $(1,1,b,b)/\|(1,1,b,b)\| \in \operatorname{ext}(S_{V^4})$. Thus, a_0 is the only norming point for $(1,1,b_0,b_0)$. By Theorem 1.2, P^{b_0} is a minimal projection onto V^4. The proof is complete.

The question arises whether the spaces V^k above satisfy $\lambda(V^k) = \lambda(V^k, l^1)$ for all $k \geq 2$. The answer is given in Theorem 2.14 below.

LEMMA 2.13. *Let V_a^k be as in Definition 2.6. Then, for $k = 2,3,4$, the dual ball B_k^* of V_a^k is the (closed) convex hull of the k-dimensional symmetric subset of $\mathbb{R}^k$ generated by $(1,1,\ldots,1)/k$ and a vector of the form $(1/b,\, b_2,\ldots,b_k)$, where*

$$b = \frac{C_k + ka}{2^{k-1}+a}. \tag{2.43}$$

Furthermore V_a^k is isometric to the k-dimensional symmetric subspace W_b^k of L_∞ generated by the k coordinate functions of B_k^*.

Proof. We only prove the case $k = 3$, since the proofs of the other cases are similar.

For arbitrary $a \geq 0$, V_a^3 is given in (x_1,x_2,x_3)-space by the norm

$$\|(x_1,x_2,x_3)\|_{l_1^7} = |x_1+x_2+x_3| + |-x_1+x_2+x_3| + |x_1-x_2+x_3| + |x_1+x_2-x_3| + a(|x_1|+|x_2|+|x_3|).$$

Thus the ball $B(V_a^3)$ of the symmetric space V_a^3 is determined by the corner points $(1,1,0)/(4+2a)$ (set $|-x_1+x_2+x_3| = 0$ and $|x_1-x_2+x_3| = 0$) and $(1,0,0)/(4+a)$ (set $|x_2| = 0$ and $|x_3| = 0$) in the region $0 \leq x_3 \leq x_2 \leq x_1$ and B_3^* is the dual ball of V_a^3 provided that

$$\|(x_1,x_2,x_3)\|_{l_1^7} = \sup_{(x_1',x_2',x_3')\in B_3^*} (x_1,x_2,x_3)\cdot(x_1',x_2',x_3').$$

Check that this will hold when

$$\left(\frac{1}{4+a},0,0\right)\cdot\left(\frac{1}{b},0,0\right) = \frac{(1,1,0)}{4+2a}\cdot\left(\frac13,\frac13,\frac13\right), \tag{2.44}$$

where (solving (2.44) for b)

$$b = \frac{6+3a}{4+a}.$$

That is, with this value of b and $b_2 = b_3 = 0$, B_3^* is the dual ball of V_a^3. Note that $(1,1,1)/\|(1,1,1)\|$ is the average of the three points symmetric to $(1,1,0)/\|(1,1,0)\|$ in the region $x_i \geq 0$ $(i = 1,2,3)$ and thus (2.44) can be rewritten

$$\left(\frac{1}{4+a},0,0\right)\cdot\left(\frac{1}{b},0,0\right) = \frac{(1,1,1)}{6+3a}\cdot\left(\frac13,\frac13,\frac13\right).$$

Finally, it is well known and immediate that V_a^k is isometric to the k-dimensional subspace W_b^k of L_∞ with basis given by the k coordinate functions of B_k^*. The proof is complete.

REMARK 2.13a. In the general case $k \geq 2$, for arbitrary $a \geq 0$, V_a^k is given in $(x_1,\ldots,x_k)$-space by the norm

$$\|(x_1,\ldots,x_k)\|_{l_1^{(2^{k-1}+k)}} = |x_1+\ldots+x_k| + |-x_1+\ldots+x_k| + \ldots + a(|x_1|+\ldots+|x_k|).$$

It follows that the ball $B(V_a^k)$ is determined by the corner points $(1,1,0,\ldots,0)/\|(1,1,0,\ldots,0)\|$, $(1,1,1,1,0,\ldots,0)/\|(1,1,1,1,0,\ldots,0)\|,\ldots$ and $(1,0,\ldots,0)/\|(1,0,\ldots,0)\|$, and B_k^* is the dual ball of V_a^k provided that

$$\|(x_1,\ldots,x_k)\|_{l_1^{(2^{k-1}+k)}} = \sup_{(x_1',\ldots,x_k')\in B_k^*} (x_1,\ldots,x_k)\cdot(x_1',\ldots,x_k').$$

We conjecture that, just as in the cases $k = 2,3,4$ (see Lemma 2.13), this will hold when

$$\begin{aligned}\left(\frac{1}{2^{k-1}+a},0,\ldots,0\right)\cdot\left(\frac1b,b_2,\ldots,b_k\right) &= \frac{(1,\ldots,1)}{C_k+ka}\cdot\left(\frac1k,\frac1k,\ldots,\frac1k\right)\\ &= \frac{(1,1,0,\ldots,0)}{\|(1,1,0,\ldots,0)\|}\cdot\left(\frac1b,b_2,\ldots,b_k\right)\\ &= \frac{(1,1,1,1,0,\ldots,0)}{\|(1,1,1,1,0,\ldots,0)\|}\cdot\left(\frac1b,b_2,\ldots,b_k\right) = \ldots,\end{aligned} \tag{2.45}$$

where (solving (2.45) for b and $b_2,\ldots,b_k$)

$$b = \frac{C_k+ka}{2^{k-1}+a},$$

and that, with these values of b and b_i $(i = 2,\ldots,k)$, B_k^* is the (closed) convex hull of the k-dimensional symmetric set in $\mathbb{R}^k$ generated by $(1,1,\ldots,1)/k$ and $(1/b,\, b_2,\ldots,b_k)$.

THEOREM 2.14. *For $k \geq 2$, let a_k be as in Theorem 2.10. Then $\lambda(V_{a_k}^k) = \lambda(V_{a_k}^k, l_1)$ for $k = 2,3$, but $\lambda(V_{a_4}^4) > \lambda(V_{a_4}^4, l_1)$.*

Proof. For $k \geq 2$, let U^k be the symmetric subspace of $l_\infty^{(2^{k-1}+k)}$ generated by the two vectors $(1,1,\ldots,1)$ and $\sqrt{k}\,e_1$ of equal Euclidean length. As in (2.43) let

$$b_k = \frac{C_k+ka_k}{2^{k-1}+a_k}$$

and note from (2.20) that $b_k = \sqrt{k}$. Thus, since (recall (2.15) and (2.16)) $\|(1,0,\ldots,0)\| = 2^{k-1} + a_k$ and $\|(1,1,\ldots,1)\| = C_k + ka_k$, the adjoint operator $P^*_{a_k} = \sum_{i=1}^k v^i(\cdot)u^i$ is a projection from $l_\infty^{(2^{k-1}+k)}$ onto the k-dimensional subspace $U^k_{\sqrt{k}}$. But also since $k/b_k = b_k$, $U^k_{\sqrt{k}}$ is generated by $(1,1,\ldots,1)/k$ and $(1/b_k, 0,\ldots,0)$. Now note that, in the cases $k = 2,3$, $W^k_{b_k} = U^k_{\sqrt{k}}$. But, by Lemma 2.13, $W^k_{b_k}$ is isometric to $V^k_{a_k}$. Thus, since it is well known that isometric k-dimensional Banach spaces have the same projection constants and that L_∞ is a "maximal overspace" (see e.g. [WO]), we have, for $k = 2,3$, $\lambda(V^k_{a_k}) = \lambda(W^k_{b_k}) = \lambda(U^k_{\sqrt{k}}) = \|P^*_{a_k}\| = \|P_{a_k}\| = \lambda(V^k_{a_k},\ l_1^{(2^{k-1}+k)}) = \lambda(V^k_{a_k},\ l_1)$.

In the case $k = 4$, however, $W^k_{b_k} \neq U^k_{\sqrt{k}}$. In fact $W^4_{b_4} = [w_1, w_2, w_3, w_4]$ is the 4-dimensional symmetric subspace of $l_\infty^{(32)}$ generated by $(1,1,1,1)$ and $(2, 1/10, 0, 0)$, as is seen from Lemma 2.13. But now one can check (by use of symmetry and minimizing over c below, or by using the theory of [CHM2]) that the minimal projection from $l_\infty^{(32)}$ onto $W^4_{b_4}$ is given by $P = \sum_{i=1}^4 z^i(\cdot)w^i$ where z_1, z_2, z_3, z_4 is a basis for the 4-dimensional symmetric subspace of $l_1^{(32)}$ generated by $\kappa(1,1,1,1)$ and $\varrho(1,c,0,0)$ where $c = 0$, $\varrho = 5/174$, $\kappa = 19/232$ and $\|P\| = 97/58 = 1.6724\ldots > 5/3 = \lambda(V^4_{a_4}, l^1)$. The proof is complete.

COROLLARY 2.15. *For $k \geq 2$, let W^k be the symmetric subspace of L_∞ with basis given by the k coordinate functions of the dual ball B^*_k of $V^k_{a_k}$.* (It is conjectured in Remark 2.13a, and proved for $k = 2,3,4$ in Lemma 2.13, that W^k is generated by $(1,1,\ldots,1)$ and $(\sqrt{k}, kb_2, \ldots, kb_k)$, for some $b_2, \ldots, b_k$.) *Then for $k \geq 3$, $k \neq 4$, the projection constants $\lambda(W^k)$ satisfy* (1.4) *and* (1.5).

NOTE 2.16. For $k = 2,3$ the operator $P_{a_k} = \sum_{i=1}^k u^i(\cdot)v^i$ of Theorem 2.14 provides an example of a minimal projection whose adjoint is also minimal. (This occurs because the L_∞-space $[u^1, \ldots, u^k]$ is isometric to the L_1-space $[v^1, \ldots, v^k]$ and L_∞ is a maximal overspace; cf. [CHPS].)

THEOREM 2.17. *Let V^4 be as in Theorem 2.12. Then $\lambda(V^4) = \lambda(V^4, l_1)$.*

Proof. Consider the orthogonal projection $P^{b_0} = \sum_{i=1}^k u^i(\cdot)v^i$ given by (2.35) with $b = b_0$. It is shown in the proof of Theorem 2.12 that P^{b_0} is minimal. Consider the adjoint projection $(P^{b_0})^* = \sum_{i=1}^k v^i(\cdot)u^i$. As shown in the proof of Theorem 2.12, $(1,1,b_0,b_0)/\|(1,1,b_0,b_0)\|$ is the only norming point for $V_1 = (1,1,b_0,b_0)$, which is the only corner of the ball of V^4 in $x_1 \geq x_2 \geq x_3 \geq x_4 \geq 0$. Thus, by symmetry, the L_∞-subspace $U^4 = [u^1, \ldots, u^4]$ has as a basis the coordinate functions of the dual ball of V^4 and hence is isometric to the L_1-subspace $[v^1, \ldots, v^4]$. Thus the conclusion follows as above, since $\lambda(V^4) = \lambda(U^4) = \|(P^{b_0})^*\| = \|P^{b_0}\| = \lambda(V^4, l_1^{(48)})$.

COROLLARY 2.18. *Let U^4 be the symmetric subspace of $l_\infty^{(48)}$ generated by $(1,1,b_0,b_0)$, where $b_0 = \sqrt{10} - 3$ (as in Definition 2.11). Then the projection constant $\lambda(U^4)$ satisfies* (1.4).

References

[CHFG] B. L. Chalmers, C. Franchetti and M. Giaquinta, *On the self-length of two-dimensional Banach spaces*, Bull. Austral. Math. Soc. 53 (1996), 101–107.

[CHM1] B. L. Chalmers and F. T. Metcalf, *The determination of minimal projections and extensions in L^1*, Trans. Amer. Math. Soc. 329 (1992), 289–305.

[CHM2] —, —, *A characterization and equations for minimal projections and extensions*, J. Operator Theory 32 (1994), 31–46.

[CHPS] B. L. Chalmers, K. C. Pan and B. Shekhtman, *When is the adjoint of a minimal projection also minimal*, in: Approximation Theory (Memphis, Tenn., 1991), Lecture Notes in Pure and Appl. Math. 138, Dekker, 1992, 217–226.

[KS] M. I. Kadets and M. G. Snobar, *Certain functionals on the Minkowski compactum*, Mat. Zametki 10 (1971), 453–458 (in Russian); English transl.: Math. Notes 10 (1971), 694–696.

[HK] H. Koenig, *Projections onto symmetric spaces*, Quaestiones Math. 18 (1995), 199–220.

[PS] E. D. Positsel'skiĭ, *Projection constants of symmetric spaces*, Mat. Zametki 15 (1974), 719–727 (in Russian); English transl.: Math. Notes 15 (1974), 430–435.

[RU] D. Rutovitz, *Some parameters associated with finite-dimensional Banach spaces*, J. London Math. Soc. 40 (1965), 241–255.

[NT] N. Tomczak-Jaegermann, *Banach–Mazur Distances and Finite-Dimensional Operator Ideals*, Wiley, New York, 1989.

[WO] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Univ. Press, 1991.

Department of Mathematics
University of California
Riverside, California 92521
U.S.A.
E-mail: blc@math.ucr.edu

Department of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: lewicki@im.uj.edu.pl

Received October 6, 1997 (3971)
Revised version September 11, 1998 and December 4, 1998