The absence of real points of C at infinity can be ensured when almost the whole $\mathbb{R}^2$ is attracted to the real foci P_i, i.e. when b and c are even (we use the formula (4) for divergence).

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Received February 20, 1997
Revised version April 3, 1998 (3845)

STUDIA MATHEMATICA 131 (2) (1998)

Asymptotic stability in the Schauder fixed point theorem

by

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Abstract. This note presents a theorem which gives an answer to a conjecture which appears in the book *Matrix Norms and Their Applications* by Belitskiĭ and Lyubich and concerns the global asymptotic stability in the Schauder fixed point theorem. This is followed by a theorem which states a necessary and sufficient condition for the iterates of a holomorphic function with a fixed point to converge pointwise to this point.

The object of this note is to settle a conjecture raised by Belitskiĭ and Lyubich in 1984 concerning the global asymptotic stability in the Schauder fixed point theorem.

1. Conjecture of Belitskiĭ and Lyubich. Let E be a (real or complex) Banach space with a non-empty bounded convex open subset D, and let $f : \overline{D} \to \overline{D}$ ($\overline{D}$ stands for the closure of D) be a compact continuous map. The celebrated Schauder fixed point theorem [13], which is one of the fundamental theorems in nonlinear functional analysis, asserts that there exists a point $\widehat{x} \in \overline{D}$ such that $f(\widehat{x}) = \widehat{x}$. For $x \in E$, denote by $f'(x)$ the Fréchet derivative of f evaluated at x. For a bounded linear operator A on E, $r(A)$ stands for the spectral radius of A. Under the assumption that f is continuously Fréchet differentiable, Belitskiĭ and Lyubich ([1], p. 41) proposed the following conjecture in 1984 concerning the asymptotic behaviour of the fixed point in the Schauder fixed point theorem.

CONJECTURE OF BELITSKIĬ AND LYUBICH. *Let E be a (real or complex) Banach space with an open subset Ω and $f : \Omega \to E$ be compact and continuously Fréchet differentiable in Ω. Suppose D is a non-empty bounded convex open subset of E such that $f(\overline{D}) \subset \overline{D} \subset \Omega$ and $\sup_{x\in\overline{D}} r(f'(x)) < 1$. Then*

1991 *Mathematics Subject Classification*: Primary 47H10; Secondary 34D20.

Key words and phrases: fixed point, asymptotic stability, spectral radius, compact map, holomorphic map, normal family, Whitney smooth extension theorem.

Research of the first author was supported in part by the National Science Council of the Republic of China.

f has a unique fixed point $\widehat{x} \in \overline{D}$ and $\widehat{x}$ is globally asymptotically stable, i.e. the sequence $\{f^k(x)\}$ of iterates converges to $\widehat{x}$ for any $x \in \overline{D}$.

Let us remark that in the original formulation of the conjecture, the space was $\mathbb{R}^n$ and the mapping f was given in a non-empty compact convex subset K of $\mathbb{R}^n$. In the above formulation we suppose that f lives in a wider open set $\Omega \supset K$. This assumption does not affect the generality in the finite-dimensional case because of Whitney's smooth extension theorem (see, e.g., Federer [4], p. 225). Since there is no infinite-dimensional version of Whitney's theorem, the infinite-dimensional version of the conjecture of Belitskiĭ and Lyubich could be formulated as above. Furthermore, in the finite-dimensional case, the assumption $K = \overline{\operatorname{int} K}$ ($\operatorname{int} K$ stands for the interior of K) is not necessary; it is sufficient to pass to the affine hull of K. Let us also remark that the local asymptotic stability problem of the conjecture of Belitskiĭ and Lyubich was considered by Kitchen [10].

The purpose of the present note is to settle this conjecture. We show that the spectral condition "$\sup_{x\in\overline{D}} r(f'(x)) < 1$" does imply f has a unique fixed point, but we give a disproof of the asymptotic stability part for the real case and a proof of the asymptotic stability part for the complex case. The method of proof employed involves Kellogg's uniqueness theorem as well as a normal family argument for holomorphic functions.

2. Solution of the conjecture. The following theorem solves the problem of Belitskiĭ and Lyubich.

THEOREM 1. *Let $(E, \|\cdot\|)$ be a (real or complex) Banach space, let Ω be an open subset of E and $f : \Omega \to E$ be compact and continuously Fréchet differentiable in Ω. Suppose D is a non-empty bounded convex open subset of E such that $f(\overline{D}) \subset \overline{D} \subset \Omega$ and $\sup_{x\in\overline{D}} r(f'(x)) < 1$. Then:*

(i) *f has a unique fixed point $\widehat{x} \in \overline{D}$;*

(ii) *$\widehat{x}$ is not necessarily globally asymptotically stable when E is a real Banach space;*

(iii) *$\widehat{x}$ is certainly globally asymptotically stable when E is a complex Banach space.*

To prove part (i) of Theorem 1, let us recall Kellogg's uniqueness theorem [8].

KELLOGG'S UNIQUENESS THEOREM. *Let D be a non-empty bounded open connected subset of a Banach space and $f : \overline{D} \to \overline{D}$ be a compact map such that f is continuously Fréchet differentiable in D and continuous in $\overline{D}$. Suppose that 1 is an eigenvalue of $f'(x)$ for no $x \in D$, and f is fixed point free on the boundary of D. Then f has at most one fixed point.*

The proof given by Kellogg [8] was based on a parity property for compact linear operators and Leray–Schauder's degree theory, especially Leray–Schauder's index formula. A much simpler proof may be found in Berger [2, p. 268].

We also need an elementary operator-theoretic result (see Holmes [7]):

Let A be a bounded linear operator on a normed space E. Then for each $\varepsilon > 0$ there is a norm $\|\cdot\|$ on E equivalent to the original norm such that $\|A\| \le r(A) + \varepsilon$.

Proof of Theorem 1. (i) Set $\operatorname{Fix}(f) := \{x \in \overline{D} : f(x) = x\}$. By Schauder's fixed point theorem, $\operatorname{Fix}(f) \ne \emptyset$. To prove the uniqueness, we consider two cases separately.

CASE 1: $\operatorname{Fix}(f) \subset D$, i.e. f is fixed point free on the boundary ∂D. Since for each $x \in D$, $f'(x)$ is a compact linear operator on E and $r(f'(x)) < 1$, it follows that 1 is an eigenvalue of $f'(x)$ for no $x \in D$. Then Kellogg's uniqueness theorem asserts that f has a unique fixed point.

CASE 2: $\operatorname{Fix}(f) \cap \partial D \ne \emptyset$. Suppose first that $\operatorname{Fix}(f) \cap \partial D = \{\widehat{x}\}$, a singleton. By Fréchet differentiability of f at $\widehat{x}$,

$$f(x) = \widehat{x} + f'(x)(x - \widehat{x}) + N(x - \widehat{x}), \quad \text{where} \quad N(x - \widehat{x}) = o(\|x - \widehat{x}\|).$$

Since $r(f'(\widehat{x})) < 1$, by Holmes' theorem stated above there exists a norm $|||\cdot|||$ on E equivalent to $\|\cdot\|$ and $\varepsilon > 0$ such that the operator norm of $f'(\widehat{x})$ satisfies $|||f'(\widehat{x})||| < \varepsilon < 1$. As $N(x - \widehat{x}) = o(\|x - \widehat{x}\|)$, there exists a $\delta > 0$ such that $|||N(x - \widehat{x})||| \le (1 - \varepsilon)|||x - \widehat{x}|||$ whenever $x \in U \subset \Omega$, where

$$U := \{x \in E : |||x - \widehat{x}||| < \delta\}.$$

Then for each $x \in U$, we have

$$|||f(x) - \widehat{x}||| \le \alpha|||x - \widehat{x}|||, \quad \text{where} \quad \alpha \equiv |||f'(\widehat{x})||| + 1 - \varepsilon < 1.$$

Consequently, U is invariant under f. Since $r(f'(\widehat{x})) < 1$ and f' is continuous, U can be chosen so that it contains no other fixed point of f and $r(f'(x)) < 1$ for all $x \in \overline{U}$. Then $f(\overline{D} \cup \overline{U}) \subset \overline{D} \cup \overline{U}$ and 1 is an eigenvalue of $f'(x)$ for no $x \in \overline{D} \cup \overline{U}$. As D and U are open convex and $D \cap U \ne \emptyset$, by a convexity argument it is not difficult to prove that $\partial(\overline{D} \cup \overline{U}) = \partial(D \cup U)$ (and hence $\operatorname{Int}(\overline{D} \cup \overline{U}) = D \cup U$). Thus f is fixed point free on $\partial(\overline{D} \cup \overline{U})$. Since $D \cup U$ is open connected, Kellogg's uniqueness theorem implies that f has a unique fixed point in $\overline{D}$. By making use of the spectral condition of the hypothesis and the inverse function theorem, we can see that each fixed point on the boundary of D is isolated. Therefore, repeating the above argument for the case of a single point on the boundary shows that the case when there are more than one fixed point on the boundary of D *cannot* occur. This proves (i).

(ii) Let $E=\mathbb{R}^2$. Define a C^1 real-valued function φ on $[-1,1]$ by
$$\varphi(x):=0 \text{ if } |x|\le 1/2, \quad \varphi(1):=\phi(-1):=1,$$
$$|\varphi(x)|\le 1 \quad (|x|\le 1).$$
Let $D:=\{(x,y)\in\mathbb{R}^2 : |x|+|y|<1\}$ and Ω an open set containing $\overline{D}$. Define $f:\Omega\to\mathbb{R}^2$ by
$$f(x,y):=(\varphi(y),\varphi(x)) \quad ((x,y)\in\Omega).$$
Then $\overline{D}$ is invariant under f, $(0,0)$ is the unique fixed point of f in $\overline{D}$, and
$$f'(x,y)=\begin{pmatrix} 0 & \varphi'(y)\\ \varphi'(x) & 0\end{pmatrix} \quad ((x,y)\in\overline{D}).$$
It is readily seen that $r(f'(x,y))=\sqrt{|\varphi'(x)\varphi'(y)|}=0$ $(|x|+|y|\le 1)$. Hence
$$\sup_{(x,y)\in\overline{D}} r(f'(x,y))=0.$$
However, $f(0,1)=(1,0)$ and $f(1,0)=(0,1)$, so that $\{f^k(0,1)\}$ cannot converge to the fixed point $(0,0)$. This proves (ii).

(iii) By (i), f has a unique fixed point $\widehat{x}$ in $\overline{D}$. Since $r(f'(\widehat{x}))<1$, by the same argument as in the proof of (i) we can find a norm $|||\cdot|||$ on E equivalent to $\|\cdot\|$ and an open ball
$$U=\{x\in E: |||x-q|||<\delta\}\subset\overline{D}$$
such that f is a *strict contraction* on U relative to the norm $|||\cdot|||$. Thus there exists a positive number $\alpha<1$ such that
$$|||f(x)-\widehat{x}|||\le\alpha|||x-\widehat{x}||| \quad (x\in U).$$
Therefore
$$(*) \qquad f^k(x)\to\widehat{x} \quad \text{as } k\to\infty \text{ for any } x\in U.$$
Since $\overline{D}$ is bounded, $\{f^k\}$ is uniformly bounded on $\overline{D}$. Thus $\{f^k\}$ is equicontinuous on $\overline{D}$ (see, e.g., [9], p. 98). Since f is compact, by Arzelà–Ascoli's theorem for compact maps (see, e.g., [3], p. 267), there exists a subsequence $\{f^{k_i}\}$ of $\{f^k\}$ such that $\{f^{k_i}\}$ converges uniformly on each compact subset of $\overline{D}$. Let
$$g(x):=\lim_{i\to\infty} f^{k_i}(x) \quad (x\in\overline{D}).$$
Then g is holomorphic on $\overline{D}$ (see, e.g., [5], p. 99). By $(*)$, $g(x)=\widehat{x}$ for all $x\in U$. Since U is open, $g(x)=\widehat{x}$ for all $x\in\overline{D}$ by the identity theorem. (In a complex Banach space, the identity theorem can be proved, with no substantial change, as in $\mathbb{C}^n$.) Therefore we can associate to each $x\in\overline{D}$ a positive integer $k(x)$ such that $f^{k(x)}\in U$. Hence for each $l=1,2,\ldots$, by $(*)$ we have
$$f^{l+k(x)}(x)=f^l(f^{k(x)}(x))\to\widehat{x} \quad \text{as } l\to\infty \ (x\in\overline{D}).$$
This proves (iii).

This concludes the proof of Theorem 1.

Note that in the complex case, the spectral condition $\sup_{x\in\overline{D}} r(f'(x))<1$ can be sharpened to $r(f'(x))<1$ for all $x\in\mathrm{Fix}(f)$. We mention here that part (iii) of Theorem 1 should be compared with the uniqueness theorem of H. Cartan (see [6]; see also [5], p. 75, [12], p. 23).

The following result is a converse of part (iii) of Theorem 1 and indicates that the asymptotic behaviour of the iterates of a holomorphic function is similar to that of the iterates of a bounded linear operator.

THEOREM 2. *Let Ω be a non-empty bounded domain in a complex Banach space E, $f:\Omega\to E$ be compact and holomorphic, and $f(\widehat{x})=\widehat{x}$. Then $\widehat{x}$ is globally asymptotically stable if and only if $r(f'(\widehat{x}))<1$.*

Proof. For "$\Leftarrow$" see the proof of (iii) in Theorem 1. We prove "$\Rightarrow$". Since Ω is bounded and f is holomorphic, by the normal family theorem there exists a subsequence $\{f^{k_i}\}$ of $\{f^k\}$ that converges uniformly to the constant function $g(x)\equiv\widehat{x}$ on each compact subset of Ω. Denote by Δ the open unit disk in the complex plane. Fix $z_0\in E$. Then the map $\lambda\mapsto f(\widehat{x}+\lambda z_0)$ is holomorphic in Δ. Let $0<\delta<1$ and $A:=f'(\widehat{x})$. By Cauchy's integral formula,
$$Az_0=\frac{d}{d\lambda}f(\widehat{x}+\lambda z_0)\Big|_{\lambda=0}=\frac{1}{2\pi i}\int_{|\lambda|=\delta}\frac{f(\widehat{x}+\lambda z_0)}{\lambda^2}\,d\lambda.$$
As $f(\widehat{x})=\widehat{x}$, we have
$$A^{k_i}z_0=\frac{d}{d\lambda}f^{k_i}(\widehat{x}+\lambda z_0)\Big|_{\lambda=0},$$
so that
$$A^{k_i}z_0=\frac{1}{2\pi i}\int_{|\lambda|=\delta}\frac{f^{k_i}(\widehat{x}+\lambda z_0)}{\lambda^2}\,d\lambda \quad (i=1,2,\ldots).$$
Since $\{\widehat{x}+\lambda z_0 : |\lambda|=\delta\}$ is compact, we have
$$\lim_{i\to\infty}A^{k_i}z_0=\frac{1}{2\pi i}\int_{|\lambda|=\delta}\lim_{i\to\infty}\frac{f^{k_i}(\widehat{x}+\lambda z_0)}{\lambda^2}\,d\lambda$$
$$=\Big(\frac{1}{2\pi i}\int_{|\lambda|=\delta}\frac{d\lambda}{\lambda^2}\Big)\widehat{x}=0 \quad (i=1,2,\ldots).$$
Since $z_0\in E$ was arbitrary, we conclude that $\lim_{i\to\infty}A^{k_i}z=0$ for all $z\in E$. Since f is compact, $A=f'(\widehat{x})$ is a compact linear operator. Hence there exists a complex number λ in the spectrum of A such that $|\lambda|=r(A)$ and $Av=\lambda v$ for some non-zero $v\in E$. Then $A^{k_i}v=\lambda^{k_i}v\to 0$ as $i\to\infty$, so that $|\lambda|=r(A)<1$.

This completes the proof.

Neither the sufficient condition nor the necessary condition are valid in the real case, as the following one-dimensional examples show, respectively.

EXAMPLE 1. $f(x) = 4x^3(1-x^2)$, $|x| < 1$.

EXAMPLE 2. $f(x) = x - x^3$, $|x| < 1$.

Let us remark that the analogue of Theorem 2 for differential equations was earlier proved by Yu. I. Lyubich [11] and the same method of proof is applicable to iterations. The proof given in Theorem 2 is somewhat different and, formally, Theorem 2 is an infinite-dimensional version.

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Received April 16, 1997
Revised version April 4, 1998 (3902)

STUDIA MATHEMATICA 131 (2) (1998)

The uniform zero-two law for positive operators in Banach lattices

by

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Dedicated to Shaul Foguel upon his retirement

Abstract. Let T be a positive power-bounded operator on a Banach lattice. We prove: (i) If $\inf_n \|T^n(I-T)\| < 2$, then there is a $k \ge 1$ such that $\lim_{n\to\infty} \|T^n(I-T^k)\| = 0$. (ii) $\lim_{n\to\infty} \|T^n(I-T)\| = 0$ if (and only if) $\inf_n \|T^n(I-T)\| < \sqrt{3}$.

In their ground breaking paper [OSu], Ornstein and Sucheston proved that if T is a positive contraction of L_1, then $\sup_{\|f\|_1\le 1} \lim_n \|T^n(I-T)f\|$ is 0 or 2, and coined the term *zero-two* law. Using their method, Foguel [F] proved that if T is a positive contraction of L_1, then $\lim_n \|T^n(I-T)\|$ is 0 or 2 (the *uniform* zero-two law). This easily implies that if T is a positive contraction of $C(K)$ with K compact Hausdorff, then $\lim_n \|T^n(I-T)\|$ is 0 or 2.

Using the regular norm (the norm of the modulus), Zaharopol [Z_1] restated [F] as

$$(*)\qquad \inf_n \|T^{n+1} - T^n\|_r < 2 \Rightarrow \lim_n \|T^{n+1} - T^n\| = 0.$$

He proved $(*)$ for positive contractions of L_p spaces ($1 < p < \infty$), $p \ne 2$. Katznelson and Tzafriri [KT] removed the restriction $p \ne 2$ of [Z_1], and proved $(*)$ for a larger class of Banach lattices. Finally, Schaefer [S_2] proved $(*)$ for a positive contraction T in any Banach lattice.

The reverse implication in $(*)$ is false: a positive contraction in L_p can satisfy $\lim_n \|T^{n+1} - T^n\| = 0$ and $\inf_n \|T^{n+1} - T^n\|_r = 2$ (see [W_2]). For certain Banach lattices, a stronger version of $(*)$, in which the conclusion is $\lim_n \|T^{n+1} - T^n\|_r = 0$, was later proved in [$W_2$], [$Z_2$], [Sc].

In this note we prove that for a power-bounded positive operator T in a Banach lattice, $\inf_n \|T^n(I-T)\| < \sqrt{3}$ implies $\lim_{n\to\infty} \|T^n(I-T)\| = 0$. For contractions in L_p this follows from [W_1] (see also [M]).

1991 *Mathematics Subject Classification*: Primary 47A35.