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Fragmentability and compactness in $C(K)$-spaces

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Abstract. Let K be a compact Hausdorff space, $C_{\mathrm{p}}(K)$ the space of continuous functions on K endowed with the pointwise convergence topology, $D \subset K$ a dense subset and $t_{\mathrm{p}}(D)$ the topology in $C(K)$ of pointwise convergence on D. It is proved that when $C_{\mathrm{p}}(K)$ is Lindelöf the $t_{\mathrm{p}}(D)$-compact subsets of $C(K)$ are fragmented by the supremum norm of $C(K)$. As a consequence we obtain some Namioka type results and apply them to prove that if K is separable and $C_{\mathrm{p}}(K)$ is Lindelöf, then K is metrizable if, and only if, there is a countable and dense subset $D \subset K$ such that $(C(K), t_{\mathrm{p}}(D))$ is analytic. We also show that if K is a separable Rosenthal compact space, then K is metrizable if, and only if, $C_{\mathrm{p}}(K)$ is Lindelöf. We complete our study by showing that if K does not contain a copy of $\beta\mathbb{N}$, then convex $t_{\mathrm{p}}(D)$-compact subsets of $C(K)$ have the weak Radon–Nikodym property.

1. Introduction and results. Throughout, $(X, \| \|)$ will be a real Banach space, X^* its dual and B_X (resp. B_{X^*}) the unit ball of X (resp. of X^*). A subset B of the dual unit ball B_{X^*} is said to be *norming* if $\|x\| = \sup\{|x^*(x)| : x^* \in B\}$ for every $x \in X$. K denotes a compact Hausdorff space and $C(K)$ the Banach space of continuous real-valued functions on K endowed with the supremum norm. If F is a subset of K, we denote by $t_{\mathrm{p}}(F)$ the topology in $C(K)$ of pointwise convergence on F. D will always be a dense subset of K (in this case $t_{\mathrm{p}}(D)$ is a Hausdorff locally convex topology in $C(K)$). $C_{\mathrm{p}}(K)$ stands for $C(K)$ endowed with the topology $t_{\mathrm{p}}(K)$.

The notion of fragmentability as stated below was introduced by Jayne and Rogers.

Definition 1 ([20]). Let (X, τ) be a topological space and ϱ a metric on X. We say that (X, τ) is *fragmented by* ϱ (or *ϱ-fragmented*) if for each

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non-empty subset A of X and for each $\varepsilon > 0$ there exists a non-empty τ-open subset U of X such that $U \cap A \neq \emptyset$ and $\operatorname{diam}_\varrho(U \cap A) \leq \varepsilon$.

The weak compact subsets of a Banach space are norm fragmented and the weak* compact subsets of a dual Banach space X^* are norm fragmented if, and only if, X^* has the Radon–Nikodym Property (briefly, RNP) [24]. A compact Hausdorff space is said to be *Radon–Nikodym compact* if it is homeomorphic to a weak* compact subset of a dual Banach space with the RNP. The paper [24] is a good source of results about fragmented compact spaces.

In this paper we go on with our previous work, [12, 11], about compact subsets of Banach spaces X endowed with topologies of the kind $\sigma(X, B)$, where B is a norming subset of the dual unit ball B_{X^*}, focusing now on the norm fragmentability of $\sigma(X,B)$-compact subsets and its consequences. A space of the kind $(X, \sigma(X,B))$, B norming, can always be realized as a subspace of a suitable space $C(K)$ endowed with the topology $t_{\mathrm{p}}(D)$ on a dense subset $D \subset K$. Our subsequent study will be done in the context of $C(K)$-spaces.

Our main result is the following

Theorem A. *Let K be a compact Hausdorff space such that $C_{\mathrm{p}}(K)$ is Lindelöf, and $D \subset K$ a dense subset. Then every $t_{\mathrm{p}}(D)$-compact subset of $C(K)$ is fragmented by the supremum norm and so it is a Radon–Nikodym compact space.*

In other words, as it is well known that pointwise compact subsets of $C(K)$ are "rather" good (they are Eberlein), we are saying that when $C_{\mathrm{p}}(K)$ is Lindelöf, $t_{\mathrm{p}}(D)$-compact subsets of $C(K)$ are not very "bad". Indeed, being Radon–Nikodym compact spaces they contain metrizable G_δ-dense subsets [24, Theorem 5.2] and so they are sequentially compact [24, Corollary 5.4]. Moreover, combining Theorem A and Theorem 10 of [12], we deduce that when $C_{\mathrm{p}}(K)$ is Lindelöf, $t_{\mathrm{p}}(D)$-compact convex subsets of $C(K)$ have the usual RNP.

Let us recall that an important class of compact spaces K for which $C_{\mathrm{p}}(K)$ is Lindelöf (see [3]) is the class of Corson compact spaces, that is, those compact spaces which are homeomorphic to pointwise compact subsets of Σ-products of real lines. For Corson compact spaces Theorem A was proved in [12].

The Banach counterpart of Theorem A is

Theorem B. *Let $(X, \|\ \|)$ be a weakly Lindelöf Banach space and $B \subset B_{X^*}$ a norming subset. Then every $\sigma(X,B)$-compact subset of X is fragmented by the norm and so it is a Radon–Nikodym compact space.*

Observe that Theorem B applied to a dual Banach space $X = Y^*$ and $B = B_Y$ gives us the well-known result saying that Y^* has the RNP when it is weakly Lindelöf [13].

Theorems A and B have some interesting consequences. As usual, given a topological space T and a Banach space $(X, \|\ \|)$ we denote by $B_1(T, X)$ the space of Baire-1 functions from T into X, that is, the space of functions $f : T \to X$ which are the pointwise-norm limits of sequences of norm continuous functions from T into X.

Corollary C. *Let K be a compact Hausdorff space such that $C_{\mathrm{p}}(K)$ is Lindelöf, and $D \subset K$ a dense subset. If T is a complete metric space and $f : T \to C(K)$ is a $t_{\mathrm{p}}(D)$-continuous function, then $f \in B_1(T, C(K))$.*

Let us remark here that for K metrizable this corollary was proved in [2]. For non-separable Banach spaces it is known that if T is a metric space, X^* has the RNP and $f : T \to X^*$ is a weak* continuous function, then $f \in B_1(T, X^*)$ [20, Theorem 8].

Corollary D. *Let K be a separable compact Hausdorff space such that $C_{\mathrm{p}}(K)$ is Lindelöf. The following are equivalent:*

(i) *K is metrizable.*

(ii) *For every countable dense subset D of K, the space $C(K)$ is $t_{\mathrm{p}}(D)$-analytic.*

(iii) *There is a countable dense subset D of K such that $C(K)$ is $t_{\mathrm{p}}(D)$-analytic.*

Given a compact Hausdorff and separable space K such that $C_{\mathrm{p}}(K)$ is Lindelöf it is undecidable whether K is metrizable or not. Indeed, Reznichenko's result [5, p. 32] states that under Martin's Axiom and $\neg$CH such a compact K is metrizable, but under CH Kunen gave an example [5, p. 31] of a non-metrizable compact space K such that K^n is hereditarily separable for every $n \in \mathbb{N}$ and $C_{\mathrm{p}}(K)$ is hereditarily Lindelöf. Observe that given K such that $C_{\mathrm{p}}(K)$ is Lindelöf if we take $D \subset K$ countable and dense then $(C(K), t_{\mathrm{p}}(D))$ is always metrizable and separable and, as said before, we cannot decide the metrizability of K. Bearing this in mind and by Corollary D, we can say that the gap to get the metrizability of K appears as the gap between separable metrizable spaces and analytic spaces.

From a different point of view Corollary D is telling us that given a compact space K such that $C_{\mathrm{p}}(K)$ is Lindelöf, separable compact subsets of K are metrizable if, and only if, for every countable subset $A \subset K$ the space $(C(\bar{A}), t_{\mathrm{p}}(A))$ is analytic.

A natural class to apply Corollary D is the class of Rosenthal compact spaces, that is, the compact spaces which are homeomorphic to pointwise compact subsets of the space of Baire-1 functions on a Polish space [15], for which we obtain

COROLLARY E. *A separable Rosenthal compact space K is metrizable if, and only if, $C_p(K)$ is Lindelöf.*

This result improves a previous one of [16], where the metrizability of K was obtained under the hypothesis of $C(K)$ being weakly Lindelöf. Let us mention that Corollary E that appears as a consequence of Theorem A is in fact equivalent to it.

COROLLARY F. *Let K be a compact Hausdorff space such that $C_p(K)$ is Lindelöf, and $D \subset K$ a dense subset. If T contains a dense Čech-complete subspace and $f : T \to C(K)$ is a $t_p(D)$-continuous function, then f is norm continuous at each point of a dense G_δ subset of T.*

By a classical result by Namioka [23], when $D = K$ in the previous corollary the hypothesis $C_p(K)$ Lindelöf can be avoided.

The topological spaces which contain a dense Čech-complete subspace are Baire spaces. For Corson compact spaces Corollary F is true for any Baire space T [9, Proposition 1.5]. In [1], the completely regular topological spaces that contain a G_δ dense Čech-complete subspace are called *almost Čech-complete spaces*. The class of almost Čech-complete spaces contains the Baire spaces which are K-analytic or more generally the Baire spaces which are Čech-analytic [10, Remarks 2.5].

2. Fragmentability. In this section we prove Theorems A and B. We will use, among other things, the proposition below for which a proof can be straightforwardly adapted from the one of Theorem 3.4 in [24] to which we refer the interested reader.

PROPOSITION 1. *Let H be a $t_p(D)$-compact subset of $C(K)$. Then $(H, t_p(D))$ is fragmented by the norm of $C(K)$ if, and only if, for each countable subset A of D, the set $R_{\overline{A}}(H) = \{f|_{\overline{A}} : f \in H\}$ of restrictions is separable in the Banach space $C(\overline{A})$.*

In view of this proposition our starting point to get fragmentability must be the study of the case $(C(K), t_p(D))$ where $D \subset K$ is dense and countable. Observe that in this case if $H \subset C(K)$ is a $t_p(D)$-compact subset then $(H, t_p(D))$ is metrizable and so in particular H is $t_p(D)$-separable. To get the fragmentability of H we need to lift the $t_p(D)$-separability of H to its $t_p(K)$-separability ($\Leftrightarrow$ norm separability). So, our original fragmentability problem can be reduced to the topological one of "lifting" the separability of a compact metrizable space, say (Z, τ), to the separability of the space endowed with a finer topology, say $(Z, \mathcal{G})$, which is far from being obvious in the known positive cases and unfortunately not true in general: in this topological setting it is known that $(Z, \mathcal{G})$ is separable when $(Z, \mathcal{G})$ is countably determined [27, Theorem 2.4], and that the separability is not lifted even when $(Z, \mathcal{G})$ is assumed to be Lindelöf.

As a consequence of these considerations, we can already conclude that Theorem A easily follows from Proposition 1 and [27, Theorem 2.4] when $C_p(K)$ is assumed to be countably determined, and that the general Lindelöf case stated there will have to involve some more intricate techniques.

Given a $t_p(D)$-compact subset H of $C(K)$ we will frequently look at elements of K as functions on H: For each point x in K we will denote by $\widehat{x}$ the restriction to H of the "point mass" at x, that is, $\widehat{x}(f) := f(x)$ for every $f \in H$. It is clear that $\widehat{D} = \{\widehat{d} : d \in D\}$ is a pointwise bounded set of continuous functions on the compact space $(H, t_p(D))$, and $\widehat{K} = \{\widehat{x} : x \in K\}$ is a pointwise bounded set of continuous functions on the topological space $(H, t_p(K))$. Obviously, the closure of $\widehat{D}$ in $\mathbb{R}^H$ is the compact set $\widehat{K}$.

The concept of an independent sequence of functions as appears below was introduced by Rosenthal [25] (see also [7]).

DEFINITION 2. A sequence (f_n) of functions in $\mathbb{R}^\Omega$ is called *independent on $A \subset \Omega$* if there are numbers $s < t$ such that for each pair of finite disjoint subsets $P, Q \subset \mathbb{N}$ we have

$$\Big[\bigcap_{n \in P} \omega \in A : f_n(\omega) < s\Big] \cap \Big[\bigcap_{n \in Q} \omega \in A : f_n(\omega) > t\Big] \neq \emptyset.$$

LEMMA 1. *Let K be a separable compact Hausdorff space, $D \subset K$ a countable dense subset and H a $t_p(D)$-compact subset of $C(K)$. If H is $t_p(K)$-Lindelöf and $\widehat{D}$ does not contain sequences independent on H, then H is norm fragmented (equivalently, norm separable).*

Proof. As $H = \bigcup_{n=1}^\infty H_n$, where $H_n = \{h \in H : \|h\| \le n\}$ we can and do assume that H is uniformly bounded. The proof will be done in two steps.

STEP 1. $\mathrm{Borel}(H, t_p(D)) = \mathrm{Baire}(H, t_p(K))$.

Here, as usual, Borel and Baire stand respectively for the σ-algebras of Borel and Baire sets. Since $(H, t_p(D))$ is metrizable, it is clear that

$$\mathrm{Borel}(H, t_p(D)) = \mathrm{Baire}(H, t_p(D)) \subset \mathrm{Baire}(H, t_p(K)).$$

On the other hand, the fact that $(H, t_p(K))$ is Lindelöf allows us to obtain, from a result of [22],

$$\mathrm{Baire}(H, t_p(K)) = H \cap \mathrm{Baire}(C(K), t_p(K)). \tag{1}$$

As $\mathrm{Baire}(C(K), t_p(K))$ is the σ-algebra generated by the "point mass" functionals $\{\delta_x : x \in K\}$ (see [13]), equation (1) implies that $\mathrm{Baire}(H, t_p(K))$ is the σ-algebra generated by $\widehat{K} = \{\delta_x|_H : x \in K\}$. On the other hand, the hypothesis that $\widehat{D} \subset C(H, t_p(D))$ does not contain independent sequences on H implies, by [7, Theorem 2F and Proposition 1E], that $\widehat{D}$ is relatively compact in $B_1(H, t_p(D))$ endowed with its pointwise convergence topology. Thus $\widehat{K} \subset B_1(H, t_p(D))$ and hence $\mathrm{Baire}(H, t_p(K)) \subset \mathrm{Baire}(H, t_p(D))$, which finishes this first step.

Step 2. *For every Radon probability μ on $(H, t_p(D))$ there is a norm separable set $S \in \mathrm{Borel}(H, t_p(D))$ with $\mu(S) = 1$.*

By our first step we can look at μ as a measure on $\mathrm{Baire}(H, t_p(K))$. Being a Baire measure on the Lindelöf space $(H, t_p(K))$, μ is τ-smooth [32, Cor. 4, p. 175], and so it has a canonical Borel extension that has a $t_p(K)$-closed non-empty support $S \subset H$ (see [13]). The map

$$\phi : (\widehat{K}|_S, t_p(S)) \to (L^1(\mu), \|\ \|_1), \quad \widehat{x}|_S \to \widehat{x},$$

is a homeomorphism onto its image. Indeed, since $(\widehat{K}|_S, t_p(S))$ is compact and ϕ is injective it suffices to prove that ϕ is continuous, that is, that for every $A \subset K$ if $\widehat{x}|_S$ is $t_p(S)$-adherent to $\widehat{A}|_S$, then $\widehat{x}$ is $\|\ \|_1$-adherent to $\widehat{A}$. Given $\widehat{x}|_S$, $t_p(S)$-adherent to $\widehat{A}|_S$, the compactness of $\widehat{K}$ allows us to assume that $\widehat{x}$ is in fact $t_p(H)$-adherent to $\widehat{A}$. As said before, $\widehat{K}$ is a compact subset of the space $B_1(H, t_p(D))$ endowed with the pointwise convergence topology, which is an angelic space [7]. So, there exists a sequence $(a_n) \subset A$ such that $\widehat{x}(h) = \lim_{n\to\infty} \widehat{a}_n(h)$ for every $h \in H$ and now the boundedness of H allows us to use the Lebesgue Convergence Theorem to deduce that $\lim_{n\to\infty} \|\widehat{a}_n - \widehat{x}\|_1 = 0$, which implies that $\widehat{x}$ is $\|\ \|_1$-adherent to $\widehat{A}$ and the proof of the continuity of ϕ is finished. Since ϕ is a homeomorphism, $(\widehat{K}|_S, t_p(S))$ is metrizable, which implies that S is $t_p(K)$-separable and hence norm separable. As S is both norm closed and norm separable, S is actually a Borel set for $(H, t_p(D))$ because

$$S = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (\overline{B(f_m, 1/n)} \cap H)$$

where $\overline{B(f,r)}$ stands for the norm closed ($t_p(D)$-closed!) ball centered at f with radius r, and (f_m) is a dense sequence in S. It is clear that $\mu(S) = 1$ and this second step is finished.

Step 3. *$(H, t_p(D))$ is norm fragmented.*

The proof is by contradiction. Suppose $(H, t_p(D))$ is not fragmented by the norm. Then by [18] there is a $t_p(D)$-compact subset C of H and a continuous map p from C onto $2^{\mathbb{N}}$ and $\varepsilon > 0$ such that whenever $\sigma, \sigma' \in 2^{\mathbb{N}}$, $\sigma \neq \sigma'$, then $p^{-1}(\sigma) \neq p^{-1}(\sigma')$ are separated by the norm distance $\geq \varepsilon$. Since $(C, t_p(D))$ is Polish, by [31, Theorem 1.3.6] there is a subset $A \subset C$ homeomorphic to the Cantor space such that $p|_A$ is a homeomorphism from A onto $p(A)$. Since A is $t_p(K)$-closed in H, A is $t_p(K)$-Lindelöf and $\widehat{D}$ does not contain sequences independent on A. So we can assume that $H = A$ in what follows.

Let μ be the Haar measure of $2^{\mathbb{N}}$ transferred onto H via p. Let $S \subset H$ be the norm separable set ensured by our second step. Since the elements of H are separated by at least ε in norm, S must be countable. Hence $\mu(S) = 0$, a contradiction that finishes the proof. ∎

In the next lemma we establish a useful relation between bounded independent sequences of continuous functions defined on a compact space and the Stone–Čech compactification of the natural numbers, $\beta\mathbb{N}$. We will look at $\beta\mathbb{N}$ as the set of all ultrafilters in $\mathbb{N}$ endowed with the topology whose open sets are $\{V(E) : E \subset \mathbb{N}\}$ where $V(E) = \{\alpha \in \beta\mathbb{N} : E \in \alpha\}$. The injection of $\mathbb{N}$ in $\beta\mathbb{N}$ is given by

$$i : \mathbb{N} \to \beta\mathbb{N}, \quad n \to \alpha_n,$$

where α_n consists of all subsets of $\mathbb{N}$ that contain n. Given a compact space K and a mapping $f : \mathbb{N} \to K$ its continuous extension $\bar{f} : \beta\mathbb{N} \to K$ is simply obtained by taking limits along ultrafilters, that is,

$$\bar{f}(\alpha) := \alpha\text{-}\lim_n f(n).$$

Lemma 2. *Let K be a compact Hausdorff space, $D \subset K$ a dense subset, H a $t_p(D)$-compact subset of $C(K)$ and $(d_n)_{n\in\mathbb{N}}$ a sequence in D. If the sequence $\{\widehat{d}_n : n \in \mathbb{N}\}$ is independent on H, then the closure of $\{d_n : n \in \mathbb{N}\}$ in K is homeomorphic to $\beta\mathbb{N}$.*

Proof. If we consider the map

$$f : \mathbb{N} \to K, \quad n \to d_n,$$

then its continuous extension $\bar{f} : \beta\mathbb{N} \to K$ satisfies $\bar{f}(\beta\mathbb{N}) = \overline{\{d_n : n \in \mathbb{N}\}}$. We prove that $\bar{f}$ is actually a homeomorphism onto F. To show this it is enough to check that $\bar{f}$ is injective, and this last fact follows from the independence of $\{\widehat{d}_n : n \in \mathbb{N}\}$ on H. Indeed, the $t_p(D)$-compactness of H and the independence of the $t_p(D)$-continuous functions $\{\widehat{d}_n : n \in \mathbb{N}\}$ imply the existence of real numbers $s < t$ such that for every pair of disjoint subsets P and Q of $\mathbb{N}$ we have

$$(2) \qquad \Big[\bigcap_{n\in P} \{h \in H : h(d_n) < s\}\Big] \cap \Big[\bigcap_{n\in Q} \{h \in H : h(d_n) > t\}\Big] \neq \emptyset.$$

Given two different ultrafilters $\alpha, \beta \in \beta\mathbb{N}$ there are two disjoint subsets $P, Q \subset \mathbb{N}$ such that $P \in \alpha$ and $Q \in \beta$. Applying (2) to P and Q we get a function $h \in H$ such that for the limits along the corresponding ultrafilters we have

$$\text{(3)} \qquad \alpha\text{-}\lim_n h(d_n) \le s \quad \text{and} \quad \beta\text{-}\lim_n h(d_n) \ge t.$$

Since h is continuous on K and

$$\bar{f}(\alpha) = \alpha\text{-}\lim_n d_n \quad \text{and} \quad \bar{f}(\beta) = \beta\text{-}\lim_n d_n$$

(3) gives us

$$h(\bar{f}(\alpha)) \le s \quad \text{and} \quad h(\bar{f}(\beta)) \ge t,$$

which implies that $\bar{f}(\alpha) \ne \bar{f}(\beta)$ and the proof is finished. ■

A glance at the above lemma is enough to convince oneself that if H is a compact space and $(f_n) \subset C(H)$ is a pointwise bounded sequence which is independent on H, then $\overline{\{f_n : n \in \mathbb{N}\}}^{\mathbb{R}^H}$ is homeomorphic to $\beta\mathbb{N}$; see [33] for a different proof of this fact and some applications.

$\beta\mathbb{N}$ is a "big" compact space which is not sequentially compact [14, Cor. 3.6.15], which has cardinality $2^{\mathfrak{c}}$ [14, Cor. 3.6.12] and its tightness and weight are $\mathfrak{c}$ [6, p. 222]. Consequently, sequentially compact spaces, compact spaces of weight less than $\mathfrak{c}$ and compact spaces with countable tightness do not contain a copy of $\beta\mathbb{N}$. Those compact spaces K for which $(C(K), t_p(K))$ is Lindelöf have countable tightness [27, Prop. 6.3], and so we finally proceed to

Proof of Theorem A. As quoted before, the hypothesis $C_p(K)$ Lindelöf implies that K does not contain homeomorphic copies of $\beta\mathbb{N}$. To prove that a $t_p(D)$-compact subset H of $C(K)$ is fragmented it will be enough to prove, by Proposition 1, that for any countable subset $A \subset D$ the set of restrictions

$$R_{\overline{A}}(H) = \{f|_{\overline{A}} : f \in H\}$$

is separable in the Banach space $C(\overline{A})$. To prove this last fact, observe that $\widehat{A}$ does not contain sequences independent on $R_{\overline{A}}(H)$ because otherwise $\overline{A}$, and therefore K, would contain a copy of $\beta\mathbb{N}$ by Lemma 2. Moreover, $R_{\overline{A}}(H)$ is $t_p(\overline{A})$-Lindelöf and Lemma 1 can be applied to show that $R_{\overline{A}}(H)$ is norm separable and the proof is done. The fact that H is a Radon–Nikodym compact space follows from [24, Corollary 6.7] if we bear in mind that the supremum norm of $C(K)$ is $t_p(D)$-lower semicontinuous. ■

Proof of Theorem B. To prove this theorem it is not restrictive to assume that B is absolutely convex because we can, and do, replace B by its absolutely convex hull. For B absolutely convex, the hypothesis B norming implies that B is weak* dense in B_{X^*}. The hypothesis X weakly Lindelöf tells us that ℓ^∞ cannot be a quotient of X whence we conclude that (B_{X^*}, weak^*) does not contain a copy of $\beta\mathbb{N}$ (see [29]). Now, keeping in mind the natural

inclusion

$$(X, \text{weak}) \hookrightarrow C_p(B_{X^*})$$

we can finish this proof combining Lemmas 2, 1 and Proposition 1 as we did in the previous proof. ■

3. Applications and concluding remarks. We devote this section to proving the consequences of Theorems A and B that we presented in the introduction as well as to showing some other applications.

Proof of Corollary C. To prove that $f \in B_1(T, C(K))$ it is enough to prove that for every compact subset W of T the restriction $f|_W$ has a point of norm continuity [26]. Given a compact $W \subset T$, the image $f(W)$ is $t_p(D)$-compact and so norm fragmented by Theorem A. According to [24, Lemma 1.1] the identity map

$$\mathrm{id} : (f(W), t_p(D)) \to (f(W), \|\ \|)$$

has a point of continuity and thus we see that $f|_W$ has a point of norm continuity and the proof is finished. ■

Proof of Corollary D. (i)⇒(ii). If K is metrizable then $(C(K), \|\ \|)$ is a Polish space and thus its continuous image $(C(K), t_p(D))$ is analytic.

(ii)⇒(iii). This is obvious.

(iii)⇒(i). If (iii) holds, there is a Polish space P and a continuous onto map $f : P \to (C(K), t_p(D))$. Corollary C applied to f ensures the existence of a sequence $f_n : P \to C(K)$ of norm continuous functions such that $f(p) = \|\ \|\text{-}\lim_n f_n(p)$ for each $p \in P$. Every $f_n(P)$ is norm separable in $C(K)$ and so

$$C(K) = f(P) = \overline{\bigcup_{n=1}^{\infty} f_n(P)}^{\|\ \|}$$

is norm separable, which implies the metrizability of K. ■

Proof of Corollary E. If K is metrizable then $C_p(K)$ is Lindelöf. Conversely, if K is a separable Rosenthal compact space, then for every dense and countable subset $D \subset K$ the space $(C(K), t_p(D))$ is analytic by [15, Théorème 4] and so Corollary D can be applied to conclude that K is metrizable. ■

Proof of Corollary F. First assume that T is *Čech-complete*. Define

$$d(x, y) := \|f(x) - f(y)\|_\infty$$

for every $x, y \in T$. Then d is a lower semicontinuous pseudometric for which the compact subsets for the original topology of T are fragmented. Indeed, d is lower semicontinuous with respect to the topology of T because D is a dense subset of K and f is $t_p(D)$-continuous. On the other hand, if H

is a compact subset of T, then $f(H)$ is a $t_p(D)$-compact subset of $C(K)$ which is $\|\ \|_\infty$-fragmented by Theorem A, which means exactly that H is d-fragmented. Now, the proof of 4.1 in [20] can be followed for the pseudometric d and the Čech-analytic space T to deduce the following

CLAIM 1. *For every $\varepsilon > 0$ there is a sequence (T_n) of subsets of T such that*

(α) $T = \bigcup_{n=1}^\infty T_n$,
(β) *for every $n \in \mathbb{N}$, if C is a non-empty subset of T_n then there is an open set $V \subset T$ such that $C \cap V \neq \emptyset$ and* $\|\ \|$-diam$(f(C \cap V)) < \varepsilon$.

Now we make another claim:

CLAIM 2. *For every $\varepsilon > 0$ and for every non-empty open subset W of T there is a non-empty open subset V of W such that* $\|\ \|$-diam$(f(V)) \le \varepsilon$.

Indeed, if we assume that Claim 2 does not hold then there are $\varepsilon > 0$ and an open subset $W \subset T$ such that

(4) $\|\ \|$-diam$(f(V)) > \varepsilon$ for every non-empty open subset $V \subset W$.

For this $\varepsilon > 0$ take the sequence (T_n) ensured by Claim 1. As $W = \bigcup_{n=1}^\infty \overline{T}_n \cap W$ and W is a Baire space, there are $n \in \mathbb{N}$ and a non-empty open set U such that $U \subset \overline{T}_n \cap W$. We can apply property (β) in Claim 1 to the non-empty set $U \cap T_n$ to get an open set V such that $U \cap T_n \cap V \neq \emptyset$ and $\|\ \|$-diam$(f(U \cap T_n \cap V)) \le \varepsilon$. Now $U \cap V$ is a non-empty open subset of W that satisfies

$$f(U \cap V) = f(U \cap \overline{T}_n \cap V) \subset f(\overline{U \cap T_n \cap V}) \subset \overline{f(U \cap T_n \cap V)}^{t_p(D)}.$$

Clearly, $f(U \cap T_n \cap V)$ and $\overline{f(U \cap T_n \cap V)}^{t_p(D)}$ have the same $\|\ \|$-diameter and so

$$\|\ \|\text{-diam}(f(U \cap V)) \le \varepsilon,$$

which contradicts (4) and finishes the proof of the claim.

Now,

$$O_n(f) := \{V \subset T : V \text{ is open and } \|\ \|\text{-diam}(f(V)) < 1/n\}$$

is a dense open subset of T by Claim 2, and thus the set of points of norm continuity of f, $\bigcap_{n=1}^\infty O_n(f)$, is a dense G_δ-set because T is a Baire space.

We now turn to the general case: *T contains a dense Čech-complete subspace*. Let us prove that f satisfies Claim 2. Take a dense and Čech-complete subspace $T_0 \subset T$. We can use Claim 2 for $f|_{T_0}$ and so, given $\varepsilon > 0$ and a non-empty open set $W \subset T$ there is a non-empty open set $V \subset W$ such that

$$\|\ \|\text{-diam}(f(V \cap T_0)) \le \varepsilon.$$

As $f(V) \subset f(\overline{V \cap T_0}) \subset \overline{f(V \cap T_0)}^{t_p(D)}$ we have $\|\ \|$-diam$(f(V)) \le \varepsilon$, which means that f satisfies Claim 2. Now, as T is a Baire space, because it contains a dense Baire subspace, the proof can be concluded as in the previous case. ∎

The condition that K does not contain homeomorphic copies of $\beta\mathbb{N}$ has played an important role, through Lemma 2, for the results in this paper. As a consequence of Lemma 2 and [7, Theorem 2F] we readily obtain

PROPOSITION 2. *Let K be a compact space, $D \subset K$ a dense subset and H a $t_p(D)$-compact subset of $C(K)$. Then the following are equivalent:*

(i) *For every sequence (d_n) in D there is a subsequence (d_{n_j}) such that $h(d_{n_j})$ converges for every h in H.*
(ii) *For every sequence (d_n) in D there is a subsequence (d_{n_j}) such that $\overline{\{\widehat{d}_{n_j} : j \in \mathbb{N}\}}^{t_p(H)}$ is not homeomorphic to $\beta\mathbb{N}$.*
(iii) *For every sequence (d_n) in D, $(\widehat{d}_n)$ is not independent on H.*

The bounded $t_p(D)$-compact subsets of $C(K)$ with property (i) of this proposition have been called *$P(D)$-sets* in our paper [12]. In that paper we proved some results for $P(D)$-sets that can now be specified for $C(K)$-spaces as follows:

PROPOSITION 3. *Let K be a compact space, and $D \subset K$ a dense subset such that for every countable subset $C \subset D$ the closure $\overline{C}$ is not homeomorphic to $\beta\mathbb{N}$. Then*

(i) *The Krein–Shmul'yan theorem holds for each norm bounded $t_p(D)$-compact subset H of $C(K)$, i.e., $\overline{\mathrm{co}(H)}^{t_p(D)}$ is $t_p(D)$-compact and $\overline{\mathrm{co}(H)}^{\|\ \|} = \overline{\mathrm{co}(H)}^{t_p(D)}$.*
(ii) *Every norm bounded $t_p(D)$-compact convex subset H of $C(K)$ is the norm closed convex hull of its extreme points, that is, $H = \overline{\mathrm{co}(\mathrm{Ext}(H))}^{\|\ \|}$.*
(iii) *Every $t_p(D)$-compact subset H of $C(K)$ is weakly fragmented, i.e., for every non-empty $t_p(D)$-closed subset F of H and $x^* \in C(K)^*$ the restriction of x^* to $(F, t_p(D))$ has a point of continuity.*
(iv) *Every norm bounded $t_p(D)$-compact convex subset of $C(K)$ has the weak Radon–Nikodym property.*

Proof. Lemma 2 implies that D satisfies the equivalent conditions in Proposition 2. Properties (i), (ii) and (iv) follow now from Corollaries 5.2 and 5.3 and Theorems 7 and 8 of [12].

Let us prove (iii). $\widehat{D} \subset C(H)$ satisfies any of the equivalent conditions in [7, Theorem 2F]. By [7, Proposition 5I] the absolutely convex hull of $\widehat{D}$, $\Gamma(\widehat{D})$, also satisfies all the equivalent conditions in [7, Theorem 2F], which means by [7, Lemma 1C] that $B_{C(K)^*}|_H$ is made up of functions which have $t_p(D)$-points of continuity on every closed subset F of $(H, t_p(D))$. ∎

Observe that the previous proposition can be applied to compact spaces K not containing homeomorphic copies of $\beta\mathbb{N}$. Property (i) in the former proposition has been proved in [11] when $C(K)$ does not contain an isomorphic copy of $\ell^1(\mathfrak{c})$. Combining [17, Lemma 1.1] with [21, Theorem 2, p. 111] one can convince oneself that the hypothesis that K does not contain a copy of $\beta\mathbb{N}$ is equivalent to $C(K)$ not containing an isometric copy of $\ell^1(\mathfrak{c})$. Under Martin's Axiom there are compact spaces K such that $C(K)$ does not contain isometric copies of $\ell^1(\mathfrak{c})$ but does contain isomorphic copies of $\ell^1(\mathfrak{c})$ (see [28]) and thus Proposition 3 cannot be derived from the results of [12] proved for Banach spaces not containing isomorphic copies of $\ell^1(\mathfrak{c})$.

Let us also remark that there are compact spaces K with a dense subset $D \subset K$ such that for every countable subset $C \subset D$ the closure $\overline{C} \neq \beta\mathbb{N}$ but K does contain a copy of $\beta\mathbb{N}$. Indeed, take a set Γ with the cardinality of the real numbers and $K := [0,1]^\Gamma$. The set

$$D := \{(x_\gamma) \in K : \{\gamma \in \Gamma : x_\gamma \neq 0\} \text{ is countable}\}$$

is dense in K, every countable subset of D has a closure in K which is metrizable, and so it is not homeomorphic to $\beta\mathbb{N}$, but K contains a copy of $\beta\mathbb{N}$ by [29].

REMARK 1. It is natural to ask if we can replace in Theorem A our assumption on $C_p(K)$ by an assumption on H and still get its fragmentability. We are able to do this in the following cases:

(i) *K does not contain homeomorphic copies of $\beta\mathbb{N}$ and H is $t_p(K)$-Lindelöf.* Under this hypothesis the fragmentability of H is obtained as a combination of Proposition 1, Lemma 1 and Lemma 2.

(ii) *H is convex and $t_p(K)$-Lindelöf.* To prove that H is fragmented it is enough to prove that $\widehat{D}$ does not have sequences independent on H and then apply Proposition 1 and Lemma 1. Let us prove that $\widehat{D}$ does not have sequences independent on H. Indeed, if (d_n) is a sequence in D such that $(\widehat{d_n})$ is independent on H, then the map

$$\phi : H \to \ell^\infty(= C(\beta\mathbb{N})), \quad f \to (f(d_n)),$$

is $t_p(K)$-$t_p(\beta\mathbb{N})$-continuous and arguments similar to those in [30, 7-3-5] yield $\varepsilon > 0$ such that $[-\varepsilon, \varepsilon]^{\mathbb{N}} \subset \phi(H)$. This implies that $(C(\beta\mathbb{N}), t_p(\beta\mathbb{N}))$ is Lindelöf, which contradicts the fact that $\beta\mathbb{N}$ does not have countable tightness.

(iii) *H is weakly Lindelöf.* As in the previous case, to prove the fragmentability of H, it is enough to prove that there is no sequence (d_n) in D

such that $(\widehat{d_n})$ is independent on H. If there is such a (d_n) we can apply [11, Lemma B] to ensure that H contains a family $(x_\alpha)_{\alpha\in\mathfrak{c}}$ equivalent to the usual basis of $\ell^1(\mathfrak{c})$. Define $Y := \overline{\operatorname{span}}\{x_\alpha : \alpha \in \mathfrak{c}\}$ in $C(K)$ and for any $A \subset \mathfrak{c}$ countable, consider

$$R_A = \Big\{x = \sum_{\alpha\in\mathfrak{c}} \lambda_\alpha x_\alpha \in H \cap Y : \sum_{\alpha\in\mathfrak{c}} \lambda_\alpha = 1,\ \lambda_\gamma = 0 \text{ for all } \gamma \in A\Big\}.$$

The sets R_A are non-void, weakly closed, have the countable intersection property, but have void intersection, which contradicts the fact that $H \cap Y$ is weakly Lindelöf.

This result applied to dual Banach spaces tells us that weakly Lindelöf weak* compact convex subsets have the RNP, a result that can be found in [8].

We naturally arrive at the following

PROBLEM 1. *Let H be a $t_p(D)$-compact subset of $C(K)$. If H is $t_p(K)$-Lindelöf, is H fragmented by the norm of $C(K)$?*

Observe that we always have $\widehat{D} \subset \widehat{K} \subset C(H, t_p(K))$. A glance at the proof of Lemma 2 allows us to conclude that if we assume that there is a sequence $(d_n) \subset D$ such that $(\widehat{d_n})$ is independent on H then $\widehat{K}$ contains a copy of $\beta\mathbb{N}$ and hence so does $C(H, t_p(K))$. So, by Lemma 1, Problem 1 will have an affirmative solution if the problem below has an affirmative solution.

PROBLEM 2. *Is it true that $\beta\mathbb{N}$ cannot be embedded as a subspace of some $C_p(T)$ for T Lindelöf?*

Topologists use the term "co-Lindelöf spaces" for those topological spaces Y for which there is a Lindelöf space T such that Y is homeomorphic to a subspace of $C_p(T)$. With this terminology we are asking if $\beta\mathbb{N}$ is not a co-Lindelöf space. If the Proper Forcing Axiom holds then every co-Lindelöf compact space has countable tightness [4], and so, in this case, $\beta\mathbb{N}$ is not a co-Lindelöf space and both Problems 1 and 2 have affirmative answers. As the Proper Forcing Axiom is consistent with the axioms in ZF we conclude that there are no counterexamples to Problems 1 and 2 in ZF. ∎

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