[4] G. Szegő, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. 23, Providence, R.I., 1975.
[5] R. Szwarc, *An analytic series of irreducible representations of the free group*, Ann. Inst. Fourier (Grenoble) 38 (1988), no. 1, 87–110.
[6] —, *A short proof of the Grigorchuk-Cohen cogrowth theorem*, Proc. Amer. Math. Soc. 106 (1989), 663–665.
[7] S. Wagon, *Elementary problem E 3226*, Amer. Math. Monthly 94 (1987), 786–787.
[8] W. Woess, *Cogrowth of groups and simple random walks*, Arch. Math. (Basel) 41 (1983), 363–370.

Institute of Mathematics
Wrocław University
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: szwarc@math.uni.wroc.pl

Received November 3, 1997 (3988)

STUDIA MATHEMATICA 131 (1) (1998)

The minimum diagonal element of a positive matrix

by

M. R. F. SMYTH (Belfast) and T. T. WEST (Dublin)

Abstract. Properties of the minimum diagonal element of a positive matrix are exploited to obtain new bounds on the eigenvalues thus exhibiting a spectral bias along the positive real axis familiar in Perron–Frobenius theory.

The (i,j)th entry of an $n\times n$ matrix T is written $[T]_{ij}$. The matrix T is *positive* $(T\ge 0)$ if $[T]_{ij}\ge 0$ $(\forall i,j)$ while T is *strictly positive* $(T>0)$ if $[T]_{ij}>0$ $(\forall i,j)$. The spectrum, or set of eigenvalues of T, is denoted $\sigma(T)$, the spectral radius $r(T)$ and the peripheral spectrum

$$\operatorname{Per}\sigma(T)=\{\lambda\in\mathbb{C}:\lambda\in\sigma(T),\ |\lambda|=r(T)\}.$$

The trace of T will be written $\operatorname{tr}(T)$, and the complex field $\mathbb{C}$.

This paper combines properties of the minimum diagonal element $\varepsilon(T)$ of a positive matrix T,

$$\varepsilon(T)=\min_{1\le i\le n}[T]_{ii},$$

with elementary spectral theory to show that $\sigma(T)$ lies inside the disc centred at $(\varepsilon(T),0)$ with radius $r(T)-\varepsilon(T)$ (Proposition 6 and Figure 1) generalising a result known for stochastic matrices ([2], III.3.4.1). Various improvements of this result are then considered.

We start with the elementary properties of $\varepsilon(T)$ for positive T, and we note that $S\ge T\ge 0$ implies that $r(S)\ge r(T)$.

LEMMA 1. (i) *If* $T\ge 0$ *then* $\varepsilon(T)\le n^{-1}\operatorname{tr}(T)\le r(T)$.

(ii) *If* $S,T\ge 0$ *then* $\varepsilon(ST)\ge\varepsilon(S)\varepsilon(T)$.

Proof. Property (i) follows from the fact that the trace of T is the sum of the eigenvalues of T repeated according to multiplicity. For (ii) we have

$$[ST]_{ii}=\sum_{k=1}^{n}[S]_{ik}[T]_{ki}\ge[S]_{ii}[T]_{ii}\qquad(\forall i),$$

hence $[ST]_{ii}\ge\varepsilon(S)\varepsilon(T)$, giving $\varepsilon(ST)\ge\varepsilon(S)\varepsilon(T)$.

1991 *Mathematics Subject Classification*: Primary 15A48.

PROPOSITION 2. *If $T \geq 0$ and $\varepsilon(T) > 0$ then $r(T) > 0$ and*

$$\operatorname{Per}\sigma(T) = \{r(T)\}.$$

Proof. Put $\varepsilon(T) = \varepsilon > 0$. Then $r(T) \geq \varepsilon > 0$ and $T \geq T - \varepsilon I \geq 0$. Hence $r(T) \geq r(T - \varepsilon I)$, therefore

$$(1) \qquad \sigma(T - \varepsilon I) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r(T)\}.$$

But, by the spectral mapping theorem,

$$\sigma(T - \varepsilon I) = \sigma(T) - \varepsilon.$$

Now suppose $\pi/2 \leq \theta \leq 3\pi/2$ and that $\mu = e^{i\theta} r(T) \in \sigma(T)$. The point $\mu - \varepsilon$ lies strictly outside the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq r(T)\}$ contradicting (1). It follows that

$$\operatorname{arc}\{e^{i\theta} r(T) : \pi/2 \leq \theta \leq 3\pi/2\} \cap \sigma(T) = \emptyset.$$

If now $|\mu| = r(T)$ but $\mu \neq r(T)$ then for some positive integer m,

$$\mu^m \in \operatorname{arc}\{e^{i\theta} r(T^m) : \pi/2 \leq \theta \leq 3\pi/2\}.$$

But $T^m \geq 0$ and, by Lemma 1,

$$\varepsilon(T^m) \geq \varepsilon(T)^m > 0.$$

Hence $\mu^m \notin \sigma(T^m)$, therefore $\mu \notin \sigma(T)$, which completes the proof.

COROLLARY 3. *If $T > 0$ then $\operatorname{Per}\sigma(T) = \{r(T)\}$.*

COROLLARY 4. *If $T \geq 0$ then $r(T) \in \operatorname{Per}\sigma(T)$.*

Proof. Suppose that $T \geq 0$ and $\delta > 0$. Then

$$\varepsilon(T + \delta I) \geq \delta > 0,$$

hence, by Proposition 2,

$$\operatorname{Per}\sigma(T + \delta I) = \{r(T + \delta I)\}$$

and $r(T + \delta I) \geq r(T)$. Thus $T + \delta I$ has a real eigenvalue not less than $r(T)$, hence T has a real eigenvalue not less than $r(T) - \delta$ for $0 \leq \delta \leq r(T)$. Since δ is arbitrary, $r(T) \in \operatorname{Per}\sigma(T)$.

It is now clear that if $T \geq 0$ and $\delta \geq 0$ then $r(T + \delta I) = r(T) + \delta$. This result can be extended.

LEMMA 5. *If $T \geq 0$ then $r(T + \delta I) = r(T) + \delta$ for $\delta \geq -\varepsilon(T)$.*

Proof. $T + \delta I \geq 0$ for $\delta \geq -\varepsilon(T)$ so, by Corollary 4, $r(T + \delta I)$ is the maximum point on the real axis in $\sigma(T + \delta I)$ for this range of values of δ. Similarly $r(T)$ is the maximum point on the real axis in $\sigma(T)$. But now

$$r(T) + \delta \in \sigma(T) + \delta = \sigma(T + \delta I).$$

Thus $r(T) + \delta$ is the maximum point on the real axis in $\sigma(T + \delta I)$. Hence

$$r(T + \delta I) = r(T) + \delta \qquad \text{for } \delta \geq -\varepsilon(T).$$

The next proposition establishes the existence of an eccentric spectral disc (Figure 1) for a positive matrix T which, if $\varepsilon(T) > 0$, is strictly contained in the spectral disc $\{\lambda \in \mathbb{C} : |\lambda| \leq r(T)\}$. This generalises a result known for stochastic matrices ([2], III.3.4.1).

PROPOSITION 6. *If $T \geq 0$ then*

$$\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda - \varepsilon(T)| \leq r(T) - \varepsilon(T)\}.$$

Proof. Let $T \geq 0$ and $\mu \in \sigma(T)$. Then if $\varepsilon(T) = \varepsilon$,

$$\mu - \varepsilon \in \sigma(T) - \varepsilon = \sigma(T - \varepsilon I),$$

hence $|\mu - \varepsilon| \leq r(T - \varepsilon I) = r(T) - \varepsilon$ by Lemma 5.

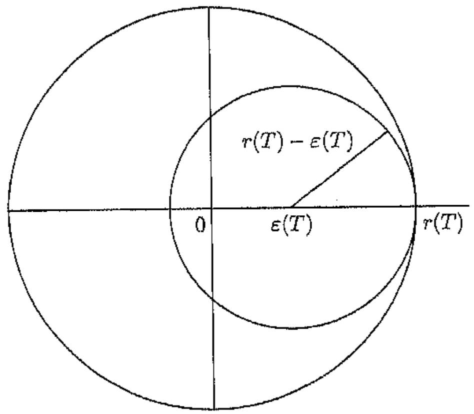

Fig. 1

This result can often be improved upon. Consider the positive matrix

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\varepsilon(T) = 0$ the two spectral discs of Figure 1 coincide giving nothing new. Observe however that, by the invariance of the trace and of the determinant under similarity, T is similar to the positive matrix

$$S = \begin{bmatrix} a & b \\ c & 1 - a \end{bmatrix},$$

where $0 \leq a \leq 1, 0 \leq b, c$ and $bc = a(1 - a)$; and that all positive matrices similar to T have this form. Further

$$\varepsilon(S) = \min_{0 \leq a \leq 1} \{a, 1 - a\},$$

and the maximum value which $\varepsilon(S)$ can take is $1/2$. Since the spectrum is invariant under similarity we see that for each $S \geq 0$ and similar to T we can replace $\varepsilon(T)$ by $\varepsilon(S)$ in Proposition 6 and Figure 1.

This suggests that, for $T \geq 0$, we introduce the subset of the positive real axis

$$\Delta(T) = \{\varepsilon(S) : S \geq 0 \text{ and similar to } T\}$$

and put $\eta(T) = \sup \Delta(T)$. Note that since the trace is invariant under similarity it follows from Lemma 1 that

$$\eta(T) \leq n^{-1}\operatorname{tr}(T).$$

Observe that, in the previous example,

$$\Delta(T) = [0, \tfrac{1}{2}] \quad \text{and} \quad \eta(T) = \tfrac{1}{2} = \tfrac{1}{2}\operatorname{tr}(T).$$

Our improved eccentric disc theorem states that $\varepsilon(T)$ can be replaced by $\eta(T)$ in Proposition 6 and Figure 1.

PROPOSITION 7. *If* $T \geq 0$ *then*

$$\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda - \eta(T)| \leq r(T) - \eta(T)\}.$$

Proof. Let $\mu \in \sigma(T)$, and suppose that

$$|\mu - \eta(T)| > r(T) - \eta(T).$$

Then there exists $\delta > 0$ such that

$$|\mu - \eta(T)| = r(T) - \eta(T) + \delta. \tag{2}$$

Choose $S \geq 0$ and similar to T such that

$$\eta(T) - \varepsilon(S) < \delta/3. \tag{3}$$

Then, by Proposition 6,

$$\begin{aligned}|\mu - \eta(T)| &\leq |\mu - \varepsilon(S)| + |\eta(T) - \varepsilon(S)| \leq r(S) - \varepsilon(S) + \delta/3\\ &= r(T) - \eta(T) + \eta(T) - \varepsilon(S) + \delta/3 \leq r(T) - \eta(T) + 2\delta/3\end{aligned}$$

by (3). This contradicts equation (2), therefore our original assumption is false.

COROLLARY 8. *If* $T \geq 0$ *and* $\operatorname{Per}\sigma(T)$ *contains two distinct eigenvalues of* T *then*

$$\varepsilon(T) = \eta(T) = 0.$$

It is easy to see that the converse of Corollary 8 is false. Consider the positive matrix

$$T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then $\operatorname{tr}(T) = 0$ so $\eta(T) = 0$ but $\sigma(T) = \{-1, 2\}$.

If S is a positive matrix we know that $\eta(S) \leq n^{-1}\operatorname{tr}(S)$. The next example shows that this inequality may be strict. Consider

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\operatorname{Per}\sigma(S) = \{-1, 1\}$, thus, by Corollary 8, $\eta(S) = 0$. But $\operatorname{tr}(S) = 1$ so $\eta(S) \neq \frac{1}{3}\operatorname{tr}(S)$.

PROBLEM. For $T \geq 0$ is $\Delta(T)$ a closed subinterval of the positive real axis?

REMARK. The minimum diagonal element of a real matrix is an object which hitherto seems to have attracted surprisingly little attention. The class $\mathcal{K}$, introduced by Fiedler and Pták [1], consists of matrices whose off-diagonal elements are negative (≤ 0) and whose real eigenvalues are strictly positive; their inverses lie within the class of positive matrices ([1], 4.2).

Let $\mathcal{N}$ denote the class of negative inverses matrices ($N \in \mathcal{N} \Leftrightarrow [N]_{ij} \leq 0\ \forall i,j$). Then a strong duality exists between classes $\mathcal{N}$ and $\mathcal{K}$ based on shifts (adding real multiples of the identity). When a sufficiently large positive shift is added to a matrix in $\mathcal{N}$ the sum is in $\mathcal{K}$. Conversely, a big enough negative shift added to a member of $\mathcal{K}$ yields a member of $\mathcal{N}$. More precisely, for real λ,

(i) if $T \in \mathcal{N}$ then $T + \lambda I \in \mathcal{K} \Leftrightarrow \lambda > r(T)$; while
(ii) if $T \in \mathcal{K}$ then $T - \lambda I \in \mathcal{N} \Leftrightarrow \lambda \geq \max_{1\leq i\leq n}[T]_{ii}$.

Now Perron–Frobenius theory for positive matrices may be employed to derive results for matrices in class $\mathcal{K}$. For example our Proposition 6 quickly leads to 4.8 of [1].

References

[1] M. Fiedler and V. Pták, *On matrices with non-positive off-diagonal elements and positive principal minors*, Czechoslovak Math. J. 12 (87) (1962), 382–400.

[2] M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Dover, Mineola, N.Y., 1992.

Northern Ireland Department of Health and Social Services
Belfast, U.K.
E-mail: rsmyth@dis.n-i.nhs.uk

Trinity College
Dublin, Ireland
E-mail: mathdep@maths.tcd.ie

Received November 12, 1997 (3996)