[H-R] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol. I, Springer, Berlin, 1963.

[I] A. Iwanik, *Toeplitz flows with pure point spectrum*, Studia Math. 118 (1996), 27–35.

[I-L] A. Iwanik and Y. Lacroix, *Some constructions of strictly ergodic non-regular Toeplitz flows*, ibid. 110 (1994), 191–203.

[J-K] K. Jacobs and M. Keane, *0-1 sequences of Toeplitz type*, Z. Wahrsch. Verw. Gebiete 13 (1969), 123–131.

[K-L] J. Kwiatkowski and Y. Lacroix, *Rank and weak closure theorem, II*, preprint.

[L] M. Lemańczyk, *Toeplitz Z_2-extensions*, Ann. Inst. H. Poincaré 24 (1988), 1–43.

[M-P] N. Markley and M. E. Paul, *Almost automorphic symbolic minimal sets without unique ergodicity*, Israel J. Math. 34 (1979), 259–272.

[O] J. C. Oxtoby, *Ergodic sets*, Bull. Amer. Math. Soc. 58 (1952), 116–136.

[W] B. Weiss, *Strictly ergodic models for dynamical systems*, ibid. 13 (1985), 143–146.

[Wi] S. Williams, *Toeplitz minimal flows which are not uniquely ergodic*, Z. Wahrsch. Verw. Gebiete 67 (1984), 95–107.

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Two-sided estimates for the approximation numbers of Hardy-type operators in L^∞ and L^1

by

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Abstract. In [2] and [3] upper and lower estimates and asymptotic results were obtained for the approximation numbers of the operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ defined by $(Tf)(x) := v(x)\int_0^\infty u(t)f(t)\,dt$ when $1 < p < \infty$. Analogous results are given in this paper for the cases $p = 1, \infty$ not included in [2] and [3].

1. Introduction. In [2] and [3] the operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ defined by

$$Tf(x) = v(x)\int_0^\infty u(t)f(t)\,dt \tag{1.1}$$

was studied in the case $1 < p < \infty$, with u, v real-valued functions and $u \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^+)$, $v \in L^p(\mathbb{R}^+)$, $p' = p/(p-1)$. Estimates for the approximation numbers $a_n(T)$ of T were obtained in [2], but the procedure for extracting the upper and lower bounds from the results is rather cumbersome to apply. This deficiency was overcome in [3] where asymptotic bounds for the approximation numbers which are easy to check in practice were determined. Specifically, it was proved that

$$\lim_{n\to\infty} na_n(T) = \frac{1}{\pi}\int_0^\infty |u(t)v(t)|\,dt \tag{1.2}$$

when $p = 2$; and when $p \neq 2$,

$$\frac{1}{4}\alpha_p \int_0^\infty |u(t)v(t)|\,dt \le \liminf_{n\to\infty} na_n(T) \le \limsup_{n\to\infty} na_n(T) \le \alpha_p \int_0^\infty |u(t)v(t)|\,dt \tag{1.3}$$

for some constant α_p depending on p. Further in [3], two-sided estimates

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are given for the l^α and weak l^α norms of $\{a_n(T)\}$ when $\alpha > 1$; in the case $p = 2$, these results recover those in [5].

The analysis in [3] is no longer valid when $p = \infty$ or 1, and, indeed, the result itself has to be modified in the following way: when $p = \infty$, the function v in the integrals in (1.2), (1.3) is replaced by

$$v_s(t) = \lim_{\varepsilon\to 0+} \|v\|_{L^\infty(t-\varepsilon,t+\varepsilon)},$$

while if $p = 1$, then u is replaced by u_s. Three critical ingredients of the proof in [2] and [3] are no longer available in these cases. The first is that the operator P defined by the integral mean over an interval $I \subset \mathbb{R}^+$, namely

$$Pf := \frac{1}{|I|}\int_I f\,dx,$$

where $|I|$ denotes the length of I, is such that the distance from T to the one-dimensional operators on $L^p(I)$ is comparable to $\|T - P \mid L^p(I) \to L^p(I)\|$. The second concerns the basic strategy which relies on a partition of $\mathbb{R}^+$ into intervals I_k which are defined by means of a continuous set function $L(I)$ which, with $I = (c,d)$, is decreasing as c increases and increasing as d increases. In the L^∞ and L^1 cases the analogue of L is no longer continuous and an alternative function, and technique, have to be found. Finally, the fact that the step functions are not dense in L^∞ causes difficulties, and indeed, it is this which dictates the form of the result noted above.

It is just as easy to consider a general interval (a,b) instead of $\mathbb{R}^+$, so that in this paper

$$Tf(x) := v(x)\int_a^x u(t)f(t)\,dt, \qquad a < x < b; \tag{1.4}$$

this simple extension will have a useful consequence when T is considered as an operator on L^1, as we can then simply translate the dual of the L^∞ result. Also, as was observed in [3], the condition on v assumed there, namely $v \in L^p(\mathbb{R}^+)$, can be weakened to $v \in L^p(x,\infty)$ for all $x > 0$, and we incorporate this fact in the present paper.

Finally, to give some insight into the significance of the function v_s in the $L^\infty(a,b)$ case, we show that, with the operator in (1.4) denoted by $T_{u,v}$, the following is possible:

$$\|T_{u,v}\| = \|T_{u,v_s}\| = \|T_{u,v} - T_{u,v-v_s}\|,$$
$$\int_a^b |u(t)v(t)|\,dt \neq \int_a^b |u(t)|v_s(t)\,dt,$$
$$\limsup na_n(T_{u,v}) \asymp \limsup na_n(T_{u,v_s}),$$
$$\liminf na_n(T_{u,v}) \asymp \liminf na_n(T_{u,v_s}),$$

where the symbol $\asymp$ indicates that the quotient of the two sides is bounded above and below by positive constants. Analogous possibilities exist in the $L^1(a,b)$ case.

2. Preliminaries. In most of the paper we shall be concerned with the operator T defined in (1.4) as a map from $L^\infty(a,b)$ into itself. The assumptions made on u, v in this case are that, for all $x \in (a,b)$,

$$u \in L^1(a,x), \tag{2.1}$$

$$v \in L^\infty(x,b). \tag{2.2}$$

The results for T acting between $L^1(a,b)$ will follow on taking duals, and for this part of the paper alternative conditions to (2.1) and (2.2) will be required.

For $I = (c,d) \subseteq (a,b)$, define

$$J(I) \equiv J(c,d) := \sup_{x\in I}\Big\{\int_c^x |u(t)|\,dt\,\|v\chi_{(x,d)}\|_\infty\Big\}, \tag{2.3}$$

where χ_S denotes the characteristic function of the set S, and $\|\cdot\|_\infty$ denotes the norm on $L^\infty(a,b)$; we shall write $\|\cdot\|_{p,I}$ for the usual norm on $L^p(I)$, $1 \le p \le \infty$, but use $\|\cdot\|_p$ when $I = (a,b)$. It is easy to see that

$$J(I) = \operatorname*{ess\,sup}_{x\in I}\Big\{\int_c^x |u(t)|\,dt\,|v(x)|\Big\}. \tag{2.4}$$

We also have

Lemma 2.1. *Suppose that* (2.1) *and* (2.2) *are satisfied. Then the function* $J(\cdot,d)$ *is continuous and non-increasing on* (a,d), *for any* $d < b$.

Proof. Given $x \in (a,b)$ and $\varepsilon > 0$, there exists

$$h = h(x,\varepsilon) \in \big(0, \min\{\tfrac12(x+a), b-x\}\big)$$

such that

$$\int_{x-h}^{x+h} |u(t)|\,dt < \min\Big(\frac{\varepsilon}{\|v\|_{\infty,((x-a)/2,d)}}, \varepsilon\Big).$$

Then

$$\begin{aligned} J(x,d) &\le J(x-h,d) \\ &= \max\Big\{\sup_{x-h<z<x}\Big[\int_{x-h}^{z} |u(t)|\,dt\,\|v\|_{\infty,(z,d)}\Big], \\ &\qquad \sup_{x<z<d}\Big[\Big(\int_{x-h}^{x} + \int_x^z\Big)|u(t)|\,dt\,\|v\|_{\infty,(z,d)}\Big]\Big\} \\ &\le \max\{\varepsilon, \varepsilon + J(x,d)\} = \varepsilon + J(x,d) \end{aligned} \tag{2.5}$$

and so $0 < J(x-h,a) - J(x,d) < \varepsilon$. Similarly, $0 < J(x) - J(x+h) < \varepsilon$ and the continuity is established. It is obvious that $J(\cdot,d)$ is non-increasing and hence the lemma is proved. ∎

The following result is known (see [4] and [6]):

PROPOSITION 2.2. *The operator T in* (1.4), *with u,v satisfying* (2.1) *and* (2.2), *is bounded as a map from $L^\infty(a,b)$ into $L^\infty(a,b)$ if and only if $J(a,b) < \infty$. It is compact if and only if $\lim_{c\to a_+} J(a,c) = \lim_{d\to b_-} J(d,b) = 0$.*

In [2], the analogue of the function J in (2.3) could have been used to construct the partition of (a,b) into the intervals I_i which feature so prominently in the analysis; see the Remark at the end of §4 in [2]. However, in the L^∞ case, for the reason given in the introduction, we need to use directly the function

$$A(I) := \begin{cases} \sup_{f\in L^\infty(I),\, f\neq 0} \inf_{\alpha\in\mathbb{R}} \|Tf-\alpha v\|_{\infty,I}/\|f\|_{\infty,I} & \text{if } v(I)>0,\\ 0 & \text{if } v(I)=0,\end{cases} \tag{2.6}$$

where $v(I) := \int_I v(t)\,dt$. If v is continuous, it can be shown that $A(\cdot,b)$ is continuous, but in general, this is not so. For, consider the example

$$v(x) = \begin{cases} 1 & \text{for } x\in(0,1)\cup(2,\infty),\\ 0 & \text{otherwise,}\end{cases} \qquad u(x) = \chi_{(1,2)}(x),$$

with $(a,b) = (0,\infty)$. Then $A(x,\infty) = 0$ for $x > 1$, but for $x < 1$,

$$A(x,\infty) \ge \inf_{\alpha\in\mathbb{R}} \Big\|\Big[\int_0^y u\,dt - \alpha\Big]v(y)\Big\|_{\infty,(x,\infty)} = \inf_{\alpha\in\mathbb{R}} \max\{|\alpha|, |1-\alpha|\} = \frac{1}{2}.$$

It is of interest to note that if (2.1) and (2.2) are satisfied and $v\notin L^\infty(a,b)$, then, since $\int_a^x u(t)f(t) \to 0$ as $x\to a_+$ for every $f\in L^\infty(a,b)$, we must have, if $\alpha\neq 0$, $\|Tf-\alpha v\|_{\infty,(a,c)} = \infty$ for $c\in(a,b]$. Hence, with $I=(a,c)$,

$$A(a,c) = \sup_{\|f\|_{\infty,I}=1} \|Tf\|_{\infty,I} = \operatorname*{ess\,sup}_{x\in I} |v(x)| \int_a^x |u(t)|\,dt = J(a,c)$$

by (2.4).

We now define, for any interval $I\subseteq(a,b)$ and $\varepsilon>0$,

$$M(I,\varepsilon) := \inf\Big\{n : I = \bigcup_{i=1}^n I_i,\ A(I_i)\le\varepsilon\Big\}. \tag{2.7}$$

Observe that if $\bar I \subset (a,b)$, then we have $M(I,\varepsilon) < \infty$. For, since $J(c,d) \le \|u\|_{1,(c,d)}\|v\|_{\infty,I}$ for any $(c,d)\subset I$ and $\|\cdot\|_1$ is absolutely continuous, it follows that the number

$$N(I,\varepsilon) := \inf\Big\{n : I = \bigcup_{i=1}^n I_i,\ J(I_i)\le\varepsilon\Big\} \tag{2.8}$$

is finite, and

$$\begin{aligned} A(I) &\le \sup_{f\in L^\infty(I),\, f\neq0} \frac{\|Tf\|_{\infty,I}}{\|f\|_{\infty,I}} \\ &\le \sup_{f\in L^\infty(I),\, f\neq0} \operatorname*{ess\,sup}_I \frac{[|v(x)|\int_a^x |u(t)|\,|f(t)|\,dt]}{\|f\|_{\infty,I}} \le J(I)\end{aligned} \tag{2.9}$$

by (2.4); thus $M(I,\varepsilon) \le N(I,\varepsilon) < \infty$. If $I = (a,b)$, we still have $M(I,\varepsilon) < \infty$ if

$$\lim_{x\to b_-} J(x,b) = \lim_{x\to a_+} J(a,x) = 0$$

since $N(I,\varepsilon)<\infty$ and (2.9) remains valid.

LEMMA 2.3. *Suppose that* (2.1) *and* (2.2) *are satisfied and let $M(I,\varepsilon) = m < \infty$ for $I\subseteq(a,b)$ and $\varepsilon>0$. Then we have:*

(i) *if $m=2n$, there exist intervals J_i, $i=1,\ldots,n$, such that $I=\bigcup_{i=1}^n J_i$ and $A(J_i)>\varepsilon$;*

(ii) *if $m=2n+1$, there exist intervals J_i, $i=1,\ldots,n+1$, such that $I=\bigcup_{i=1}^{n+1} J_i$, $A(J_i)>\varepsilon$, $i=1,\ldots,n$, and $A(J_{n+1})\le\varepsilon$.*

Proof. From the definition of $M(I,\varepsilon)$ in (2.7) there exist I_i, $i=1,\ldots,m$, such that $A(I_i)\le\varepsilon$ and $A(I_i\cup I_{i+1})>\varepsilon$. Now set $J_1 = I_1\cup I_2$, $J_2 = I_3\cup I_4,\ldots,$ with $J_{n+1} = I_m$ in case (ii). ∎

The final preliminary result is the following critical lemma which will yield a one-dimensional approximation to T on I.

LEMMA 2.4. *There exists $\omega_I\in\{L^\infty(I)\}^*$ such that $\omega_I(1)=1$, $\|\omega_I\|_{\{L^\infty(I)\}^*} = 1$ and, for all $f\in L^\infty(I)$,*

$$\inf_{\alpha\in\mathbb{R}} \|(f-\alpha)v\|_{\infty,I} \le \|(f-\omega_I(f))v\|_{\infty,I} \le 2\inf_{\alpha\in\mathbb{R}}\|(f-\alpha)v\|_{\infty,I}, \tag{2.10}$$

Proof. For $0<\gamma<\|v\|_{\infty,I}$ and $A_\gamma := \{x : v(x)>\gamma\}$, define $\omega_\gamma\in\{L^\infty(I)\}^*$ by

$$\omega_\gamma(f) := \frac{1}{|A_\gamma|}\int_{A_\gamma} f(x)\,dx, \quad f\in L^\infty(I).$$

Then $\omega_\gamma(1) = 1$, $\|\omega_\gamma\|_{\{L^\infty(I)\}^*} = 1$ and

$$|\omega_\gamma(f)| \le \frac{1}{\gamma}\|fv\|_{\infty,I}. \tag{2.11}$$

The set $W := \{W_\beta : 0<\beta<\|v\|_{\infty,I}\}$, where $W_\beta = \{\omega_\gamma : \gamma>\beta\}$, is a filter base whose members W_β are subsets of the unit ball in $\{L^\infty(I)\}^*$.

Hence, by the weak* compactness of this unit ball, W has an adherent point, ω_I say. It follows that $\omega_I(1) = 1$, $\|\omega_I\|_{\{L^\infty(I)\}^*} = 1$ and, from (2.11), for all $\beta \in (0, \|v\|_{\infty,I})$,

$$|\omega_I(f)| \le \frac{1}{\beta}\|fv\|_{\infty,I}, \qquad f \in L^\infty(I).$$

Consequently, for any $\delta \in \mathbb{R}$,

$$\begin{aligned}\inf_{\alpha\in\mathbb{R}} \|(f-\alpha)v\|_{\infty,I} &\le \|(f-\omega_I(f))v\|_{\infty,I}\\ &\le \|(f-\delta)v\|_{\infty,I} + \|\omega_I(f-\delta)v\|_{\infty,I}\\ &\le \|(f-\delta)v\|_{\infty,I}\Big\{1 + \frac{\|v\|_{\infty,I}}{\beta}\Big\}.\end{aligned}$$

Since $\delta \in \mathbb{R}$ and $\beta \in (0, \|v\|_{\infty,I})$ are arbitrary, the lemma follows. ∎

3. Bounds for the approximation numbers. We recall that, given any $m \in \mathbb{N}$, the mth approximation number of a bounded operator T, $a_m(T)$, is defined by

$$a_m(T) := \inf \|T - F\|,$$

where the infimum is taken over all bounded linear maps $F : L^\infty(a,b) \to L^\infty(a,b)$ with rank less than m. General information on approximation numbers may be found in [3]. Since $L^\infty(a,b)$ has the approximation property, T is compact if and only if $a_m(T) \to 0$ as $m \to \infty$.

The first two lemmas of this section give estimates for $a_m(T)$ which are the analogues of those obtained in [2]. Hereafter, until §7, we shall always assume (2.1) and (2.2).

LEMMA 3.1. *Suppose that $T : L^\infty(a,b) \to L^\infty(a,b)$ is bounded. Let $\varepsilon > 0$ and suppose that there exist $N \in \mathbb{N}$ and numbers c_k, $k = 0,1,\ldots,N$, with $a = c_0 < c_1 < \ldots < c_N = b$, such that $A(I_k) \le \varepsilon$ for $k = 0,1,\ldots,N-1$, where $I_k = (c_k, c_{k+1})$. Then $a_{N+1}(T) \le 2\varepsilon$.*

Proof. Let $f \in L^\infty(a,b)$ be such that $\|f\|_\infty = 1$, and write

$$Pf := \sum_{i=0}^{N-1} P_{I_k} f$$

where the P_{I_k} are the one-dimensional operators

$$P_{I_k} f(x) := \chi_{I_k}(x)v(x)\widehat{\omega}_{I_k}\Big(\int_a^x uf\,dt\Big), \qquad k = 0,1,\ldots,N-1,$$

and

$$\widehat{\omega}_{I_k}\Big(\int_a^x uf\,dt\Big) = \int_a^{c_k} uf\,dt + \omega_{I_k}\Big(\int_{c_k}^x uf\,dt\Big),$$

with $\omega_{I_k} \in \{L^\infty(I_k)\}^*$ the functionals in Lemma 2.4.

It is obvious that P_k, $k = 1,\ldots,N-2$, are bounded. With $k = 0$ or $N-1$ we have on $I = (a, c_1)$ or (c_N, b),

$$\Big|v(x)\omega_I\Big(\int_{c_k}^x uf\,dt\Big)\Big| \le \|\omega_I\|_{\{L^\infty(I)\}^*}|v(x)| \int_{c_k}^x |u(t)|\,dt\,\|f\|_{\infty,I}$$

and hence P is bounded in view of Proposition 2.2 and (2.4). We have

$$\begin{aligned}\|Tf - Pf\|_\infty &= \sup_{k\in\{0,1,\ldots,N-1\}} \|Tf - P_{I_k}f\|_{\infty,I_k}\\ &= \sup_{k\in\{0,1,\ldots,N-1\}} \Big\|v(x)\Big[\int_{c_k}^x uf\,dt - \omega_{I_k}\Big(\int_{c_k}^x uf\,dt\Big)\Big]\Big\|_{\infty,I_k}\\ &\le 2 \sup_{k\in\{0,1,\ldots,N-1\}} A(I_k)\|f\|_{\infty,I_k} \le 2\varepsilon\|f\|_{\infty,I}.\end{aligned}$$

by Lemma 2.4. Since rank $P \le N$, the lemma follows. ∎

LEMMA 3.2. *Suppose that $T : L^\infty(a,b) \to L^\infty(a,b)$ is bounded. Let $\varepsilon > 0$ and suppose that there exist $N \in \mathbb{N}$ and numbers d_k, $k = 0,1,\ldots,K$, with $a = d_0 < d_1 < \ldots < d_K \le b$ such that $A(I_k) \ge \varepsilon$ for $k = 0,1,\ldots,K-1$, where $I_k = (d_k, d_{k+1})$. Then $a_K(T) \ge \varepsilon$.*

Proof. Let $\lambda \in (0,1)$. From the definition of $A(I_k)$ we see that there exists $\phi_k \in L^\infty(I_k)$ with $\|\phi_k\|_{\infty,I_k} = 1$ and such that

$$\inf_{\alpha\in\mathbb{R}} \|T\phi_k - \alpha v\|_{\infty,I_k} > \lambda A(I_k) \ge \lambda\varepsilon. \tag{3.1}$$

Set $\phi_k(x) = 0$ for $x \notin I_k$. Let $P : L^\infty(a,b) \to L^\infty(a,b)$ be bounded and rank $P \le K-1$. Then there are constants $\lambda_0,\ldots,\lambda_{n-1}$, not all zero, such that

$$P\Big(\sum_{k=0}^{K-1} \lambda_k\phi_k\Big) = 0.$$

Put $\phi = \sum_{k=0}^{K-1} \lambda_k\phi_k$. Then

$$\begin{aligned}&\|T\phi - P\phi\|_\infty\\ &\quad= \|T\phi\|_\infty\\ &\quad\ge \sup_{k\in\{0,1,\ldots,K-1\}} \Big\|v(x)\Big(\int_{c_k}^x \lambda_k\phi_k(t)u(t)\,dt + \int_a^{c_k} \phi(t)u(t)\,dt\Big)\Big\|_{\infty,I_k}\\ &\quad= \sup_{k\in\{0,1,\ldots,K-1\}} |\lambda_k|\|T\phi_k + \alpha_k v\|_{\infty,I_k} \quad \text{where } \alpha_k = \lambda_k^{-1}\textstyle\int_a^{c_k}\phi(t)u(t)\,dt,\\ &\quad\ge \sup_{k\in\{0,1,\ldots,K-1\}} \inf_{\alpha\in\mathbb{R}} |\lambda_k|\,\|T\phi_k - \alpha v\|_{\infty,I_k}\\ &\quad\ge \sup_{k\in\{0,1,\ldots,K-1\}} \lambda|\lambda_k|\varepsilon = \lambda\varepsilon\|\phi\|_\infty\end{aligned}$$

by (3.1). This implies that $a_K(T) \geq \lambda\varepsilon$, whence the result since $\lambda \in (0,1)$ is arbitrary. ■

COROLLARY 3.3. *Suppose that T is compact* (see Proposition 2.2). *Then, for $\varepsilon \in (0, A(a,b))$,*

$$a_{M(\varepsilon)+1}(T) \leq 2\varepsilon, \quad a_{[M(\varepsilon)/2]-1}(T) > \varepsilon,$$

where $M_\varepsilon \equiv M((a,b),\varepsilon)$ is defined in (2.7) *and* $[\cdot]$ *denotes integer part.*

Proof. This is an immediate consequence of Lemmas 3.1 and 3.2. ■

4. Local asymptotic results. We need some preliminary results and the functions v_s mentioned in §1, namely

$$v_s(x) := \lim_{\varepsilon\to 0+} \|v\|_{\infty,(x-\varepsilon,x+\varepsilon)}$$

for $x \in (a,b)$.

LEMMA 4.1. *For any $I \subseteq (a,b)$, we have $J(I;u,v) = J(I;u,v_s)$ and $A(I;u,v) = A(I;u,v_s)$, where $J(I;u,v)$ and $A(I;u,v)$ are the functions defined in* (2.3) *and* (2.6) *respectively.*

Proof. For any continuous function ϕ, it is readily shown that $\|v_s\phi\|_{\infty,I} = \|v\phi\|_{\infty,I}$, and this fact yields the lemma. ■

LEMMA 4.2. *Let $\overline{I} \subset (a,b)$, and let $\vartheta_n = \{I_i^n\}_{i=1}^{l(n)}$ be a partition of I by intervals I_i^n such that each $I_i^{(n+1)} \in \vartheta_{n+1}$ is a subinterval of some $I_j^{(n)} \in \vartheta_n$, and $|I_i^n| \to 0$ as $n \to \infty$. Define*

$$v_s^n(t) := \sum_{i=1}^{l(n)} \chi_{I_i^n}(t)c_i^n, \quad c_i^n = \|v_s\|_{\infty,I_i^n}.$$

Then for a.e. $t \in I$,

(i) $\|v_s\|_{\infty,I} \geq v_s^n(t) \geq v_s(t)$,
(ii) $v_s^n(t) \searrow v_s(t)$ *as* $n \to \infty$,
(iii) $\lim_{n\to\infty} \int_I u(t)[v_s^n(t) - v_s(t)]\,dt = 0$.

Proof. Since v_s is upper semi-continuous and bounded, it is known that it can be approximated from above by a decreasing sequence of step functions. However, we shall give a proof of the lemma for completeness and subsequent reference.

If $t \in \operatorname{int} I_i^n$, the interior of I_i^n, then $v_s^n(t) = \|v_s\|_{\infty,I_i^n}$ satisfies

$$v_s(t) \leq v_s^n(t) \leq \|v_s\|_{\infty,I}.$$

This establishes (i), the exceptional set being $S = \bigcup_{n\in\mathbb{N}} S_n$, where S_n is the set of end-points of the intervals $I_i^n \in \vartheta_n$. If $t \in \operatorname{int} I_{i(n+1)}^{n+1} \subset \operatorname{int} I_{i(n)}^n$ say, we

have $c_{i(n+1)}^{n+1} \leq c_{i(n)}^n$ and so $v_s^{n+1}(t) \leq v_s^n(t)$ for $t \in I \setminus S$. Also, if $t \in \operatorname{int} I_{i(n)}^n$,

$$v_s^n(t) = \|v_s\|_{\infty,I_{i(n)}^n} = \|v\|_{\infty,I_{i(n)}^n} \geq v(t)$$

as observed in the proof of Lemma 4.1. Moreover, given $\delta > 0$ there exists $\varepsilon_0 > 0$ such that

$$v_s(t) > \|v\|_{\infty,(t-\varepsilon_0,t+\varepsilon_0)} - \delta.$$

Now choose N such that for all $n \geq N$,

$$t \in \operatorname{int} I_{i(n)}^n \subset (t-\varepsilon_0, t+\varepsilon_0).$$

Then we have, for all $n \geq N$,

$$0 < v_s^n(t) - v_s(t) < \delta$$

and hence $v_s^n(t) \to v_s(t)$ for all $t \in I \setminus S$.

Finally, (iii) follows by the dominated convergence theorem since $u \in L^1(I)$ and $\|v_s^n\|_{\infty,I} = \|v_s\|_{\infty,I} = \|v\|_{\infty,I} < \infty$. ■

LEMMA 4.3. *Let u, v be constant on $I \subset\subset (a,b)$. Then*

$$A(I) = \tfrac{1}{2}|u|\,|v|\,|I|. \tag{4.1}$$

Proof. We have, if $I = (c,d)$,

$$A(I) \geq |u|\,|v| \inf_\alpha \|x - c - \alpha\|_{\infty,I} = |u|\,|v|\,\big\|x - c - \tfrac{1}{2}(d-c)\big\|_{\infty,I} = \tfrac{1}{2}|u|\,|v|\,|I|.$$

Let $f \in L^\infty(I)$ and set $F(x) = \int_c^x f\,dt$. Then there exist $x_0, x_1 \in [c,d]$ such that

$$F(x_0) \leq F(x) \leq F(x_1), \quad x \in [a,b],$$

and hence

$$\inf_\alpha \|F - \alpha\|_{\infty,I} \leq \big\|F - \tfrac{1}{2}(F(x_0) + F(x_1))\big\|_{\infty,I} = \frac{1}{2}(F(x_1) - F(x_0)) = \frac{1}{2}\int_{x_0}^{x_1} f\,dt.$$

This yields

$$A(I) \leq \sup_{\|f\|_{\infty,I}=1} \left\{\frac{1}{2}\int_{x_0}^{x_1} f\,dt\right\} \leq \frac{1}{2}|I|$$

and the lemma is proved. ■

LEMMA 4.4. *Let $I \subset\subset (a,b)$ and $u_1, u_2 \in L^1(I)$. Then*

$$|A(I;u_1,v) - A(I;u_2,v)| \leq \|u_1 - u_2\|_{1,I}\|v\|_{\infty,I}. \tag{4.2}$$

Proof. We have

$$|A(I;u_1,v) - A(I;u_2,v)| \leq \sup_{\|f\|_{\infty,I}=1} \left| \inf_\alpha \left\|v(x)\Big(\int_a^x u_1 f\,dt - \alpha\Big)\right\|_{\infty,I} - \inf_\alpha \left\|v(x)\Big(\int_a^x u_2 f\,dt - \alpha\Big)\right\|_{\infty,I}\right|.$$

Suppose f is such that

$$(4.3)\qquad \inf_\alpha \Big\| v(x)\Big(\int_a^x u_1 f\,dt - \alpha\Big)\Big\|_{\infty,I} \ge \inf_\alpha \Big\| v(x)\Big(\int_a^x u_2 f\,dt - \alpha\Big)\Big\|_{\infty,I}.$$

Given $\varepsilon > 0$ there exists $\alpha_0 \in \mathbb{R}$ such that

$$\inf_\alpha \Big\| v(x)\Big(\int_a^x u_2 f\,dt - \alpha\Big)\Big\|_{\infty,I} > \Big\| v(x)\Big(\int_a^x u_2 f\,dt - \alpha_0\Big)\Big\|_{\infty,I} - \varepsilon.$$

Hence

$$\begin{aligned}
0 &\le \inf_\alpha \Big\| v(x)\Big(\int_a^x u_1 f\,dt - \alpha\Big)\Big\|_{\infty,I} - \inf_\alpha \Big\| v(x)\Big(\int_a^x u_2 f\,dt - \alpha\Big)\Big\|_{\infty,I}\\
&\le \Big\| v(x)\Big(\int_a^x u_1 f\,dt - \alpha_0\Big)\Big\|_{\infty,I} - \Big\| v(x)\Big(\int_a^x u_2 f\,dt - \alpha_0\Big)\Big\|_{\infty,I} + \varepsilon\\
&\le \Big\| v(x)\int_a^x (u_1 - u_2) f\,dt\Big\|_{\infty,I} + \varepsilon\\
&\le \|v(x)\|_{\infty,I}\|u_1 - u_2\|_{1,I}\|f\|_{\infty,I} + \varepsilon.
\end{aligned}$$

This remains valid if the inequality (4.3) is reversed, and so

$$|A(I;u_1,v) - A(I;u_2,v)| \le \|v(x)\|_{\infty,I}\|u_1 - u_2\|_{1,I} + \varepsilon$$

Since ε is arbitrary, the lemma is proved. ∎

In the next lemma g^* denotes the non-increasing rearrangement of a function g on an interval I: g^* is the generalised inverse of the non-increasing distribution function g_* of g, namely

$$(4.4)\qquad g^*(x) := \inf\{t : g_*(t) \ge x\}$$

where

$$(4.5)\qquad g_*(t) := |\{x \in I : g(x) \ge t\}|.$$

Note that since we have $\ge$ in the definitions above, g_* and g^* are left-continuous functions.

LEMMA 4.5. *Let $I \subset\subset (a,b)$ and $\gamma,\delta \in \mathbb{R}$ with $\delta \ge v_s(t) \ge 0$ on I. Then*

$$(4.6)\qquad A(I;\gamma,\delta) \ge A(I;\gamma,v_s) \ge \tfrac12|\gamma|\,\|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)}.$$

Proof. The first inequality in (4.6) is obvious. The set

$$M_\beta := \{y \in I : v_s(y) \ge \beta\}$$

is relatively closed in $\bar I$. For if $\{y_n\} \subset M_\beta$ and $y_n \to y \in \bar I$ as $n \to \infty$, then given $\varepsilon > 0$ there exists N such that $(y-\varepsilon, y+\varepsilon) \supset \big(y_n - \frac12\varepsilon, y_n + \frac12\varepsilon\big)$ for $n > N$. Hence

$$\|v\|_{\infty,(y-\varepsilon,y+\varepsilon)} \ge \|v\|_{\infty,(y_n-\frac12\varepsilon,y_n+\frac12\varepsilon)} \ge v_s(y_n) \ge \beta$$

whence $v_s(y) \ge \beta$ and $y \in M_\beta$. From the observed left continuity of (4.4) and (4.5), we have

$$\|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)} = \max_{(0,|I|]} |(v_s\chi_I)^*(t)t| = |(v_s\chi_I)^*(t_0)t_0|$$

for some $t_0 \in (0,|I|]$, and there exists $\beta > 0$ such that $|M_\beta| = t_0$. Choose the optimal c_0, d_0 such that $M_\beta \subseteq [c_0,d_0] \subseteq \bar I$. Then, with $I = (c,d)$,

$$\begin{aligned}
A(I;\gamma,v_s) &\ge |\gamma| \inf_\alpha \Big\| v_s(y)\Big(\int_c^y dt - \alpha\Big)\Big\|_{\infty,I}\\
&\ge |\gamma| \inf_\alpha \|\beta\chi_{M_\beta}(y)(y - c - \alpha)\|_{\infty,I}\\
&= \beta|\gamma|\,\big\|y - c - \tfrac12(c_0 + d_0 - 2c)\big\|_{\infty,M_\beta}\\
&= \tfrac12\beta|\gamma|(d_0 - c_0) \ge \tfrac12\beta|\gamma|\,|M_\beta|\\
&= \tfrac12|\gamma|\,[(v_s\chi_I)^*(t_0)t_0] = \tfrac12|\gamma|\,\|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)}.
\end{aligned}$$

The lemma is therefore proved. ∎

LEMMA 4.6. *Let $I \subset\subset (a,b)$ and $\gamma,\delta \in \mathbb{R}$ with $\delta \ge v_s(t) \ge 0$ on I. Then, for any $\alpha > 1$,*

$$(4.7)\qquad A(I;\gamma,\delta) - A(I;\gamma,v_s) \le \frac{\alpha}{2}\int_I |\gamma|(\delta - v_s(t))\,dt + \frac{|\gamma|\delta|I|}{2\alpha}.$$

Proof. We first observe that

$$(4.8)\qquad (v_s\chi_I)^*(t) \ge v_0(t) := \Big(\delta - \frac{V\alpha}{|\gamma|\,|I|}\Big)\chi_{(0,|I|-|I|/\alpha)}$$

where $V = |\gamma|\int_I(\delta - v_s(t))\,dt$. For, with $S := \{x : v_s(x) < \delta - V\alpha/(|\gamma|\,|I|)\}$,

$$\frac{V}{|\gamma|} > \int_S \Big(\delta - \delta + \frac{V\alpha}{|\gamma|\,|I|}\Big)dt = \frac{V\alpha}{|\gamma|\,|I|}|S|,$$

which implies that

$$\Big|\Big\{x : v_s(x) > \delta - \frac{V\alpha}{|\gamma|\,|I|}\Big\}\Big| > |I| - \frac{|I|}{\alpha}$$

and hence (4.8). Note that (4.8) is trivially true if $\delta - V\alpha/(|\gamma|\,|I|) < 0$. On using (4.1) and (4.6),

$$\begin{aligned}0 &\le A(I;\gamma,\delta)-A(I;\gamma,v_s)\le \frac{1}{2}|\gamma|\delta|I|-\frac{1}{2}|\gamma|\,\|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)}\\
&\le \frac{1}{2}|\gamma|\delta|I|-\frac{1}{2}\max_{(0,|I|]}(tv_0(t))\\
&= \frac{1}{2}|\gamma|\delta|I|-\frac{1}{2}|\gamma|\left(\delta-\frac{V\alpha}{|\gamma|\,|I|}\right)\left(|I|-\frac{|I|}{\alpha}\right)\\
&= \frac{\alpha V}{2}+\frac{|\gamma|\delta|I|}{2\alpha}-\frac{V}{2}\\
&\le \frac{\alpha}{2}\int_I |\gamma|(\delta-v_s(t))\,dt+\frac{|I|}{2\alpha}|\gamma|\delta,\end{aligned}$$

which is (4.7). ∎

THEOREM 4.7. *For any $I\subset\subset(a,b)$,*

$$\begin{aligned}(4.9)\qquad \frac{1}{2}\int_I |u(t)|v_s(t)\,dt &\le \liminf_{\varepsilon\to 0_+}\varepsilon M(I,\varepsilon)\\
&\le \limsup_{\varepsilon\to 0_+}\varepsilon M(I,\varepsilon)\le \int_I |u(t)|v_s(t)\,dt.\end{aligned}$$

Proof. On using Lemma 4.2, we infer that for each $\eta>0$ there exist step functions u_η, v_η on I such that

$$\|u-u_\eta\|_{1,I}<\eta,\qquad \int_I |u(t)|(v_\eta(t)-v_s(t))\,dt<\eta$$

and

$$\|v_s\|_{\infty,I}\ge v_\eta(t)\ge v_s(t)$$

on I. We may assume that

$$u_\eta=\sum_{j=1}^m \xi_j\chi_{W(j)},\qquad v_\eta=\sum_{j=1}^m \eta_j\chi_{W(j)},$$

where the $W(j)$ are disjoint subintervals of I, and $\eta_j\ge 0$.

Let $\varepsilon>0$, $M\equiv M(I,\varepsilon)$, and let $c_k\equiv c_k(\varepsilon)$, $k=1,\dots,M+1$, be the end-points of the intervals in (2.7): with $I=[c,d]$ and $I_k\equiv I_k(\varepsilon)=[c_k,c_{k+1}]$, we have $c=c_1<\dots<c_{M+1}=d$ and

$$\begin{aligned}A(I_k)\equiv A(I_k;u,v)&\le\varepsilon,\qquad k=1,\dots,M,\\
A(I_k\cup I_{k+1})&>\varepsilon,\qquad k=1,\dots,M-1.\end{aligned}$$

Then

$$\begin{aligned}(4.10)\qquad \Big|\int_I |u(t)|v_s(t)\,dt-\int_I |u_\eta(t)|v_\eta(t)\,dt\Big|
&\le \int_I |u(t)|(v_\eta(t)-v_s(t))\,dt+\int_I |u(t)-u_\eta(t)|v_\eta(t)\,dt\\
&<\eta(1+\|v_\eta\|_{\infty,I})\le \eta(1+\|v_s\|_{\infty,I}).\end{aligned}$$

Next, let $\mathbf{K}:=\{k:\text{there exist } j \text{ such that } I_{2k}\cup I_{2k+1}\subset W(j)\}$. Then $\#\mathbf{K}\ge[M/2]-2m\ge M/2-1-2m$, and, by Lemmas 4.4 and 4.6,

$$\begin{aligned}\left(\frac{M}{2}-1-2m\right)\varepsilon &\le \sum_{k\in\mathbf{K}} A(I_{2k}\cup I_{2k+1};u,v)\\
&\le \sum_{k\in\mathbf{K}}\{A(I_{2k}\cup I_{2k+1};u_\eta,v_\eta)\\
&\quad+(A(I_{2k}\cup I_{2k+1};u,v_s)-A(I_{2k}\cup I_{2k+1};u_\eta,v_s))\\
&\quad+(A(I_{2k}\cup I_{2k+1};u_\eta,v_s)-A(I_{2k}\cup I_{2k+1};u_\eta,v_\eta))\}\\
&\le \frac{1}{2}\sum_j |\xi_j|\eta_j|W(j)|\\
&\quad+\sum_j\Big\{\|u-u_\eta\|_{1,W(j)}\|v_s\|_{\infty,W(j)}\\
&\quad+\frac{\alpha}{2}\int_{W(j)}|\xi_j|(v_\eta-v_s)\,dt+\frac{|\xi_j|\eta_j}{2\alpha}|W(j)|\Big\}\\
&\le \frac{1}{2}\int_I |u_\eta|v_\eta\,dt+\|u-u_\eta\|_{1,I}\|v_s\|_{\infty,I}\\
&\quad+\frac{\alpha}{2}\int_I |u_\eta|(v_\eta-v_s)\,dt+\frac{1}{2\alpha}\int_I |u_\eta|v_\eta\,dt\\
&\le \frac{1}{2}\int_I |u_\eta|v_\eta\,dt+K\left(\alpha\eta+\frac{1}{\alpha}\right)\\
&\le \frac{1}{2}\int_I |u(t)|v_s(t)\,dt+K\left(\alpha\eta+\frac{1}{\alpha}\right)\end{aligned}$$

by (4.10), for some constant K independent of ε. We therefore conclude that

$$\limsup_{\varepsilon\to 0_+}\varepsilon M(I,\varepsilon)\le \int_I |u(t)|v_s(t)\,dt+K\left(\alpha\eta+\frac{1}{\alpha}\right)$$

and the right-hand inequality in (4.9) follows since $\eta>0$ and $\alpha>1$ are arbitrary.

For the left-hand inequality in (4.9), we add the end-points of the intervals $W(j)$, $j=1,\ldots,m$, to the c_k, $k=1,\ldots,M-1$, to form the partition $c=e_1<\ldots<e_n=d$, say, where $n\le M+1+m$. Note that each interval $J_i:=[e_i,e_{i+1}]$ is a subinterval of some $W(j)$ and hence u_η, v_η have constant values on each J_i. We again use Lemmas 4.3, 4.4 and 4.6 to get

$$\begin{aligned}\frac{1}{2}\int_I |u_\eta|v_\eta\,dt &= \sum_{j=1}^m \sum_{J_i\subseteq W(j)} A(J_i;u_\eta,v_\eta)\\ &\le \sum_{i=1}^n \Big\{A(J_i;u,v_s)+\|u-u_\eta\|_{1,J_i}\|v_s\|_{\infty,J_i}\\ &\quad+\frac{\alpha}{2}\int_{J_i}|u_\eta|(v_\eta-v_s)\,dt+\frac{1}{2\alpha}\int_{J_i}|u_\eta|v_\eta\,dt\Big\}\\ &\le (M+1+m)\varepsilon+K\Big(\alpha\eta+\frac{1}{\alpha}\Big).\end{aligned}$$

Hence, from (4.10),

$$\frac{1}{2}\int_I |u(t)|v_s(t)\,dt\le (M+1+m)\varepsilon+K\Big(\alpha\eta+\frac{1}{\alpha}\Big)$$

and the left-hand inequality in (4.9) follows. ■

5. The main result. With $U(x):=\int_a^x|u(t)|\,dt$, we define $\xi_k\in\mathbb{R}^+$ by

$$U(\xi_k)=2^k; \tag{5.1}$$

if $u\notin L^1(a,b)$, then k may be any integer, but if $u\in L^1(a,b)$, then $2^k\le\|u\|_1$. For each admissible k we set

$$\sigma_k:=\|uv\|_{\infty,Z_k},\qquad Z_k=(\xi_k,\xi_{k+1}), \tag{5.2}$$

so that

$$2^k\|v\|_{\infty,Z_k}\le\sigma_k\le 2^{k+1}\|v\|_{\infty,Z_k}. \tag{5.3}$$

For non-admissible k we set $\sigma_k=0$. The sequence $\{\sigma_k\}$ is the analogue of that defined in [3, §3], which in turn was motivated by a similar sequence introduced in [5].

The following technical lemma has a central role in this section.

LEMMA 5.1. *Let $k_0,k_1,k_2\in\mathbb{Z}$ with $k_0<k_1<k_2$, and let $I_j=(a_j,b_j)$ $(j=0,1,\ldots,l)$ be intervals in (a,b) which are non-overlapping and such that $I_j\subset Z_{k_2}$ $(j=1,\ldots,l)$, $a_0\in Z_{k_0}$, $b_0\in Z_{k_2}$. Let $x_j\in I_j$ $(j=0,1,\ldots,l)$ and $x_0\in Z_{k_1}$. Then, if $\alpha\ge1$,*

$$S:=\sum_{j=0}^l\Big(\int_{a_j}^{x_j}|u(t)|\,dt\Big)^\alpha\|v\|^\alpha_{\infty,(x_j,b_j)}\le(2^\alpha+1)\max_{k_0\le n\le k_2}\sigma_n^\alpha. \tag{5.4}$$

Proof. On using Jensen's inequality, we have

$$\begin{aligned}S&\le\Big(\int_{\xi_{k_0}}^{\xi_{k_1+1}}|u(t)|\,dt\Big)^\alpha\|v\|^\alpha_{\infty,(\xi_{k_1},\xi_{k_2+1})}+\sum_{j=1}^l\Big(\int_{I_j}|u(t)|\,dt\Big)^\alpha\|v\|^\alpha_{\infty,I_j}\\ &\le\Big\{(2^{k_1+1}-2^{k_0})\max_{k_1\le n\le k_2}\frac{\sigma_n}{2^n}\Big\}^\alpha+\Big(\int_{Z_{k_2}}|u(t)|\,dt\Big)^\alpha\|v\|^\alpha_{\infty,Z_{k_2}}\quad\text{by (5.3)},\\ &\le\{2\max_{k_1\le n\le k_2}\sigma_n\}^\alpha+\Big\{2^{k_2}\frac{\sigma_{k_2}}{2^{k_2}}\Big\}^\alpha,\end{aligned}$$

whence (5.4). ■

LEMMA 5.2. *The quantity $J(a,b)$ defined in (2.3) satisfies*

$$\tfrac{1}{3}J(a,b)\le\sup_k\sigma_k\le 2J(a,b). \tag{5.5}$$

Proof. From (2.4) and Lemma 5.1,

$$J(a,b)\le 3\sup_k\sigma_k.$$

Also,

$$\sigma_k\le 2^{k+1}\|v\|_{\infty,Z_k}\le 2\int_a^{\xi_k}|u(t)|\,dt\,\|v\|_{\infty,(\xi_k,b)}\le 2J(a,b).\ ■$$

COROLLARY 5.3. *The operator $T:L^\infty(a,b)\to L^\infty(a,b)$ is bounded if and only if the sequence $\{\sigma_k\}$ is bounded, in which case their norms are equivalent:*

$$\|T\|\asymp\|\{\sigma_k\}\|_\infty. \tag{5.6}$$

Also, T is compact if and only if $\lim_{k\to\pm\infty}\sigma_k=0$.

Proof. The first part is an immediate consequence of Proposition 2.2 and Lemma 5.2. We also have from Lemma 5.2, as in its proof,

$$\tfrac{1}{3}J(a,\xi_{k_2})\le\max_{n\le k_2}\sigma_n\le 2J(a,\xi_{k_2+1})$$

and

$$\tfrac{1}{3}J(\xi_{k_0},b)\le\max_{n\ge k_0}\sigma_n\le 2J(\xi_{k_0-1},b).$$

Since $\xi_{k_2}\to a$ if and only if $k_2\to-\infty$, and ξ_{k_0} tends to b if and only if k_0 tends to ∞ in the case $u\notin L^1(a,b)$ and otherwise to the largest admissible value of k in the definition of σ_k, the corollary follows. ■

The main result is

THEOREM 5.4. *Suppose that* (2.1) *and* (2.2) *are satisfied*, T *is compact*, *and that* $\sum_{n\in\mathbb{Z}}\sigma_n$ *is convergent. Then*

$$\frac{1}{4}\int_a^b |u(t)|v_s(t)\,dt \le \liminf_{n\to\infty} na_n(T) \le \limsup_{n\to\infty} na_n(T) \le 2\int_a^b |u(t)|v_s(t)\,dt. \tag{5.7}$$

Proof. Let $I=[c,d]\subset(a,b)$ and suppose that $c\in[\xi_{k_0},\xi_{k_0+1}]$ and $d\in[\xi_{k_1},\xi_{k_1+1})$. With $I_j(\varepsilon)$, $j=1,\ldots,M(\varepsilon)$, the covering of (a,b) in (2.7), where $M(\varepsilon)\equiv M((a,b),\varepsilon)$, let

$$m_0(\varepsilon)=\#\{j:I_j(\varepsilon)\subset[a,c]\},\qquad m_1(\varepsilon)=\#\{j:I_j(\varepsilon)\subset[a,d]\}.$$

Then

$$m_1(\varepsilon)-m_0(\varepsilon)\le M(I,\varepsilon)+1$$

and

$$\begin{aligned}
&\frac{\varepsilon}{2}(M(\varepsilon)-M(I,\varepsilon)-9)\\
&\quad\le \varepsilon([m_0(\varepsilon)/2]+[M(\varepsilon)/2]-[m_1(\varepsilon)/2]-2)\\
&\quad\le \sum_{j=1}^{[m_0(\varepsilon)/2]} A(I_{2j-1}\cup I_{2j};u,v)+\sum_{j=[m_1(\varepsilon)/2]+2}^{[M(\varepsilon)/2]} A(I_{2j-1}\cup I_{2j};u,v)\\
&\quad\le \sum_{j=1}^{[m_0(\varepsilon)/2]} J(I_{2j-1}\cup I_{2j};u,v)+\sum_{j=[m_1(\varepsilon)/2]+2}^{[M(\varepsilon)/2]} J(I_{2j-1}\cup I_{2j};u,v)\\
&\quad\le 3\sum_{n\le k_0}\sigma_n+3\sum_{n\ge k_1}\sigma_n
\end{aligned}$$

on using (2.9) and (5.5).

It follows from Theorem 4.7 that

$$\limsup_{\varepsilon\to0+}\varepsilon M(\varepsilon)\le\int_{\xi_{k_0}}^{\xi_{k_1+1}}|u(t)|v_s(t)\,dt+3\Big(\sum_{n\le k_0}\sigma_n+\sum_{n\ge k_1}\sigma_n\Big),$$

which yields

$$\limsup_{\varepsilon\to0+}\varepsilon M(\varepsilon)\le\int_a^b|u(t)|v_s(t)\,dt.$$

On setting $n=M(\varepsilon)+1$ in Corollary 3.3, we get $\varepsilon\ge\frac12 a_n(T)$ and hence

$$\limsup_{n\to\infty} na_n(T)\le 2\int_a^b|u(t)|v_s(t)\,dt.$$

Similarly, from Theorem 4.7,

$$\liminf_{\varepsilon\to0+}\varepsilon M(\varepsilon)\ge\frac12\int_a^b|u(t)|v_s(T)\,dt$$

and from Corollary 3.3,

$$\liminf_{n\to\infty} na_n(T)\ge\frac14\int_a^b|u(t)|v_s(T)\,dt. \ \blacksquare$$

6. l^q and weak-l^q estimates. In this section we show that the sequences $\{a_n(T)\}_{n\in\mathbb{N}}$ and $\{\sigma_n\}_{n\in\mathbb{Z}}$ belong to l^q and weak-l^q sequence spaces with the same exponent q, and have equivalent norms. We first need some preparatory results.

LEMMA 6.1. *Let* $I=[c,d]\subset(a,b)$ *and, for* $\varepsilon>0$, *suppose that*

$$\sigma(\varepsilon):=\{k\in\mathbb{Z}:Z_k\subset I,\ \sigma_k>\varepsilon\}$$

has at least 4 *distinct elements. Then* $A(I)>\varepsilon/8$.

Proof. Let $Z_{k_1},Z_{k_2},Z_{k_3},Z_{k_4}$, with $k_1<k_2<k_3<k_4$, be distinct members of $\sigma(\varepsilon)$, and set $I_1=(\xi_{k_1},\xi_{k_2})$, $I_2=(\xi_{k_2+1},\xi_{k_4})$. Then, with $f_0=\chi_{I_1}+\chi_{I_2}$,

$$\begin{aligned}
A(I)&\ge\inf_\alpha\Big\|v(x)\Big(\int_c^x|u(t)|f_0(t)\,dt-\alpha\Big)\Big\|_{\infty,I}\\
&\ge\inf_\alpha\max\Big\{\|v\|_{\infty,Z_{k_2}}\Big|\int_{I_1}|u(t)|\,dt-\alpha\Big|;\|v\|_{\infty,Z_{k_4}}\Big|\int_{I_1\cup I_2}|u(t)|\,dt-\alpha\Big|\Big\}\\
&=\inf_\alpha\max\{\|v\|_{\infty,Z_{k_2}}|2^{k_2}-2^{k_1}-\alpha|;\\
&\qquad\qquad\|v\|_{\infty,Z_{k_4}}|2^{k_2}-2^{k_1}+2^{k_4}-2^{k_2+1}-\alpha|\}\\
&\ge\inf_\alpha\max\Big\{\frac{\varepsilon}{2^{k_2+1}}|2^{k_2}-2^{k_1}-\alpha|;\frac{\varepsilon}{2^{k_4+1}}|2^{k_2}-2^{k_1}+2^{k_4}-2^{k_2+1}-\alpha|\Big\}\\
&\ge\frac{\varepsilon}{2^{k_4}+1}\cdot\frac12(2^{k_4}-2^{k_2+1})\ge\frac{\varepsilon}{8}. \ \blacksquare
\end{aligned}$$

LEMMA 6.2. *Let* $\varepsilon>0$ *and* $M(\varepsilon)=M((a,b),\varepsilon)$. *Then*

$$\#\{k\in\mathbb{Z}:\sigma_k>8\varepsilon\}\le5M(\varepsilon)+3. \tag{6.1}$$

Proof. Clearly, with $I_i = (c_i, c_{i+1})$ the intervals in (2.7) when $I = (a, b)$,

$$\#\{k \in \mathbb{Z} : c_i \in \overline{Z}_k \text{ for some } i \in \{1, \ldots, M(\varepsilon)\}\} \le 2M(\varepsilon).$$

Also, for every $k \in \mathbb{Z}$ not included in the above set, we have $\overline{Z}_k \subset I_i$ for some $i \in \{1, \ldots, M(\varepsilon)\}$. Hence, by Lemma 6.1,

$$\#\{k \in \mathbb{Z} : \sigma_k > 8\varepsilon\} \le 2M(\varepsilon) + 3(M(\varepsilon)+1) = 5M(\varepsilon) + 3. \blacksquare$$

LEMMA 6.3. *For all $t > 0$,*

$$\#\{k \in \mathbb{Z} : \sigma_k > t\} \le 10\#\{k \in \mathbb{N} : a_k(T) > t/8\} + 23. \tag{6.2}$$

Proof. By Corollary 3.3,

$$\#\{k \in \mathbb{N} : a_k(T) > \varepsilon\} \ge \frac{M(\varepsilon)}{2} - 2.$$

Hence, by Lemma 6.2,

$$\#\{k \in \mathbb{Z} : \sigma_k > t\} \le 5M(t/8) + 3 \le \#\{k \in \mathbb{N} : a_k(T) > t/8\} + 23. \blacksquare$$

LEMMA 6.4. *For all $q > 0$,*

$$\|\{\sigma_k\}\|^q_{l^q(\mathbb{Z})} \le 10 \cdot 8^q \|\{a_k(T)\}\|^q_{l^q(\mathbb{N})} + 23\|\{\sigma_k\}\|^q_{l^\infty(\mathbb{Z})}. \tag{6.3}$$

Proof. Let $\lambda = \|\{\sigma_k\}\|_{l^\infty(\mathbb{Z})}$. Then, by Lemma 6.3,

$$\begin{aligned}
\|\{\sigma_k\}\|^q_{l^q(\mathbb{Z})} &= q\int_0^\lambda t^{q-1}\#\{k \in \mathbb{Z} : \sigma_k > t\}\, dt \\
&\le 10q\int_0^\lambda t^{q-1}\#\{k \in \mathbb{N} : a_k(T) > t/8\}\, dt + 23\lambda^q \\
&\le 10 \cdot 8^q \|\{a_k(T)\}\|^q_{l^q(\mathbb{N})} + 23\lambda^q. \blacksquare
\end{aligned}$$

COROLLARY 6.5. *For any $q > 0$ there exists a constant $C > 0$ such that*

$$\|\{\sigma_k\}\|_{l^q(\mathbb{Z})} \le C\|\{a_k(T)\}\|_{L^q(\mathbb{N})}. \tag{6.4}$$

Proof. By (5.6),

$$\|\{\sigma_k\}\|_{l^\infty(\mathbb{Z})} \le C\|T\| = Ca_1(T) \le C\|\{a_k(T)\}\|_{l^q(\mathbb{N})}.$$

The result then follows from Lemma 6.4. ■

THEOREM 6.6. *For $q \in (1, \infty)$, we have $\{a_k(T)\} \in l^q(\mathbb{N})$ if and only if $\{\sigma_k\} \in l^q(\mathbb{Z})$, and*

$$\|\{\sigma_k\}\|_{l^q(\mathbb{Z})} \asymp \|\{a_k(T)\}\|_{l^q(\mathbb{N})}.$$

Proof. Let I_i, $i = 1, \ldots, N(\varepsilon)$, be the intervals in (2.8) with $I = (a, b)$ and $N(\varepsilon) \equiv N((a, b), \varepsilon)$; note that in view of Lemma 2.1, we have $J(I_i) = \varepsilon$. We group the intervals I_i into families $\mathbf{F}_j$, $j = 1, 2, \ldots$, such that each $\mathbf{F}_j$ consists of the maximal number of those intervals satisfying the hypothesis of Lemma 5.1; they lie within (ξ_{k_0}, ξ_{k_2+1}) for some k_0, k_2, and the next interval

I_k intersects Z_{k_2+1}. Hence, by Lemma 5.1, there is a positive constant c such that

$$\varepsilon\#\mathbf{F}_j \le c \max_{k_0 \le n \le k_2} \sigma_n = c\sigma_{k_j},$$

say. It follows that, with $n_j = [c\sigma_{k_j}/\varepsilon]$,

$$\begin{aligned}
N(\varepsilon) &= \sum_j \#\mathbf{F}_j \le \sum_j \sum_{n=1}^{n_j} 1 = \sum_{n=1}^\infty \sum_{j : n_j \ge n} 1 \\
&= \sum_{n=1}^\infty \#\{j : c\sigma_{k_j}/\varepsilon \ge n\} \le \sum_{n=1}^\infty \#\{k : \sigma_k \ge n\varepsilon/c\}.
\end{aligned} \tag{6.5}$$

Thus, if $\{\sigma_k\} \in l^q(\mathbb{Z})$ for some $q \in (1, \infty)$, then

$$\begin{aligned}
q\int_0^\infty t^{q-1}N(t)\, dt &\le q\int_0^\infty \sum_{n=1}^\infty t^{q-1}\#\{k : \sigma_k > nt/c\}\, dt \\
&= qc^q \int_0^\infty \sum_{n=1}^\infty n^{-q}s^{q-1}\#\{k : \sigma_k > s\}\, ds \\
&\preceq \|\{\sigma_k\}\|^q_{l^q(\mathbb{Z})}
\end{aligned} \tag{6.6}$$

where $\preceq$ stands for less than or equal to a constant multiple of what follows. From Corollary 3.3, $a_{M(\varepsilon)+1}(T) \le 2\varepsilon$ and so

$$\#\{k \in \mathbb{N} : a_k(T) > t\} \le M(t/2) + 1 \le N(t/2) + 1.$$

This yields

$$\begin{aligned}
\|\{a_k(T)\}\|^q_{l^q(\mathbb{N})} &= q\int_0^\infty t^{q-1}\#\{k \in \mathbb{N} : a_k(T) > t\}\, dt \\
&\le q\int_0^{\|T\|} t^{q-1}[N(t/2) + 1]\, dt \\
&\preceq \|\{\sigma_k\}\|^q_{l^q(\mathbb{Z})} + \|T\|^q \preceq \|\{\sigma_k\}\|^q_{l^q(\mathbb{Z})}
\end{aligned}$$

by (6.6) and since $\|T\| \preceq \|\{\sigma_k(T)\}\|_{l^\infty(\mathbb{Z})} \le \|\{\sigma_k\}\|_{l^q(\mathbb{Z})}$, by (5.6). The theorem follows from (6.4). ■

The final result in this section concerns the weak l^q spaces, which we denote by l^q_ω ($l^{q,\infty}$ in the Lorentz scale). Recall that $l^q_\omega(\mathbb{Z})$ is the space of sequences $x = \{x_k\}$ such that

$$\|x\|_{l^q_\omega(\mathbb{Z})} := \sup_{t>0}\{t[\#\{k \in \mathbb{Z} : |x_k| > t\}]^{1/q}\} < \infty.$$

The space $l^q_\omega(\mathbb{N})$ is defined analogously.

THEOREM 6.7. *For $q \in (1,\infty)$, we have $\{a_k(T)\} \in l^q_\omega(\mathbb{N})$ if and only if $\{\sigma_k\} \in l^q_\omega(\mathbb{Z})$, and*

$$\|\{\sigma_k\}\|_{l^q_\omega(\mathbb{Z})} \asymp \|\{a_k(T)\}\|_{l^q_\omega(\mathbb{N})}.$$

Proof. Suppose $\{\sigma_k\} \in l^q_\omega(\mathbb{Z})$. From Corollary 3.3 and (6.5),

$$\|\{a_k(T)\}\|^q_{l^q_\omega(\mathbb{N})} \preceq \sup_{t>0}\{t^q M(t)\} \le \sup_{t>0}\{t^q N(t)\} \le \sum_{n=1}^{\infty} t^q \#\{k : \sigma_k \ge nt/c\}$$
$$\le \sum_{n=1}^{\infty} \|\{\sigma_k\}\|^q_{l^q_\omega(\mathbb{Z})}(c/n)^q \preceq \|\{\sigma_k\}\|^q_{l^q_\omega(\mathbb{Z})}.$$

Now suppose that $\{a_k(T)\} \in l^q_\omega(\mathbb{N})$. From Lemma 6.3,

$$\sup_{t>0}(t^q \#\{k \in \mathbb{Z} : \sigma_k > t\}) \preceq \sup_{t>0}(t^q(\#\{k \in \mathbb{N} : a_k(T) > t/8\} + 1)).$$

Since

$$\#\{k \in \mathbb{N} : a_k(T) > t/8\} \ge \frac{M(t/8)}{2} - 2 \ge 1$$

for sufficiently small t, we conclude that

$$\sup_{t>0}(t^q \#\{k \in \mathbb{Z} : \sigma_k > t\}) \preceq \sup_{t>0}(t^q \#\{k \in \mathbb{N} : a_k(T) > t/8\}).$$

This implies that $\{\sigma_k\} \in l^q_\omega(\mathbb{Z})$ and $\|\{\sigma_k\}\|_{l^q_\omega(\mathbb{Z})} \preceq \|\{a_k(T)\}\|_{l^q_\omega(\mathbb{N})}$. The theorem is therefore proved. ■

7. The operator T on L^1. In this case the assumptions (2.1) and (2.2) on u and v are replaced by

$$u \in L^\infty(a,x), \tag{7.1}$$
$$v \in L^1(x,b), \tag{7.2}$$

for all $x \in (a,b)$. On setting $a = -B$, $b = -A$, $\widehat{f}(x) = f(-x)$, and similarly for u, v in (1.4), we see that

$$T\widehat{f}(x) = \widehat{v}(x)\int_x^B \widehat{u}(t)\widehat{f}(t)\,dt, \quad A \le x \le B.$$

But this is the adjoint of the map $S : L^\infty(A,B) \to L^\infty(A,B)$ defined by

$$Sg(x) = \widehat{u}(x)\int_A^x \widehat{v}(t)g(t)\,dt, \quad A \le x \le B.$$

Hence, T and S have the same norms and their approximation numbers are equal if one, and hence both, are compact (see [1; Proposition II.2.5]). The results for $T : L^1(a,b) \to L^1(a,b)$ therefore follow from those proved for the $L^\infty(a,b)$ case on interchanging u and v. Before stating the results, we need some new terminology.

Let $\eta_k \in \mathbb{R}^+$ be defined by

$$V(x) := \int_x^b |v(t)|\,dt, \quad V(\eta_k) = 2^k, \tag{7.3}$$

where $k \in \mathbb{Z}$ if $v \in L^1(a,b)$, but otherwise $2^k \le \|v\|_1$. Set

$$\zeta_k := \|uv\|_{\infty,W_k}, \quad W_k = (\eta_k, \eta_{k+1}),$$

with $\zeta_k = 0$ if $v \in L^1(a,b)$ and $2^k > \|v\|_1$.

THEOREM 7.1. *Suppose that* (7.1) *and* (7.2) *are satisfied. Then*

(i) *T in* (1.4), *as a map from $L^1(a,b)$ into $L^1(a,b)$, is bounded if and only if $\{\zeta_k\} \in l^\infty(\mathbb{Z})$, in which case*

$$\|T\| \asymp \|\{\zeta_k\}\|_{l^\infty(\mathbb{Z})};$$

(ii) *T is compact if and only if $\lim_{k\to\pm\infty}\zeta_k = 0$;*

(iii) *if $\{\zeta_k\} \in l^1(\mathbb{Z})$ then*

$$\frac{1}{4}\int_a^b u_s(t)|v(t)|\,dt \le \liminf_{n\to\infty} n a_n(T) \le \limsup_{n\to\infty} n a_n(T) \le 2\int_a^b u_s(t)|v(t)|\,dt;$$

(iv) *for $q \in (1,\infty)$, we have $\{a_k(T)\} \in l^q(\mathbb{N})$ if and only if $\{\zeta_k\} \in l^q(\mathbb{Z})$ and*

$$\|\{\zeta_k\}\|_{l^q(\mathbb{Z})} \asymp \|\{a_k(T)\}\|_{l^q(\mathbb{N})};$$

(v) *for $q \in (1,\infty)$, we have $\{a_k(T)\} \in l^q_\omega(\mathbb{N})$ if and only if $\{\zeta_k\} \in l^q_\omega(\mathbb{Z})$ and*

$$\|\{\zeta_k\}\|_{l^q_\omega(\mathbb{Z})} \asymp \|\{a_k(T)\}\|_{l^q_\omega(\mathbb{N})}.$$

REMARK 7.2. Let M be a dense subset of $(0,1)$ with measure $|M| = \alpha < 1$ and let $u = 1$, $v = \chi_M$. Then $v_s = 1$, $(v - v_s)_s = 1$ on $(0,1)$ and so

$$\|v\|_{\infty,(x,1)} = \|v_s\|_{\infty,(x,1)} = \|v - v_s\|_{\infty,(x,1)}$$

for any $x \in (0,1)$. Since

$$\|T_{u,v} \mid L^\infty(0,1) \to L^\infty(0,1)\| = \sup_{0<x<1}\Big\{\int_0^x dt\,\|v\|_{\infty,(x,1)}\Big\}$$

(see [6]), where $T_{u,v}$ denotes the operator in (1.4), it follows that

$$\|T_{u,v}\| = \|T_{u,v_s}\| = \|T_{u,v} - T_{u,v-v_s}\|,$$

for the operator norms from $L^\infty(0,1)$ to $L^\infty(0,1)$. Also,

$$\int_0^1 |u(t)v(t)|\,dt = |M| < 1 = \int_0^1 |u(t)|v_s(t)\,dt.$$

The choice $u = \chi_M$, $v = 1$ gives an analogous example in the $L^1(0,1)$ case.

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References

[1] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Oxford Univ. Press, Oxford, 1987.

[2] D. E. Edmunds, W. D. Evans and D. J. Harris, *Approximation numbers of certain Volterra integral operators*, J. London Math. Soc. (2) 37 (1988), 471–489.

[3] —, —, —, *Two-sided estimates of the approximation numbers of certain Volterra integral operators*, Studia Math. 124 (1997), 59–80.

[4] D. E. Edmunds, P. Gurka and L. Pick, *Compactness of Hardy-type integral operators in weighted Banach function spaces*, ibid. 109 (1994), 73–90.

[5] J. Newman and M. Solomyak, *Two-sided estimates of singular values for a class of integral operators on the semi-axis*, Integral Equations Operator Theory 20 (1994), 335–349.

[6] B. Opic and A. Kufner, *Hardy-type Inequalities*, Pitman Res. Notes Math. Ser. 219, Longman Sci. & Tech., Harlow, 1990.

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Corrigendum and addendum: "On the axiomatic theory of spectrum II"

by

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Abstract. The main purpose of this paper is to correct the proof of Theorem 15 of [4], concerned with the stability of the class of quasi-Fredholm operators under finite rank perturbations, and to answer some open questions raised there.

Recall some notations and terminology from [4].

For closed subspaces M, L of a Banach space X we write $M \overset{e}{\subset} L$ (M is *essentially contained* in L) if there is a finite-dimensional subspace $F \subset X$ such that $M \subset L + F$. Equivalently, $\dim M/(M \cap L) = \dim(M+L)/L < \infty$. Similarly we write $M \overset{e}{=} L$ if $M \overset{e}{\subset} L$ and $L \overset{e}{\subset} M$.

For a (bounded linear) operator $T \in \mathcal{L}(X)$ write $R^\infty(T) = \bigcap_{n=0}^\infty R(T^n)$ and $N^\infty(T) = \bigcup_{n=0}^\infty N(T^n)$.

An operator $T \in \mathcal{L}(X)$ is called *semiregular* (*essentially semiregular*) if $R(T)$ is closed and $N(T) \subset R^\infty(T)$ ($N(T) \overset{e}{\subset} R^\infty(T)$, respectively). Further, T is called *quasi-Fredholm* if there exists $d \ge 0$ such that $R(T^{d+1})$ is closed and $R(T) + N(T^d) = R(T) + N^\infty(T)$ (equivalently, $N(T) \cap R(T^d) = N(T) \cap R^\infty(T)$).

The proof of Theorem 15 of [4] relies on the following statement (where d is the integer whose existence is postulated in the definition of quasi-Fredholm operators):

If T is quasi-Fredholm and F of rank 1 then $N(T) \cap R(T^d) \subset R^\infty(T+F)$.

This, however, need not be satisfied.

COUNTEREXAMPLE. Let H be the Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots\}$. Define $T, F \in \mathcal{L}(H)$ by

$$Te_1 = 0, \quad Te_n = e_{n-1} \quad (n \ge 2), \qquad Fe_2 = -e_1, \quad Fe_n = 0 \quad (n \ne 2).$$

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