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The symmetric tensor product of a direct sum of locally convex spaces

by

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Abstract. An explicit representation of the n-fold symmetric tensor product (equipped with a natural topology τ such as the projective, injective or inductive one) of the finite direct sum of locally convex spaces is presented. The formula for $\bigotimes^n_{\tau,s}(F_1 \oplus F_2)$ gives a direct proof of a recent result of Díaz and Dineen (and generalizes it to other topologies τ) that the n-fold projective symmetric and the n-fold projective "full" tensor product of a locally convex space E are isomorphic if E is isomorphic to its square E^2.

1. Symmetric tensor products

1.1. If $E_1, \ldots, E_n, E$ and F are vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ we denote by $L(E_1, \ldots, E_n; F)$ the space of n-linear maps $E_1 \times \ldots \times E_n \to F$. We write briefly $L(^nE; F) := L(E, \ldots, E; F)$ and $L_s(^nE; F)$ for the space of n-linear symmetric maps $E \times \ldots \times E \to F$; the space of n-homogeneous polynomials $E \to F$ is denoted by $P^n(E; F)$ (they are the restrictions to the diagonal of $E \times \ldots \times E$ of elements in $L(^nE; F)$). The polarization formula gives a natural isomorphism $P^n(E; F) = L_s(^nE; F)$. If the underlying spaces are locally convex we denote by $\mathcal{L}(E_1, \ldots, E_n; F)$ or $\mathcal{L}(^nE; F)$ if $E = E_1 = \ldots = E_n$, $\mathcal{L}_s(^nE; F)$ and $\mathcal{P}^n(E; F)$ the spaces of continuous n-linear, continuous n-linear symmetric mappings and continuous n-homogeneous polynomials respectively. Moreover, we use $L(^nE) := L(^nE; \mathbb{K})$ and, similarly, $L_s(^nE), P^n(E), \mathcal{L}(^nE)$, $\mathcal{L}_s(^nE)$ and $\mathcal{P}^n(E)$ in the case of $F = \mathbb{K}$. We shall write $E \cong F$ if the two locally convex spaces E and F are topologically isomorphic.

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1.2. The n-fold tensor product $\bigotimes_{j=1}^n E_j = E_1 \otimes \ldots \otimes E_n$ together with the n-linear map $\otimes : E_1 \times \ldots \times E_n \to \bigotimes_{j=1}^n E_j$ is, up to isomorphism, uniquely defined by the following universal property: For every vector space F and every $\varphi \in L(E_1, \ldots, E_n; F)$ there is a unique "linearization" $\varphi^L \in L(\bigotimes_{j=1}^n E_j; F)$ with $\varphi = \varphi^L \circ \otimes$:

$$L(E_1, \ldots, E_n; F) = L(\bigotimes_{j=1}^n E_j; F), \qquad \varphi \rightsquigarrow \varphi^L.$$

1.3. If $E = E_1 = \ldots = E_n$ and $\eta \in S_n$ (the group of permutations of $\{1, \ldots, n\}$) the n-linear map

$$E \times \ldots \times E \ni (x_1, \ldots, x_n) \rightsquigarrow x_{\eta^{-1}(1)} \otimes \ldots \otimes x_{\eta^{-1}(n)} \in \bigotimes^n E$$

has a linearization $\bigotimes^n E \to \bigotimes^n E$ which is denoted by $z \rightsquigarrow z^\eta$. It is easy to see that $(z^\eta)^\sigma = z^{\sigma \circ \eta}$; this relation justifies the use of η^{-1} instead of η in the definition (see also [13]), but this definition will also be convenient for our calculations in Section 3. The symmetrization map $\sigma_E^n : \bigotimes^n E \to \bigotimes^n E$ is defined by

$$\sigma_E^n(z) := \frac{1}{n!} \sum_{\eta \in S_n} z^\eta .$$

Using the polarization formula it is not difficult to see that

$$\bigotimes_s^n E := \sigma_E^n(\bigotimes^n E) = \operatorname{span}\{\bigotimes^n x := \underbrace{x \otimes \ldots \otimes x}_{n \text{ times}} \mid x \in E\};$$

moreover, an element $z \in \bigotimes^n E$ is in $\bigotimes_s^n E$ if and only if $z = z^\eta$ for all $\eta \in S_n$. This is why the elements in $\bigotimes_s^n E$ may be called *symmetric* tensors. The pair $(G, \varphi_0) := (\bigotimes_s^n E, \sigma_E^n \circ \otimes)$ has the universal mapping property for symmetric n-linear maps: For every F and every $\varphi \in L_s({}^nE; F)$ there is a unique $T \in L(G; F)$ with $\varphi = T \circ \varphi_0$. It follows that for two such pairs (G_j, φ_0^j) there is an isomorphism $I : G_1 \to G_2$ with $\varphi_0^2 = I \circ \varphi_0^1$ and $\varphi_0^1 = I^{-1} \circ \varphi_0^2$. The pair $(\bigotimes_s^n E, \sigma_E^n \circ \otimes)$ is therefore called the *symmetric n-fold tensor product* of E; the mapping σ_E^n is a projection of the "full" n-fold tensor product $\bigotimes^n E$ onto the symmetric n-fold tensor product $\bigotimes_s^n E$. Note that $\bigotimes_s^1 E = \bigotimes^1 E = E$.

1.4. The universal property of the symmetric tensor product gives

$$P^n(E; F) = L_s({}^nE; F) = L(\bigotimes_s^n E; F), \qquad q \rightsquigarrow \check{q} \rightsquigarrow (\check{q})^L|_{\bigotimes_s^n E},$$

and the map

$$L({}^nE; F) \to L_s({}^nE; F), \qquad \varphi \rightsquigarrow \varphi^L \circ \sigma_E^n \circ \otimes,$$

is the symmetrization of n-linear maps.

1.5. If $E_1, \ldots, E_n$ are locally convex, the projective (locally convex) topology π on $\bigotimes_{j=1}^n E_j$ (notation: $\bigotimes_{\pi,j=1}^n E_j$) is uniquely determined by the property that, for each locally convex space F, every $\varphi \in L(E_1, \ldots, E_n; F)$ is continuous if and only if $\varphi^L : \bigotimes_{\pi,j=1}^n E_j \to F$ is continuous:

$$\mathcal{L}(E_1, \ldots, E_n; F) = \mathcal{L}(\bigotimes_{\pi,j=1}^n E_j; F).$$

If $E = E_1 = \ldots = E_n$, then each mapping $z \rightsquigarrow z^\eta$ from 1.3 is continuous with respect to the projective topology and hence so is the projection σ_E^n onto $\bigotimes_s^n E$. It follows that $\bigotimes_s^n E$ equipped with the induced topology (notation: $\bigotimes_{\pi,s}^n E$) is a complemented topological subspace of $\bigotimes_\pi^n E$ and that

$$\mathcal{P}^n(E; F) = \mathcal{L}_s({}^nE; F) = \mathcal{L}(\bigotimes_{\pi,s}^n E; F).$$

In Section 3 we shall investigate other topologies. Note that, if E is normed, the "natural" norm on $\bigotimes_\pi^n E$ induces a norm on $\bigotimes_s^n E$ which is equivalent (but in general not equal) to the "natural" projective norm on $\bigotimes_s^n E$. For more details on symmetric tensor products see [16] and [9].

1.6. If $F := \bigoplus_{j=1}^n E_j$, and I_j and P_j are the natural injections and projections then (see [5], [7], [3] and [1])

$$\mathrm{id}_{\bigotimes_{j=1}^n E_j} : \bigotimes_{j=1}^n E_j \xrightarrow{I_1 \otimes \ldots \otimes I_n} \bigotimes^n F \xrightarrow{\sigma_F^n} \bigotimes_s^n F \hookrightarrow \bigotimes^n F \xrightarrow{n!(P_1 \otimes \ldots \otimes P_n)} \bigotimes_{j=1}^n E_j,$$

hence $\bigotimes_{j=1}^n E_j$ is isomorphic to a complemented subspace of $\bigotimes_s^n F$. In particular: If E is a locally convex space which is topologically isomorphic to E^n then the projective full tensor product $\bigotimes_\pi^n E$ is topologically isomorphic to a complemented subspace of the symmetric tensor product $\bigotimes_{\pi,s}^n E$—and the same holds for any symmetric n-tensor topology such as the injective topology ε or the hypocontinuous topologies of L. Schwartz such as the inductive topology ι (see Section 3 for the definitions).

1.7. It follows from 1.5 and 1.6 that if E is isomorphic to E^n (or even to E^2), then the spaces $\bigotimes_{\pi,s}^n E$ and $\bigotimes_\pi^n E$ are isomorphic to complemented subspaces of each other. If, in particular, E is a Banach space, the same happens for the Banach spaces $\widetilde{\bigotimes}_{\pi,s}^n E$ and $\widetilde{\bigotimes}_\pi^n E$ (completions). Though these two spaces are even isomorphic if $E \cong E^2$ (this is a result of Díaz and Dineen [8] which we shall prove differently) it is in general not true that two Banach spaces which are complemented in each other are isomorphic: this follows from Gowers' results in [12]. This observation is taken from [8]. The case where E is only isomorphic to E^n (for $n > 2$—these spaces need not be isomorphic to E^2 by Gowers' results) remains open.

2. A formula for the symmetric tensor product of a direct sum

2.1. If $A \in L(E; F)$ and $k \in \mathbb{N}$ we denote by $\bigotimes^k A$ the mapping

$$A \otimes \ldots \otimes A : \bigotimes^k E \to \bigotimes^k F.$$

We shall need for every $k = 0, \ldots, n$ the sets

$$S_n^k := \{\eta \in S_n \mid \eta|_{\{1,\ldots,k\}} \text{ and } \eta|_{\{k+1,\ldots,n\}} \text{ are increasing}\},$$
$$T_n := \{f \mid f : \{1,\ldots,n\} \to \{1,2\}\},$$
$$T_n^k := \{f \in T_n \mid \operatorname{card} f^{-1}(1) = k\}.$$

Note that $S_n^0 = S_n^n = \{\mathrm{id}\}$ and T_n^0 (resp. T_n^n) consists just of the constant mapping 2 (resp. 1).

2.2. Take $E = F_1 \oplus F_2$ with projections $P_j : E \to F_j$ and define for every $k = 0, \ldots, n$,

$$Q_k : \bigotimes^n E \xrightarrow{[\bigotimes^k P_1]\otimes[\bigotimes^{n-k} P_2]} [\bigotimes^k F_1] \otimes [\bigotimes^{n-k} F_2] \xrightarrow{\sigma_{F_1}^k \otimes \sigma_{F_2}^{n-k}} [\bigotimes_s^k F_1] \otimes [\bigotimes_s^{n-k} F_2]$$

with the obvious meaning for $k = 0$ and $k = n$ (i.e. just omit $\bigotimes_s^0 F_j$ and $\bigotimes^0 F_j$). Since $[\bigotimes_s^k F_1] \otimes [\bigotimes_s^{n-k} F_2] \subset [\bigotimes^k F_1] \otimes [\bigotimes^{n-k} F_2] \subset \bigotimes^n E$ we can make all calculations in $\bigotimes^n E$. Observe that

$$(*) \qquad Q_k(\bigotimes^n x) = [\bigotimes^k P_1(x)] \otimes [\bigotimes^{n-k} P_2(x)].$$

It is clear that for every $f \in T_n^k$ there is a unique $\eta \in S_n^k$ with

$$(**) \qquad P_{f(1)}x \otimes \ldots \otimes P_{f(n)}x = [Q_k(\bigotimes^n x)]^\eta$$

for every $x \in E$ and vice versa. It follows that

$$\bigotimes^n x = \bigotimes^n [P_1(x) + P_2(x)] = \sum_{f \in T_n} P_{f(1)}x \otimes \ldots \otimes P_{f(n)}x = \sum_{k=0}^n \sum_{\eta \in S_n^k} [Q_k(\bigotimes^n x)]^\eta$$

and hence

$$(***) \qquad z = \sum_{k=0}^n \sum_{\eta \in S_n^k} [Q_k(z)]^\eta$$

for every $z \in \bigotimes_s^n E$.

THEOREM. *The linear mapping*

$$Q : \bigotimes_s^n (F_1 \oplus F_2) \longrightarrow \bigoplus_{k=0}^n [\bigotimes_s^k F_1] \otimes [\bigotimes_s^{n-k} F_2]$$

defined by $Q(z) := Q_0(z) \oplus \ldots \oplus Q_n(z)$ *is an isomorphism; its inverse is*

$$Q^{-1}(w_0 \oplus \ldots \oplus w_n) = \sum_{k=0}^n \sum_{\eta \in S_n^k} [w_k]^\eta$$

for $w_k \in [\bigotimes_s^k F_1] \otimes [\bigotimes_s^{n-k} F_2]$.

Proof. It follows from $(***)$ that Q is injective. To show that it is onto take $w_k \in [\bigotimes_s^k F_1] \otimes [\bigotimes_s^{n-k} F_2]$ for $k = 0, \ldots, n$ and define

$$z := \sum_{k=0}^n \sum_{\eta \in S_n^k} [w_k]^\eta \in \bigotimes^n E;$$

we have to show that (a) $Q_l(z) = w_l$ for all $l = 0, \ldots, n$ and (b) z is symmetric, i.e. $z \in \bigotimes_s^n E$.

(a) Since for every $k = 0, \ldots, n$ we have

$$[\bigotimes^k P_1] \otimes [\bigotimes^{n-k} P_2](w_k) = w_k$$

and $P_1 \circ P_2 = P_2 \circ P_1 = 0$ it follows that

$$\begin{aligned}
&[\bigotimes^l P_1] \otimes [\bigotimes^{n-l} P_2]([w_k]^\eta) \\
&\quad = [\bigotimes^l P_1] \otimes [\bigotimes^{n-l} P_2]([\bigotimes^k P_1] \otimes [\bigotimes^{n-k} P_2](w_k))^\eta \\
&\quad = \begin{cases} w_k & \text{if } k = l \text{ and } \eta = \mathrm{id}, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

hence $Q_l(w_k^\eta) = 0$ if either $k \neq l$ or $\eta \neq \mathrm{id}$ and $Q_l(w_l) = w_l$ since $\sigma_{F_1}^l \otimes \sigma_{F_2}^{n-l}(w_l) = w_l$.

(b) To show that $z \in \bigotimes_s^n E$ observe first that $w_0, w_n \in \bigotimes_s^n E$. Therefore it is enough, by linearity, to prove that for each $k = 1, \ldots, n-1$ and each $x_j \in F_j$,

$$u := \sum_{\eta \in S_n^k} ([\bigotimes^k x_1] \otimes [\bigotimes^{n-k} x_2])^\eta = \sum_{f \in T_n^k} x_{f(1)} \otimes \ldots \otimes x_{f(n)}$$

(by $(**)$) is symmetric: For this take $\sigma \in S_n$ and $f \in T_n^k$; then

$$[x_{f(1)} \otimes \ldots \otimes x_{f(n)}]^\sigma = x_{f(\sigma^{-1}(1))} \otimes \ldots \otimes x_{f(\sigma^{-1}(n))};$$

since the mapping $T_n^k \to T_n^k$ defined by $f \rightsquigarrow f \circ \sigma^{-1}$ is clearly bijective we obtain

$$u^\sigma = \sum_{f \in T_n^k} x_{f(\sigma^{-1}(1))} \otimes \ldots \otimes x_{f(\sigma^{-1}(n))} = \sum_{g \in T_n^k} x_{g(1)} \otimes \ldots \otimes x_{g(n)} = u$$

for all $\sigma \in S_n$, which proves $u \in \bigotimes_s^n E$ (see 1.3). ∎

2.3. If we write the theorem as

$$\bigotimes_s^n (F_1 \oplus F_2) = \bigoplus_{k+l=n} [\bigotimes_s^k F_1] \otimes [\bigotimes_s^l F_2]$$

an easy induction gives

COROLLARY.

$$\bigotimes_s^n[\bigoplus_{j=1}^m F_j] = \bigoplus_{\substack{l_1+\ldots+l_m=n\\ l_j\in\{0,\ldots,n\}}} \bigotimes_{j=1}^m [\bigotimes_s^{l_j} F_j].$$

2.4. Observing that $\dim \bigotimes_s^l F = \binom{l+k-1}{k-1}$ if $\dim F = k$ (this formula is well known but also follows from Corollary 2.3 applied to $F = \bigoplus_{j=1}^k \mathbb{K}$) we obtain for $m, n, k_1, \ldots, k_m \in \mathbb{N}$ and $k := \sum_{i=1}^m k_j$ the combinatorial formula

$$\binom{n+k-1}{k-1} = \sum_{\substack{l_1+\ldots+l_m=n\\ l_j\in\{0,\ldots,n\}}} \prod_{j=1}^m \binom{l_j+k_j-1}{k_j-1}.$$

This can also be proved by looking at the coefficients of the power series of

$$(1-x)^{-k} = \prod_{j=1}^m (1-x)^{-k_j}$$

around 0 (we owe this remark to P. Brumatti, Campinas).

3. Tensor topologies

3.1. In order to see for which topologies the isomorphism in Theorem 2.2 is even topological we use the following definition (for $n = 2$ see [10] and [11]): An *n-tensor topology* τ (for locally convex spaces) assigns to each n-tuple $(E_1, \ldots, E_n)$ of locally convex spaces a locally convex topology $\tau(E_1, \ldots, E_n)$ on the n-fold tensor product $E_1 \otimes \ldots \otimes E_n$ (notation: $\bigotimes_{\tau,j=1}^n E_j$ or $\bigotimes_\tau(E_1, \ldots, E_n)$) such that

(1) *The canonical mapping* $E_1 \times \ldots \times E_n \to \bigotimes_\tau(E_1, \ldots, E_n)$ *is separately continuous.*

(2) *If* $D_j \subset E_j'$ *are equicontinuous subsets then the set*

$$\{\varphi_1 \otimes \ldots \otimes \varphi_n \mid \varphi_j \in D_j\} \subset (E_1 \otimes \ldots \otimes E_n)^*$$

is τ*-equicontinuous.*

(3) (The mapping property) *If* $T_j \in \mathcal{L}(E_j; F_j)$ *then*

$$T_1 \otimes \ldots \otimes T_n : \bigotimes_\tau(E_1, \ldots, E_n) \to \bigotimes_\tau(F_1, \ldots, F_n)$$

is continuous.

For $n = 1$ we obtain $\bigotimes_\tau^1 E = E$ topologically. Examples:

(a) The projective topology π.

(b) The injective topology ε.

(c) More general: If α is a tensor norm on n-fold tensor products of normed spaces (i.e. $\varepsilon \le \alpha \le \pi$ and the metric mapping property

$$\|T_1 \otimes \ldots \otimes T_n : \bigotimes_\alpha(E_1, \ldots, E_n) \to \bigotimes_\alpha(F_1, \ldots, F_n)\| = \prod_{j=1}^n \|T_j : E_j \to F_j\|$$

holds) then one can define (as in the case of $n = 2$, see [6], §35) the locally convex tensor norm topology associated with α; this is also an n-tensor topology.

(d) The hypocontinuous topologies of L. Schwartz ([17], I, p. 18; for $n = 2$ see also [10]): If a is a cover prescription of bounded sets (i.e. for each locally convex space E the set $a(E)$ is a filtrating set of bounded absolutely convex sets with $\bigcup a(E) = E$ and $T(A) \in a(F)$ for every $A \in a(E)$ and $T \in \mathcal{L}(E; F)$) then $\varphi \in L(E_1, \ldots, E_n; F)$ is called *a-hypocontinuous* if for all $k = 1, \ldots, n$ and all $A_j \in a(E_j)$ the restriction of φ to $A_1 \times \ldots \times A_{k-1} \times E_k \times A_{k+1} \times \ldots \times A_n$ is continuous. There is a unique locally convex topology $\tau_a(E_1, \ldots, E_n)$ on $E_1 \otimes \ldots \otimes E_n$ such that

(1) the tensor map $E_1 \times \ldots \times E_n \to \bigotimes_{\tau_a}(E_1, \ldots, E_n)$ is a-hypocontinuous,

(2) $\varphi \in L(E_1, \ldots, E_n; F)$ is a-hypocontinuous if and only if its linearization $\varphi^L : \bigotimes_{\tau_a}(E_1, \ldots, E_n) \to F$ is continuous.

It is easy to see that τ_a defines an n-tensor topology. The proofs of these facts are straightforward generalizations of the case $n = 2$.

(e) In particular: the inductive topology ι of Grothendieck [14] (take $a(E) := \{A \subset E \mid A$ bounded, absolutely convex, finite-dimensional$\}$) which gives separate continuity and the topology β where $a(E) := \{A \subset E \mid A$ bounded, absolutely convex$\}$ are n-tensor topologies for all $n \ge 2$.

It is rather obvious that τ is an n-tensor topology if and only if $\varepsilon \subset \tau \subset \iota$ and τ has the mapping property.

3.2. An n-tensor topology is called *symmetric* if for each locally convex space E and each $\eta \in S_n$ the mapping

$$\bigotimes_\tau^n E \to \bigotimes_\tau^n E, \quad z \rightsquigarrow z^\eta,$$

(see 1.3) is continuous. In this case the natural projection σ_E^n onto the symmetric tensor product $\bigotimes_s^n E$ is continuous. If one equips $\bigotimes_s^n E$ with the induced topology from $\bigotimes_\tau^n E$ (notation: $\bigotimes_{\tau,s}^n E$) then $\bigotimes_{\tau,s}^n E$ is a topologically complemented subspace of $\bigotimes_\tau^n E$. The projective, injective and all hypocontinuous n-tensor topologies are symmetric. It might be worth mentioning that symmetric n-tensor topologies τ induce on $\bigotimes_s^n E$ a topology which has the following mapping property: If $T \in \mathcal{L}(E; F)$ then $\bigotimes_s^n T : \bigotimes_{\tau,s}^n E \to \bigotimes_{\tau,s}^n F$ is continuous as well. This implies, for example, that $\bigotimes_{\tau,s}^n F$ is topologically complemented in $\bigotimes_{\tau,s}^n E$ if $F \subset E$ is complemented.

3.3. A *tensor topology* for locally convex spaces is a sequence $\tau = (\tau_n)_{n\in\mathbb{N}}$ of n-tensor topologies τ_n which is *associative*: For all $n, k \in \mathbb{N}$ with $k < n$ and E_j the equality

$$(\bigotimes_{\tau_k, j=1}^{k} E_j) \otimes_{\tau_2} (\bigotimes_{\tau_{n-k}, j=k+1}^{n} E_j) = \bigotimes_{\tau_n, j=1}^{n} E_j$$

holds topologically. Notation: $\bigotimes_{\tau,j=1}^{n} E_j := \bigotimes_{\tau_n,j=1}^{n} E_j$.

Note that $E \otimes_\tau \mathbb{K} = E$ for all tensor topologies since this is true for ε and ι. A tensor topology $\tau = (\tau_n)$ is called *symmetric* if all τ_n are symmetric.

PROPOSITION. *ε, π and ι are symmetric tensor topologies.*

Proof. It remains to show the associativity: For π this is rather straightforward, for ε it was proved in ([17], I, p. 38) and for ι one uses the properties of 3.1(d). ■

We do not know whether the topology β of Schwartz (see 3.1(e)) is associative, i.e. is a tensor topology. Note that $\beta = \pi$ on $\bigotimes_{j=1}^{n} E_j$ if all E_j are metrizable (by sequential continuity) or if all E_j are gDF-spaces (by [15], p. 335); in this case $\bigotimes_{\beta,j=1}^{n} E_j$ is metrizable or gDF respectively.

3.4. Coming back to Theorem 2.1 we take a symmetric tensor topology τ and see that Q and Q^{-1} are continuous; note that Q^{-1} factors through

$$\begin{aligned}[\textstyle\bigotimes_{\tau,s}^{k} F_1] \otimes_\tau [\textstyle\bigotimes_{\tau,s}^{n-k} F_2] &\hookrightarrow [\textstyle\bigotimes_{\tau}^{k} F_1] \otimes_\tau [\textstyle\bigotimes_{\tau}^{n-k} F_2] \\ &\hookrightarrow [\textstyle\bigotimes_{\tau}^{k} E] \otimes_\tau [\textstyle\bigotimes_{\tau}^{n-k} E] = \textstyle\bigotimes_{\tau}^{n} E.\end{aligned}$$

Therefore we obtain

THEOREM. *If $F_1, \ldots, F_n$ are locally convex spaces then*

$$\bigotimes_{\tau,s}^{n} [\bigoplus_{j=1}^{m} F_j] \cong \bigoplus_{\substack{l_1+\ldots+l_m=n \\ l_j\in\{0,\ldots,n\}}} \bigotimes_{\tau,j=1}^{m} [\textstyle\bigotimes_{\tau,s}^{l_j} F_j]$$

for all symmetric tensor topologies τ.

The isomorphism was described in 2.2 and 2.3. Note that for this theorem it would be enough to have an associative tuple $(\tau_2, \ldots, \tau_n)$ of symmetric tensor topologies τ_k.

3.5. A first application of this formula (for $m = 2$) is the following: If E is isomorphic to a proper complemented subspace (such as a hyperplane), i.e. $E \cong E \oplus F$ (topologically) with $F \neq \{0\}$ then Theorem 3.4 implies that $\bigotimes_{\tau,s}^{k} E$ *is isomorphic to a topologically complemented subspace of* $\bigotimes_{\tau,s}^{n} E$ *for all $k = 1, \ldots, n-1$ and all symmetric tensor topologies τ*. This result is true, however, without the assumption $E \cong E \oplus F$: this was proved by Blasco [3], [4] for the projective topology, but his proof can be easily adapted to hold for arbitrary symmetric tensor topologies τ.

4. Stable locally convex spaces

4.1. If the locally convex space E is *stable*, i.e. topologically isomorphic to its square E^2, Díaz and Dineen [8] showed that $\mathcal{L}_s(^nE)$ is isomorphic to $\mathcal{L}(^nE)$ and deduced from this that the symmetric and full projective n-fold tensor products are isomorphic. The formula in 3.4 gives this directly—and for some other topologies as well.

THEOREM. *If E is a stable locally convex space then*

$$\bigotimes_{\tau,s}^{n} E \cong \bigotimes_{\tau}^{n} E$$

for all symmetric tensor topologies τ and all $n \in \mathbb{N}$.

Proof. We use ideas from [8]. Let $E = F_1 \oplus F_2$ with $F_j \cong E$. Then using the properties of tensor topologies gives

$$\begin{aligned} H_k : &= \textstyle\bigotimes_{\tau}^{k} E = (\bigotimes_{\tau}^{k-1} E) \otimes_\tau (F_1 \oplus F_2) \\ &= [(\textstyle\bigotimes_{\tau}^{k-1} E) \otimes_\tau F_1] \oplus [(\bigotimes_{\tau}^{k-1} E) \otimes_\tau F_2] \cong H_k^2 \end{aligned}$$

and hence $H_k \cong \ldots \cong H_k^l$ for all $l \in \mathbb{N}$.

As we have already noted, $\bigotimes_{\tau}^{1} E = E = \bigotimes_{\tau,s}^{1} E$. We proceed by induction and assume that $G_k := \bigotimes_{\tau,s}^{k} E \cong H_k$ for all $k < n$. Then, by Theorem 3.4, we obtain

$$\begin{aligned} G_n &= \textstyle\bigotimes_{\tau,s}^{n} (F_1 \oplus F_2) \\ &\cong [\textstyle\bigotimes_{\tau,s}^{n} F_1] \oplus [\bigoplus_{k=1}^{n-1} (\bigotimes_{\tau,s}^{k} F_1) \otimes_\tau (\bigotimes_{\tau,s}^{n-k} F_2)] \oplus [\bigotimes_{\tau,s}^{n} F_2] \\ &\cong G_n^2 \oplus [\textstyle\bigoplus_{k=1}^{n-1} (\bigotimes_{\tau}^{k} E) \otimes_\tau (\bigotimes_{\tau}^{n-k} E)] = G_n^2 \oplus H_n^{n-1} \cong G_n^2 \oplus H_n. \end{aligned}$$

Now $H_n = G_n \oplus V$ (topologically, see 3.2), hence the two formulas $H_n \cong H_n^l$ and $G_n \cong G_n^2 \oplus H_n$ give

$$H_n = G_n \oplus V \cong G_n^2 \oplus H_n \oplus V \cong G_n \oplus H_n^2 \cong G_n^2 \oplus H_n^3 \cong G_n^2 \otimes H_n \cong G_n,$$

which is the result. ■

For the stable Banach spaces $E = \ell_p$ (with $1 \le p < \infty$) and the projective topology π this result was proved by Arias–Farmer [2] by using Pełczyński's decomposition method.

4.2. Dually, it follows that $\mathcal{L}(\bigotimes_{\tau}^{n} E; F)$ and $\mathcal{L}(\bigotimes_{\tau,s}^{n} E; F)$ are algebraically isomorphic whenever E and F are locally convex, E is stable and τ is a symmetric tensor topology. Taking $\tau = \iota, \pi$ and ε we obtain

COROLLARY. *If E and F are locally convex spaces, and E is stable, then there are algebraic isomorphisms*

$$\{\varphi \in L(^nE;F) \mid sep.\ cont.\} = \{q \in P^n(E;F) \mid \check{q}\ sep.\ cont.\},$$
$$\mathcal{L}(^nE;F) = \mathcal{P}^n(E;F),$$
$$\{\varphi \in \mathcal{L}(^nE;F) \mid \varphi^L\ \varepsilon\text{-}cont.\} = \{q \in \mathcal{P}^n(E;F) \mid \check{q}\ \varepsilon\text{-}cont.\}.$$

Note that for $F = \mathbb{K}$ the latter equality means that the spaces of integral (in the sense of Grothendieck) n-linear forms and of integral n-homogeneous polynomials are isomorphic.

4.3. Concerning topologies on $\mathcal{L}(^nE;F)$ and $\mathcal{P}^n(E;F)$ note that for stable locally convex spaces E the constructions in 2.2 and 4.1 give an isomorphism $I : \bigotimes^n E \to \bigotimes^n_s E$ with the property that for every $A \subset E$ bounded (resp. compact) there is a $B \subset E$ bounded (resp. compact) such that

$$I(\bigotimes^n A) \subset \Gamma(\bigotimes^n_s B)$$

where $\bigotimes^n_s A := \{\bigotimes^n a \mid a \in A\}$ and $\bigotimes^n A := \{a_1 \otimes \ldots \otimes a_n \mid a_j \in A\}$ and also that for every $B \subset E$ bounded (resp. compact) there is such an $A \subset E$ with

$$I^{-1}(\bigotimes^n_s B) \subset \Gamma(\bigotimes^n A).$$

Denoting by b (resp. co) the topology of uniform convergence on all bounded (resp. compact) sets we obtain (see also [8])

COROLLARY. *If E and F are locally convex spaces, and E is stable, then*

$$\mathcal{L}_{\mathrm{b}}(^nE;F) \cong \mathcal{P}^n_{\mathrm{b}}(E;F), \qquad \mathcal{L}_{\mathrm{co}}(^nE;F) \cong \mathcal{P}^n_{\mathrm{co}}(E;F).$$

4.4. If E and F are normed spaces, then the spaces

$$\mathcal{I}(^nE;F) := \mathcal{L}(\bigotimes^n_\varepsilon E;F), \qquad \mathcal{P}^{I,n}(E;F) := \mathcal{L}(\bigotimes^n_{\varepsilon_s,s} E;F)$$

of "integral" n-linear mappings and "integral" n-homogeneous polynomials have natural norms (ε_s denotes the injective symmetric norm which is equivalent to the induced ε-norm, see [9]) as spaces of continuous mappings between normed spaces. Obviously we have

COROLLARY. *If E and F are normed spaces, and E is stable, then the normed spaces $\mathcal{I}(^nE;F)$ and $\mathcal{P}^{I,n}(E;F)$ are isomorphic.*

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