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Maximal functions and smoothness spaces in $L_p(\mathbb{R}^d)$

by

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Abstract. We study smoothness spaces generated by maximal functions related to the local approximation errors of integral operators. It turns out that in certain cases these smoothness classes coincide with the spaces $C_p^\alpha(\mathbb{R}^d)$, $0 < p \le \infty$, introduced by DeVore and Sharpley [DS] by means of the so-called sharp maximal functions of Calderón and Scott. As an application we characterize the $C_p^\alpha(\mathbb{R}^d)$ spaces in terms of the coefficients of wavelet decompositions.

1. Introduction. Maximal operators play an important role in various aspects of harmonic analysis and approximation theory, such as interpolation and differentiation. A paradigm is the so-called *sharp maximal function*, of Calderón and Scott [CS], given by

$$f_\alpha^\sharp(x) := \sup_{Q\ni x} \frac{1}{|Q|^{1+\alpha/d}} \int_Q |f - f_Q|, \quad 0 < \alpha < 1, \tag{1.1}$$

where $f_Q := |Q|^{-1}\int_Q f$ is the average of f over Q, and Q ranges over all cubes containing x. When $\alpha > 0$, $f_\alpha^\sharp$ is related to classical differentiation; for instance it is well known that

$$f \in \mathrm{Lip}_\alpha(\mathbb{R}^d) \Leftrightarrow f_\alpha^\sharp \in L_\infty(\mathbb{R}^d), \quad 0 < \alpha < 1, \tag{1.2}$$

where Lip_α is the Lipschitz space of smoothness α.

The extension of (1.1) to functions of higher smoothness was given by DeVore and Sharpley [DS]. For every $\alpha \ge 1$ they replaced the average f_Q by a best polynomial approximation from $\Pi_{[\alpha]}$ (the space of polynomials of degree at most $[\alpha]$) and they introduced the spaces $C_p^\alpha := C_p^\alpha(\mathbb{R}^d)$.

For $0 < p \le \infty$ and $\alpha > 0$, C_p^α is defined to be the collection of all functions $f \in L_p := L_p(\mathbb{R}^d)$ such that

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$$\|f\|_{C_p^\alpha} := \|f\|_{L_p} + \|f^\sharp_{\alpha,\min(1,p)}\|_{L_p} < \infty, \tag{1.3}$$

where for any $0 < q < \infty$,

$$f^\sharp_{\alpha,q}(x) := \sup_{Q\ni x} \inf_{\pi\in\Pi_{[\alpha]}} \frac{1}{|Q|^{\alpha/d}}\left(\frac{1}{|Q|}\int_Q |f-\pi|^q\right)^{1/q} \tag{1.4}$$

and the supremum is taken with respect to all cubes containing x.

Similarly to (1.2) the size of $f^\sharp_{\alpha,q}$ gives control over the smoothness of f; for $\alpha > 0$ and $0 < p \le \infty$ it is known (see [DS]) that

$$B^\alpha_{p,p} \hookrightarrow C^\alpha_p \hookrightarrow B^\alpha_{p,\infty},$$

where $B^\alpha_{p,q} := B^\alpha_q(L_p(\mathbb{R}^d))$, $0 < q \le \infty$, is the Besov space of smoothness α (see definition in §4).

The spaces C^α_p, $\alpha > 0$, $0 < p \le \infty$, have also been studied by Triebel [T] where it is shown that for $\alpha > d(1/p-1)_+$, C^α_p coincides with the so-called Triebel–Lizorkin space $F^\alpha_{p,\infty}$ (see [T] for definitions and details).

Our goal in this paper is to explore the connection between smoothness and the local errors of approximation induced by alternate approximation methods. In particular, instead of local polynomial approximation we employ families $T := (T_k)_{k\in\mathbb{Z}}$ of linear operators that are related to shift-invariant spaces and to the construction of wavelets. For every such family T and $0 < \alpha, q < \infty$, we define on $f \in L_q(\mathrm{loc})$ the maximal function

$$f^T_{\alpha,q}(x) := \sup_{Q\ni x} \frac{1}{|Q|^{\alpha/d}}\left(\frac{1}{|Q|}\int_Q |f - T_{k_Q} f|^q\right)^{1/q}, \tag{1.5}$$

where the supremum is taken over all cubes containing x and for each cube Q with edgelength $\ell(Q)$, k_Q is the unique integer with $2^{-k_Q-1} < \ell(Q) \le 2^{-k_Q}$.

Under certain assumptions on the family T we will show that the finiteness of $f^T_{\alpha,q}$ reflects directly in the smoothness of f. For instance, we will prove (see Lemma 4.8) that for any $r > \alpha$, $f \in L_1(\mathrm{loc})$ and $h \in \mathbb{R}^d$,

$$|\Delta^r_h(f,x)| \le \mathrm{const}\,|h|^\alpha \sum_{i=0}^r (f^T_\alpha + Mf)(x+ih) \quad \text{a.e.},$$

where $f^T_\alpha := f^T_{\alpha,1}$, $\Delta^r_h(f,\cdot)$ is the rth difference of f (see definition below) and M is the Hardy–Littlewood maximal operator. A similar result holds in the opposite direction as well (see Theorem 4.19).

Going even further, we will establish in Theorems 4.31 and 5.25 that for $\alpha > d(1/p-1)_+$ and $0 < p, q \le \infty$ the following are equivalent:

$$(1.6)\quad
\begin{aligned}
&\text{(i)} && \|f\|_{C^\alpha_p},\\
&\text{(ii)} && \|f\|_{L_p} + \|f^T_\alpha\|_{L_p},\\
&\text{(iii)} && \|f\|_{L_p} + \|\sup_{k\in\mathbb{Z}} 2^{k\alpha}|f - T_k f|\|_{L_p},\\
&\text{(iv)} && \|f\|_{L_p} + \|\sup_{k\in\mathbb{Z}} 2^{k\alpha}|T_k f - T_{k-1} f|\|_{L_p},\\
&\text{(v)} && \|f\|_{L_p} + \|\sup_{Q\ni x} 2^{k_Q(\alpha+d/q)}\|T_{k_Q} f - T_{k_Q-1} f\|_{L_q(Q)}\|_{L_p}.
\end{aligned}$$

We point out that the spaces C^α_p play an important role in *adaptive polynomial approximation* (see [DY]). In addition, the characterization (1.6) appears in connection with adaptive methods for the solution of elliptic PDE's, based on wavelet-type decompositions. In particular, we will prove in Theorem 4.37 that compactly supported orthogonal wavelet sets $\{\psi^e\}_{e\in E}$ (see definitions in Section 2) form unconditional bases for C^α_p, $\alpha > 0$, $1 < p < \infty$, and that if $f(x) = \sum_{k,j,e} a^e_{k,j}\psi^e_{k,j}$ is the wavelet decomposition of f, then (1.6)(v) is equivalent to

$$\|f\|_p + \Big\|\sup_{D\ni Q\ni x} 2^{k_Q(\alpha+d/2)} \sum_{e\in E} \|(a^e_{k_Q,j})_j\|_{l_q(\Lambda^e(Q))}\Big\|_{L_p} \tag{1.7}$$

(a similar result will be given for the whole range of p (Theorem 5.28)).

At this point we need to introduce some notation; we will use $\mathcal{Q}$ for the collection of all cubes in $\mathbb{R}^d$, and $\mathcal{Q}^k$, $k \in \mathbb{Z}$, for the cubes Q of sidelength $2^{-k-1} < \ell(Q) \le 2^{-k}$. Also, as is customary, we will denote by D the set of dyadic cubes in $\mathbb{R}^d$, that is, $D := \bigcup_{k\in\mathbb{Z}} D_k$, where D_k is the family of cubes with sides parallel to the coordinate axes, sidelength 2^{-k} and lower left corner at $2^{-k}\mathbb{Z}^d$.

In what follows we shall use standard multi-index notation; for every $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}^d$, we define $x^\alpha := x_1^{\alpha_1}\ldots x_d^{\alpha_d}$, $|\alpha| := \alpha_1 + \ldots + \alpha_d$ and $D^\alpha := \partial^{|\alpha|}/\partial^{\alpha_1}x_1\ldots\partial^{\alpha_d}x_d$.

Also for every $r \in \mathbb{N}$, $h \in \mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$ we denote by $\Delta^r_h(f,\cdot,\Omega)$ the rth forward difference relative to Ω, in the direction of h, defined by

$$\Delta^r_h(f,x,\Omega) := \begin{cases} \displaystyle\sum_{k=0}^r \binom{r}{k}(-1)^{r-k} f(x+kh), & x, x+h, \ldots, x+rh \in \Omega,\\ 0, & \text{otherwise.}\end{cases}$$

Finally, by $A \approx B$ and $A \lesssim B$ we mean that there exist positive constants, independent of the variables involved, such that $\mathrm{const} \le A/B \le \mathrm{const}$ and $A \le \mathrm{const}\,B$ respectively.

2. Multilevel operators and wavelet-type decompositions. Let $\phi, \widetilde{\phi}$ be bounded functions on $\mathbb{R}^d$ such that

(A1) $\operatorname{supp} \phi, \widetilde{\phi} \subset [-L, L]^d$, $L \in \mathbb{N}$.

The integer translates of ϕ and $\widetilde{\phi}$ give rise to a linear operator T_0 defined on $L_1(\mathrm{loc})$ by

$$T_0 f(\cdot) := \sum_{j\in\mathbb{Z}^d} \int f(y)\overline{\widetilde{\phi}(y-j)}\, dy\, \phi(\cdot - j),$$

while dilation and translation yields a sequence $T := (T_k)_{k\in\mathbb{Z}}$ of operators given by

$$T_k f(\cdot) := \sum_{j\in\mathbb{Z}^d} 2^{kd} \int f(y)\overline{\widetilde{\phi}(2^k y-j)}\, dy\, \phi(2^k \cdot - j). \tag{2.1}$$

We assume that the sequence $(T_k)_{k\in\mathbb{Z}}$ has the following properties:

(A2) For some $\alpha > 0$,

$$T_k \pi = \pi, \quad \pi \in \Pi_{[\alpha]},\ k \in \mathbb{Z}.$$

(A3) For every $k, \nu \in \mathbb{Z}$ with $k \ge \nu$,

$$T_k T_\nu = T_\nu.$$

(A4) $\phi \in C^{[\alpha]+1}$.

(A5) In addition, we will often need the shifts (integer translates) of ϕ to be linearly independent over $[0,1]^d$; by this we mean that the family of functions

$$\{\phi(\cdot - j) : j \in \mathbb{Z}^d \text{ and } \phi(\cdot - j) \text{ is not identically zero on } [0,1]^d\}$$

is linearly independent over $[0,1]^d$. In particular, since any two norms on a finite-dimensional space are equivalent, it is easily seen that there exist constants, depending on $0 < p, q \le \infty$ and ϕ, such that for any dyadic cube $Q \in D_k$, $k \in \mathbb{Z}$, and any sequence $\{a_j\}_{j\in\mathbb{Z}^d}$,

$$2^{kd/p}\Big\| \sum_{j\in\Lambda(Q)} a_j \phi(2^k \cdot - j)\Big\|_{L_p(Q)} \approx 2^{kd/q}\Big\| \sum_{j\in\Lambda(Q)} a_j \phi(2^k \cdot - j)\Big\|_{L_q(Q)} \approx \|a_j\|_{l_q(\Lambda(Q))}, \tag{2.2}$$

where $\Lambda(Q)$ denotes the set of $j \in \mathbb{Z}^d$ such that $\phi(2^k \cdot - j)$ is not identically zero on Q.

All these assumptions are standard and well understood because this kind of operators play a dominant role in the characterization of the approximation orders of shift-invariant spaces and the construction of wavelets.

For instance, (A2) is usually related to the approximation properties of the sequence $(T_k)_{k\in\mathbb{Z}}$ and holds if ϕ satisfies the Strang–Fix conditions of order $[\alpha]+1$, i.e.,

$$D^\beta \widehat{\phi}(2\pi j) = 0, \quad j \in \mathbb{Z}^d \setminus \{0\},\ |\beta| \le [\alpha],$$

and in addition $\widehat{\phi}(0) = 1$, $\widehat{\widetilde{\phi}}(0) = 1$, while for $1 \le |\beta| \le [\alpha]$, $D^\beta \widehat{\widetilde{\phi}}(0) = D^\beta \widehat{\phi}(0) = 0$ (for a proof see [K]).

On the other hand, the rest of the assumptions are always satisfied within the framework of compactly supported wavelet bases. Although we assume that the reader is familiar with the usual wavelet theory, we will very briefly review the construction of compactly supported wavelet bases in order to describe sufficient conditions for the assumptions (A1, 3, 5) to hold. For details we refer the reader to [D], [M]. As is customary, in what follows for any function f defined on $\mathbb{R}^d$ we adopt the notation $f_{k,j}(\cdot) := 2^{kd/2} f(2^k \cdot - j)$.

The term *wavelet* refers to a function ψ in $L_2(\mathbb{R})$ whose dilated translates $\{\psi_{k,j}\}$, $k, j \in \mathbb{Z}$, constitute an orthonormal basis for $L_2(\mathbb{R})$.

The usual construction of compactly supported wavelets on $\mathbb{R}$ starts from a compactly supported function $\varphi \in L_2(\mathbb{R})$ having orthonormal integer shifts, i.e.,

$$\int_{\mathbb{R}} \varphi(x)\overline{\varphi(x-j)}\, dx = \delta_{0,j}, \quad j \in \mathbb{Z}. \tag{2.3}$$

It is also required that for some sequence $(h_j)_j$ of complex numbers, φ satisfies the *refinement equation*

$$\varphi(\cdot) = \sum_{j\in\mathbb{Z}} h_j \varphi(2 \cdot - j). \tag{2.4}$$

The function φ is then used to construct a multiresolution analysis consisting of an ascending sequence (V_k), $k \in \mathbb{Z}$, of subspaces of $L_2(\mathbb{R})$. Each V_k, $k \in \mathbb{Z}$, is generated by the $2^{-k}\mathbb{Z}$-shifts of the function $\varphi(2^k \cdot)$; in other words,

$$V_k \subset V_{k+1}, \quad k \in \mathbb{Z}, \tag{2.5}$$

and

$$V_k := \overline{\operatorname{span}\{\varphi_{k,j} : j \in \mathbb{Z}\}}, \quad k \in \mathbb{Z}.$$

Under the assumption that φ satisfies the Strang–Fix conditions of some positive order and $\int \varphi = 1$, one can prove that

$$\bigcap_{k\in\mathbb{Z}} V_k = \{0\} \quad \text{and} \quad \overline{\bigcup_{k\in\mathbb{Z}} V_k} = L_2(\mathbb{R}), \tag{2.6}$$

and the wavelet ψ is given by

$$\psi(\cdot) = \sum_j (-1)^{j-1} h_{-j-1} \varphi(2 \cdot - j).$$

It is known that for each $k \in \mathbb{Z}$, $\psi_{k,0}$ lives in the space $W_k := V_{k+1} \ominus V_k$, and that

$$L_2(\mathbb{R}) = \bigoplus_{k\in\mathbb{Z}} W_k,$$

from which one derives that for every $f \in L_2(\mathbb{R})$,

$$f(\cdot) = \sum_{k,j\in\mathbb{Z}} c_{k,j}(f)\psi_{k,j}(\cdot), \quad c_{k,j} := \int_{\mathbb{R}} f(y)\overline{\psi_{k,j}(y)}\,dy. \tag{2.7}$$

In the multivariate case a rather simple approach is by means of tensor products. For $x := (x_1, \ldots, x_d)$ we define

$$\phi(x) := \varphi(x_1)\ldots\varphi(x_d),$$

with φ the function defined above.

It follows that ϕ satisfies the refinement equation (2.4) with coefficients $a_j := h_{j_1}\ldots h_{j_d}$, $j := (j_1, \ldots j_d)$. Moreover, the sequence

$$V_k := \overline{\operatorname{span}\{\phi_{k,j} : j \in \mathbb{Z}^d\}}, \quad k \in \mathbb{Z},$$

forms a multiresolution analysis for $L_2(\mathbb{R}^d)$.

To construct the corresponding tensor-product wavelets in the wavelet space W^0 we set $\eta_0 := \varphi$, $\eta_1 := \psi$ and we let E be the set of non-zero vertices of the unit cube $[0,1]^d$. Finally, a family of $2^d - 1$ wavelets in W^0 is given by

$$\psi^e(x) := \prod_{j=1}^{d} \eta_{e_j}(x_j), \quad e \in E. \tag{2.8}$$

The family $\{\psi^e_{k,j}\}_{e,k,j}$ is an orthonormal basis for $L_2(\mathbb{R}^d)$ (see [M]).

The orthogonal projections of $L_2(\mathbb{R}^d)$ onto V_k, $k \in \mathbb{Z}$, are given by the operators

$$P_k f(\cdot) := \sum_{j\in\mathbb{Z}^d} 2^{kd}\int f(y)\overline{\phi(2^k y - j)}\,dy\,\phi(2^k\cdot - j). \tag{2.9}$$

Finally, one can prove that for every $k \in \mathbb{Z}$,

$$f(\cdot) = P_k f(\cdot) + \sum_{\nu\ge k}(P_{\nu+1}f(\cdot) - P_\nu f(\cdot)), \tag{2.10}$$

and that

$$P_{k+1}f(\cdot) - P_k f(\cdot) = \sum_{e\in E}\sum_{j\in\mathbb{Z}^d} a^e_{k,j}\psi^e_{k,j}(\cdot),$$

where for every $k \in \mathbb{Z}$ and $j \in \mathbb{Z}^d$, the wavelet coefficients are defined by

$$a^e_{k,j} := \int f(y)\overline{\psi^e_{k,j}(y)}\,dy. \tag{2.11}$$

It is important to note that wavelet decompositions hold for a variety of spaces. For instance, (2.10) is valid for any function $f \in L_p(\mathbb{R}^d)$, $1 \le p < \infty$, while for $p = \infty$ one could resort to the Besov spaces $B^\alpha_q(L_\infty(\mathbb{R}^d))$, $\alpha, q > 0$ (see [M]).

As is readily seen, the sequence $(P_k)_{k\in\mathbb{Z}}$ is a special case of the operators $(T_k)_{k\in\mathbb{Z}}$ defined in (2.1) and it is well known that the P_k's satisfy both (A2) and (A3). (A5) is also known to hold in our setting and has been established by Lemarié and Malgouyres (see [LM]) for the univariate orthogonal scaling functions ϕ as well for the corresponding wavelets ψ.

Closing this section we note that it follows immediately that the tensor-product wavelets ψ^e, $e \in E$, also have linearly independent shifts over $[0,1]^d$. This fact implies, similarly to (2.2), that there exist constants depending on $0 < q \le \infty$ and ψ^e, $e \in E$, such that for any dyadic cube $Q \in D_k$, $k \in \mathbb{Z}$, and any integrable function f,

$$\|P_{k+1}f - P_k f\|_{L_q(Q)} \approx 2^{kd(1/2-1/q)}\sum_{e\in E}\|a^e_j\|_{l_q(\Lambda^e(Q))}, \tag{2.12}$$

where $\Lambda^e(Q)$ denotes the set of $j \in \mathbb{Z}^d$ such that $\psi^e(2^k\cdot - j)$ is not identically zero on Q.

3. Multilevel operators and maximal functions.

3. Multilevel operators and maximal functions. For the rest of the paper we assume that we have a sequence $(T_k)_{k\in\mathbb{Z}}$ of operators that satisfy (A1–4) for some $\alpha > 0$. For any $k \in \mathbb{Z}$, we define

$$\mathcal{Q}^k_L := \{Q \in \mathcal{Q} : (4L + 2\sqrt{d})2^{-k-1} < \ell(Q) \le (4L + 2\sqrt{d})2^{-k}\}.$$

Moreover, for every $x \in \mathbb{R}^d$ and $k \in \mathbb{Z}$, we denote by $Q^k_L(x)$ the cube centered at x with sidelength $4L + 2\sqrt{d}$, i.e.,

$$Q^k_L(x) = x + [-\sqrt{d} - 2L, \sqrt{d} + 2L]^d/2^k.$$

Let $x \in \mathbb{R}^d$ and assume that $Q \in \mathcal{Q}^k$, $k \in \mathbb{Z}$, contains x. It is easily seen that $Q \subset x + [-\sqrt{d}, \sqrt{d}]^d/2^k$. Since $\operatorname{supp}\phi(2^k\cdot - j) \subset ([-L, L]^d + j)/2^k$ it follows that for every $z \in Q$,

$$\begin{aligned} |T_k f(z)| &= \Big|\sum_{\{j\in\mathbb{Z}^d : 2^k z - j\in[-L,L]^d\}} 2^{kd}\int f(y)\overline{\tilde{\phi}(2^k y - j)}\,dy\,\phi(2^k z - j)\Big| \\ &\lesssim \frac{1}{|Q^k_L(x)|}\int_{Q^k_L(x)} |f|. \end{aligned} \tag{3.1}$$

Employing (A2) it follows from (3.1) that for any cube $Q \in \mathcal{Q}^k, k \in \mathbb{Z}$, containing x,

$$\begin{aligned} |f(x) - T_k f(x)| &\le \|f - f_{Q^k_L(x)} + T_k(f_{Q^k_L(x)}) - T_k f\|_{L_\infty(Q)} \\ &\le \|f - f_{Q^k_L(x)}\|_{L_\infty(Q)} + \|T_k(f - f_{Q^k_L(x)})\|_{L_\infty(Q)} \end{aligned} \tag{3.2}$$

$$\leq \|f - f_{Q_L^k(x)}\|_{L_\infty(Q_L^k(x))} + \frac{1}{Q_L^k(x)} \int_{Q_L^k(x)} |f - f_{Q_L^k(x)}|$$
$$\lesssim \|f - f_{Q_L^k(x)}\|_{L_\infty(Q_L^k(x))},$$

where $f_{Q_L^k(x)} := |Q_L^k(x)|^{-1} \int_{Q_L^k(x)} f$. From (3.2) it is easily seen that for every continuous function f,

$$\lim_{k\to\infty} T_k f(x) = f(x). \tag{3.3}$$

It is routine now to extend (3.3) to locally integrable functions, in the sense of almost everywhere convergence. For this we recall the Hardy–Littlewood maximal operator

$$Mf(x) := \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f|.$$

Let

$$Af(x) := \overline{\lim_{k\to\infty}} |f(x) - T_k f(x)|.$$

Since $|f| \leq Mf$ a.e., we deduce from (3.1) that $Af \lesssim Mf$, which in turn implies that A is an operator of weak type $(1,1)$, i.e., for any $h \in L_1$,

$$t|\{x : Ah(x) > t\}| \lesssim \|h\|_{L_1}.$$

Let Q be an open cube in $\mathbb{R}^d$ and $t > 0$; given $\varepsilon > 0$ we can choose a continuous function $g \in L_1(\mathbb{R}^d)$ such that $\|f\chi_Q - g\|_{L_1(\mathbb{R}^d)} \leq \varepsilon t$. Taking into account that $Ag = 0$ we derive that $A(f\chi_Q) \leq A(f\chi_Q - g)$. Therefore,

$$\begin{aligned}|\{x : A(f\chi_Q) > t\}| &\leq |\{x : A(f\chi_Q - g) > t\}| \\ &\lesssim \|f\chi_Q - g\|_{L_1(\mathbb{R}^d)}/t \lesssim \varepsilon.\end{aligned}$$

Since ε was arbitrary we conclude that $A(f\chi_Q) = 0$ a.e. Finally, we note that for every $x \in Q$ there exists k_0 large enough with $T_k f(x) = T_k(f\chi_Q)(x)$ for all $k \geq k_0$. We deduce that $A(f) = A(f\chi_Q) = 0$ a.e. on Q. By varying the cube Q we find that for every $f \in L_1(\mathrm{loc})$,

$$\lim_{k\to\infty} T_k f(x) = f(x) \quad \text{a.e.} \tag{3.4}$$

On the other hand, for every $f \in L_p$, $1 \leq p < \infty$, and $x \in \mathbb{R}^d$, Hölder's inequality yields

$$|T_k f(x)| \lesssim \frac{1}{|Q_L^k(x)|} \int_{Q_L^k(x)} |f(y)|\,dy \lesssim |Q_L^k(x)|^{-1/p} \|f\|_{L_p(\mathbb{R}^d)}.$$

Letting $k \to -\infty$ we get

$$\lim_{k\to-\infty} T_k f(x) = 0, \quad x \in \mathbb{R}^d. \tag{3.5}$$

Concluding this section we note that for any $f \in L_p$, $1 \leq p < \infty$, (3.3) and (3.5) imply that

$$f(x) = \sum_{k\in\mathbb{Z}^d} (T_k f(x) - T_{k-1} f(x)) \quad \text{a.e.} \tag{3.6}$$

4. Smoothness spaces in L_p, $1 < p \leq \infty$. In what follows we will frequently use a slight variation of the maximal function f_α^T defined by

$$f_\alpha^{T,L}(x) := \sup_{\substack{k\in\mathbb{Z} \\ Q_L^k \ni Q \ni x}} \frac{1}{|Q|^{1+\alpha/d}} \int_Q |f - T_k f|. \tag{4.1}$$

Our first task is to show that (1.5) and (4.1) give rise to equivalent maximal functions, i.e.,

$$f_\alpha^T(x) \approx f_\alpha^{T,L}(x), \quad x \in \mathbb{R}^d. \tag{4.2}$$

In view of this fact we will drop the superscript L from $f_\alpha^{T,L}$ and we will use f_α^T for both maximal functions indiscriminately, without any further notice.

Let $Q \in \mathcal{Q}^k$, $k \in \mathbb{Z}$, containing x. We denote by Q_L the cube in $\mathcal{Q}_L^k$ which is concentric with Q, has sides parallel to those of Q and sidelength $\ell(Q_L) = (4L + 2\sqrt{d})\ell(Q)$. Since $Q_L \supset Q$ it follows easily that

$$\frac{1}{|Q|^{1+\alpha/d}} \int_Q |f - T_k f| \lesssim \frac{1}{|Q_L|^{1+\alpha/d}} \int_{Q_L} |f - T_k f|.$$

Taking now the supremum first over all $Q_L \in \mathcal{Q}_L^k$, $k \in \mathbb{Z}$, and then over $Q \in \mathcal{Q}$ we get

$$f_\alpha^T(x) \lesssim f_\alpha^{T,L}(x). \tag{4.3}$$

Working now towards establishing the reverse inequality we note that for any cube $Q_L \in \mathcal{Q}_L^k$, $k \in \mathbb{Z}$, containing x and $\nu \leq k$, $\nu \in \mathbb{Z}$, we have

$$\begin{aligned}\frac{1}{|Q_L|^{1+\alpha/d}} &\int_{Q_L} |f - T_k f| \\ &\lesssim \frac{1}{|Q_L|^{1+\alpha/d}} \Big\{ \int_{Q_L} |f - T_\nu f| + \int_{Q_L} |T_k(T_\nu f - f)| \Big\}.\end{aligned} \tag{4.4}$$

Similarly to (3.1) we can prove that there exists a cube $\widetilde{Q}_L \supset Q_L$, centered at x having sidelength $(8L\sqrt{d} + 4L + 4d)2^{-k}$, and such that for every $z \in Q_L$,

$$|T_k f(z)| \lesssim \frac{1}{|\widetilde{Q}_L|} \int_{\widetilde{Q}_L} |f|. \tag{4.5}$$

Let $\ell < k$ be the largest integer such that $\widetilde{Q}_L \subset I$ for some cube $I \in \mathcal{Q}^\ell$ centered at x. From (4.5) we see that for some constant independent of k

and any $z \in Q_L$,

$$|T_k(T_\ell f - f)(z)| \lesssim \frac{1}{|I|}\int_I |T_\ell f - f|. \tag{4.6}$$

Using (4.6) in (4.4) (with $\nu = \ell$) we get

$$\frac{1}{|Q_L|^{1+\alpha/d}}\int_{Q_L} |f - T_k f| \lesssim \frac{1}{|I|^{1+\alpha/d}}\int_I |f - T_\ell f|$$

for some constant independent of k, ℓ. Taking now the supremum first with respect to all $I \ni x$ and then with respect to $Q_L \in \mathcal{Q}_L^k$, $k \in \mathbb{Z}$, we get

$$f_\alpha^{T,L}(x) \lesssim f_\alpha^T(x), \quad x \in \mathbb{R}^d. \tag{4.7}$$

The following very useful lemma shows that the size of f_α^T contains valuable information regarding the smoothness of a locally integrable function.

LEMMA 4.8. *Assume that the sequence $T = (T_k)_{k\in\mathbb{Z}}$ of linear operators satisfies assumptions* (A1, 3, 4) *for some $\alpha > 0$. Let also $r = [\alpha] + 1$ and $f \in L_1(\mathrm{loc})$. Then for any $h \in \mathbb{R}^d$,*

$$|\Delta_h^r(f, x)| \lesssim |h|^\alpha \sum_{i=0}^r (f_\alpha^T + Mf)(x + ih) \quad a.e. \tag{4.9}$$

In addition, if $f \in L_p(\mathbb{R}^d)$, $1 \le p < \infty$, then

$$|\Delta_h^r(f, x)| \lesssim |h|^\alpha \sum_{i=0}^r f_\alpha^T(x + ih) \quad a.e. \tag{4.10}$$

Proof. First we prove (4.10). We fix $h \in \mathbb{R}^d$ and we assume that for some $k \in \mathbb{Z}$, $2^{-k-1} < |h| \le 2^{-k}$. We define the set Ω_h by

$$\Omega_h := \{x \in \mathbb{R}^d : \lim_{\nu\to\infty} T_\nu f(x + ih) = f(x + ih),\ i = 0, 1, \ldots, r\}.$$

From (3.4) and (3.6) we know that $|\Omega_h^c| = 0$ and that for all $x \in \Omega_h$,

$$f(x + ih) = \sum_{\nu\in\mathbb{Z}} (T_\nu f(x + ih) - T_{\nu-1} f(x + ih)), \quad i = 0, 1, \ldots, r.$$

It follows that

$$\begin{aligned} |\Delta_h^r(f, x)| \le \sum_{\nu\ge k} |\Delta_h^r(T_\nu f - T_{\nu-1} f, x)| \\ + \sum_{\nu<k} |\Delta_h^r(T_\nu f - T_{\nu-1} f, x)| =: I + II. \end{aligned} \tag{4.11}$$

For the first sum we note that

$$\begin{aligned} I &\lesssim \sum_{\nu\ge k}\sum_{i=0}^r |T_\nu(f - T_{\nu-1} f)(x + ih)| \\ &\lesssim \sum_{\nu\ge k}\sum_{i=0}^r \frac{1}{|Q_L^{\nu-1}(x + ih)|}\int_{Q_L^{\nu-1}(x+ih)} |f(y) - T_{\nu-1} f(y)|\,dy \\ &= \sum_{i=0}^r \sum_{\nu\ge k-1} |Q_L^\nu(x + ih)|^{\alpha/d} \\ &\quad\times \Big(\frac{1}{|Q_L^\nu(x + ih)|^{1+\alpha/d}}\int_{Q_L^\nu(x+ih)} |f(y) - T_\nu f(y)|\,dy\Big) \\ &\lesssim \sum_{i=0}^r f_\alpha^T(x + ih) \sum_{\nu\ge k-1} |Q_L^\nu(x + ih)|^{\alpha/d} \lesssim |h|^\alpha \sum_{i=0}^r f_\alpha^T(x + ih). \end{aligned} \tag{4.12}$$

In order to estimate the second sum we will use the inequality

$$|\Delta_h^r(T_\nu f - T_{\nu-1} f, x)| \lesssim |h|^r 2^{\nu(r-\alpha)} \sum_{i=0}^r f_\alpha^T(x + ih), \quad \nu < k. \tag{4.13}$$

To prove (4.13) we note from the integral representation of the rth difference (see [BS], p. 336) that

$$\begin{aligned} &|\Delta_h^r(T_\nu f - T_{\nu-1} f, x)| \\ &\quad = \Big|\int_0^r M_r(\xi) \sum_{|\beta|=r} \frac{r!}{\beta!} D^\beta(T_\nu f - T_{\nu-1} f)(x + \xi h) h^\beta\,d\xi\Big| \\ &\quad \lesssim |h|^r \int_0^r M_r(\xi) \sum_{|\beta|=r} |D^\beta(T_\nu f - T_{\nu-1} f)(x + \xi h)|\,d\xi, \end{aligned} \tag{4.14}$$

where M_r is the B-spline of order r supported on $[0, r]$. We fix $0 \le \xi \le r$. Since $|h| < 2^{-k} \le 2^{-\nu}$ it is easily seen by the construction that there exists $i \in \{0, \ldots, r\}$ such that $x + ih \in Q_L^\nu(x + \xi h)$.

Since ϕ is compactly supported, for every $z \in \mathbb{R}^d$ and $|\beta| = r$ we have

$$\begin{aligned} &D^\beta(T_\nu(f - T_{\nu-1} f))(z) \\ &\quad = \sum_{j\in\mathbb{Z}^d} 2^{\nu(d+|\beta|)} \int (f - T_{\nu-1} f)(y) \overline{\widetilde{\phi}(2^\nu y - j)}\,dy\,(D^\beta\phi)(2^\nu z - j), \end{aligned} \tag{4.15}$$

and similarly to (3.1) we get

$$\begin{aligned} &|D^\beta(T_\nu(f - T_{\nu-1} f))(x + \xi h)| \\ &\quad \lesssim \frac{2^{\nu r}}{|Q_L^\nu(x + \xi h)|}\int_{Q_L^\nu(x+\xi h)} |f - T_{\nu-1} f(y)|\,dy \lesssim 2^{\nu(r-\alpha)} f_\alpha^T(x + ih). \end{aligned} \tag{4.16}$$

From (4.14) and (4.16) one easily gets (4.13). It follows that

$$II \lesssim |h|^r \Big(\sum_{\nu<k} 2^{\nu(r-\alpha)}\Big) \sum_{i=0}^{r} f_\alpha^T(x+ih) \qquad (4.17)$$
$$\lesssim |h|^\alpha \sum_{i=0}^{r} f_\alpha^T(x+ih).$$

To conclude the proof of (4.10) we only need to employ (4.12) and (4.17) in (4.11).

In order to establish (4.9) we assume, without loss of generality, that $|h|<1/2$. Otherwise we easily get

$$|\Delta_h^r(f,x)| \lesssim \sum_{i=0}^{r} |f(x+ih)| \lesssim |h|^\alpha \sum_{i=0}^{r} Mf(x+ih) \quad \text{a.e.}$$

Instead of (4.11) we now use the decomposition

$$|\Delta_h^r(f,x)| \le \sum_{\nu\ge k} |\Delta_h^r(T_\nu f - T_{\nu-1}f,x)| + \sum_{0<\nu<k} |\Delta_h^r(T_\nu f - T_{\nu-1}f,x)|$$
$$+ |\Delta_h^r(T_0 f,x)| =: I + II + III.$$

I is estimated as before. For II using (4.13) we get

$$II \lesssim |h|^r \sum_{0<\nu<k} 2^{\nu(r-\alpha)} \Big(\sum_{i=0}^{r} f_\alpha^T(x+ih)\Big) \lesssim |h|^\alpha \sum_{i=0}^{r} f_\alpha^T(x+ih).$$

Finally, for III we get from (4.14) and (4.16), using T_0 instead of $T_\nu - T_{\nu-1}$,

$$III \lesssim |h|^r \sum_{i=0}^{r} Mf(x+ih) \lesssim |h|^\alpha \sum_{i=0}^{r} Mf(x+ih).$$

This completes the proof of the lemma. ∎

It is well known (see [DS]) that for $\alpha>0$, the space C_∞^α coincides with the Besov space $B_\infty^\alpha(L_\infty(\mathbb{R}))^d$ (see definitions below) and more specifically that

$$\|f\|_{B_\infty^\alpha(L_\infty(\mathbb{R}))^d} \approx \|f\|_{C_\infty^\alpha}.$$

Using the previous lemma it is also easily seen that

$$\|f\|_{B_\infty^\alpha(L_\infty(\mathbb{R}))^d} \approx \|f\|_{L_\infty} + \|f_\alpha^T\|_{L_\infty}. \qquad (4.18)$$

Before we establish (4.18) we introduce the Besov spaces $B_q^\alpha(L_p(\Omega))$ where $\alpha>0$, $0<q$, $p\le\infty$ and $\Omega\subset\mathbb{R}^d$. We recall that for every f defined on Ω, $r\in\mathbb{N}$ and $t>0$ the modulus of smoothness $w_r(f,t)_p$ is given by

$$w_r(f,t)_p := \sup_{0<|h|\le t} \|\Delta_h^r(f,\cdot,\Omega)\|_{L_p(\mathbb{R}^d)}.$$

Let $r\in\mathbb{N}$ be such that $r>\alpha>0$. The Besov space $B_q^\alpha(L_p(\Omega))$, $0<q,p\le\infty$, is defined to be the family of all functions such that

$$\|f\|_{B_q^\alpha(L_p(\Omega))} := \|f\|_{L_p} + \begin{cases} \Big(\int_0^\infty [t^{-\alpha} w_r(f,t)_p]^q \frac{dt}{t}\Big)^{1/q}, & 0<q<\infty,\\ \sup_{t\ge0} t^{-\alpha} w_r(f,t)_p, & q=\infty,\end{cases}$$

is finite.

THEOREM 4.19. *Let $\alpha>0$ and assume that $T=(T_k)_{k\in\mathbb{Z}}$ satisfies assumptions* (A1–4). *Then for every $f\in L_\infty$,*

$$\|f\|_{B_\infty^\alpha(L_\infty(\mathbb{R}))^d} \approx \|f\|_{L_\infty} + \|f_\alpha^T\|_{L_\infty}.$$

Proof. One direction is trivial; taking the supremum over $x\in\mathbb{R}^d$ and $|h|<t$ in (4.9) we get

$$\|f\|_{B_\infty^\alpha(L_\infty(\mathbb{R}))^d} \le \|f\|_{L_\infty(\mathbb{R}^d)} + \|f_\alpha^T\|_{L_\infty} + \|Mf\|_{L_\infty(\mathbb{R}^d)}$$
$$\le \|f\|_{L_\infty(\mathbb{R}^d)} + \|f_\alpha^T\|_{L_\infty}.$$

For the other direction we let $x\in\mathbb{R}^d$ and assume that Q is a cube in $\mathcal{Q}^k$, $k\in\mathbb{Z}$, containing x. For any polynomial $P\in\Pi_{[\alpha]}$ we have

$$\int_Q |f-T_kf| \le \int_Q |f-P| + \int_Q |T_k(f-P)|. \qquad (4.20)$$

From (3.1) we know that for every $z\in Q$,

$$|T_k(f-P)(z)| \lesssim \frac{1}{|Q_L^k(x)|} \int_{Q_L^k(x)} |f-P|. \qquad (4.21)$$

It follows from (4.20) that

$$\int_Q |f-T_kf| \lesssim \int_{Q_L^k(x)} |f-P|. \qquad (4.22)$$

Let now $P\in\Pi_{[\alpha]}$ be a polynomial of best $L_\infty(Q_L^k(x))$ approximation to f. It is known (see [DP]) that

$$\|f-P\|_{L_\infty(Q_L^k(x))} \lesssim |Q_L^k(x)|^{\alpha/d} \|f\|_{B_\infty^\alpha(L_\infty(Q_L^k(x)))}. \qquad (4.23)$$

It follows from (4.22) that for every $x\in\mathbb{R}^d$ and $\mathcal{Q}^k\ni Q\ni x$,

$$\frac{1}{|Q|^{1+\alpha/d}} \int_Q |f-T_kf| \lesssim \|f\|_{B_\infty^\alpha(L_\infty(Q_L^k(x)))} \lesssim \|f\|_{B_\infty^\alpha(L_\infty(\mathbb{R}^d))}.$$

Taking the supremum first with respect to $Q\ni x$ and then applying the L_∞ norm yields the result. ∎

Next we want to investigate (1.6) for $\alpha>0$ and $1<p<\infty$. For this we will establish the equivalence between the maximal functions f_α^T and $f_\alpha^\sharp$.

THEOREM 4.24. *If $T=(T_k)_{k\in\mathbb{Z}}$ is a sequence of linear operators satisfying* (A1–4) *for some $\alpha>0$, then for any $f\in L_p$, $1\le p<\infty$,*

$$f_\alpha^T(x)\approx f_\alpha^\sharp(x),\qquad x\in\mathbb{R}^d. \tag{4.25}$$

Proof. First we prove that for every $x\in\mathbb{R}^d$, $f_\alpha^T(x)\lesssim f_\alpha^\sharp(x)$. Let $x\in\mathbb{R}^d$ and assume that Q is a cube in $\mathcal{Q}^k$, $k\in\mathbb{Z}$, containing x. We know from (4.22) that for any polynomial $P\in\Pi_{[\alpha]}$,

$$\int_Q |f-T_kf|\lesssim \int_{Q_L^k(x)}|f-P|.$$

Taking now the infimum with respect to all $P\in\Pi_{[\alpha]}$, dividing by $|Q|^{1+\alpha/d}$ and then taking the supremum with respect to $Q\ni x$ we get

$$f_\alpha^T\lesssim f_\alpha^\sharp.$$

For the other direction, for any $x\in\mathbb{R}^d$ and any cube Q containing x we will find a polynomial P_Q such that

$$\int_Q |f-P_Q|\lesssim\int_Q|f-T_{k_Q}f|+|Q|^{1+\alpha/d}f_\alpha^T(x). \tag{4.26}$$

If this is the case then

$$f_\alpha^\sharp(x)\le\sup_{Q\ni x}\frac{1}{|Q|^{1+\alpha/d}}\int_Q|f-P_Q|\lesssim f_\alpha^T(x). \tag{4.27}$$

Towards this end let $Q\in\mathcal{Q}^k$, $k\in\mathbb{Z}$, be such that $Q\ni x$. On Q we define the polynomial P_Q by

$$P_Q(y):=\sum_{|\beta|\le[\alpha]}\binom{[\alpha]}{\beta}D^\beta(T_kf)(x)(y-x)^\beta.$$

It is easily seen that

$$\int_Q|f-P_Q|\le\int_Q|f-T_kf|+\int_Q|T_kf-P_Q|=:I+II. \tag{4.28}$$

For II we note that for every $y\in Q$,

$$|T_kf(y)-P_Q(y)|=\Big|T_kf(y)-\sum_{|\beta|\le[\alpha]}\binom{[\alpha]}{\beta}D^\beta(T_kf)(x)(y-x)^\beta\Big| \tag{4.29}$$
$$\lesssim|Q|^{([\alpha]+1)/d}\sup_{w\in Q}\sum_{|\beta|=[\alpha]+1}|D^\beta(T_kf)(w)|.$$

From (3.5) we see that for any $x\in\mathbb{R}^d$,

$$T_kf(x)=\sum_{\nu\le k}(T_\nu f(x)-T_{\nu-1}f(x)).$$

Moreover, similarly to the proof of (3.5), for every β with $|\beta|\le[\alpha]+1$,

$$|D^\beta(T_kf)(x)|\lesssim|Q_L^k(x)|^{-1/p-|\beta|/d}\|f\|_{L_p},$$

which shows that

$$D^\beta(T_kf)(x)=\sum_{\nu\le k}D^\beta(T_\nu f-T_{\nu-1}f)(x), \tag{4.30}$$

because the series on the right hand side of (4.30) converges uniformly.

Let now $w\in Q$, and $|\beta|=[\alpha]+1$. Using (4.30) and an argument similar to (4.16) we get

$$|D^\beta(T_kf)(w)|\le\sum_{\nu\le k}|D^\beta(T_\nu f-T_{\nu-1}f)(w)|\lesssim\sum_{\nu\le k}2^{\nu([\alpha]+1-\alpha)}f_\alpha^T(x)$$
$$\lesssim|Q|^{(\alpha-[\alpha]-1)/d}f_\alpha^T(x).$$

It follows from (4.29) that for every $y\in Q$,

$$|T_kf(y)-P_Q(y)|\lesssim|Q|^{\alpha/d}f_\alpha^T(x),$$

which implies that for every $Q\ni x$,

$$\int_Q|T_kf-P_Q|\lesssim|Q|^{1+\alpha/d}f_\alpha^T(x).$$

Plugging this last inequality in (4.28) gives (4.26). ∎

THEOREM 4.31. *Let $1<p\le\infty$, and $0<q\le\infty$. If $T=(T_k)_{k\in\mathbb{Z}}$ satisfies* (A1–4) *for some $\alpha>0$ then, for every $f\in C_p^\alpha$, the following are equivalent:*

$$\begin{aligned}
&\text{(i)}\quad N_1(f):=\|f\|_{C_p^\alpha},\\
&\text{(ii)}\quad N_2(f):=\|f\|_{L_p}+\|f_\alpha^T\|_{L_p},\\
&\text{(iii)}\quad N_3(f):=\|f\|_{L_p}+\|\sup_{k\in\mathbb{Z}}2^{k\alpha}|f-T_kf|\|_{L_p},\\
&\text{(iv)}\quad N_4(f):=\|f\|_{L_p}+\|\sup_{k\in\mathbb{Z}}2^{k\alpha}|T_kf-T_{k-1}f|\|_{L_p},\\
&\text{(v)}\quad N_5(f):=\|f\|_{L_p}+\|\sup_{Q\ni x}2^{k_Q\alpha}\|T_{k_Q}f-T_{k_Q-1}f\|_{L_\infty(Q)}\|_{L_p}.
\end{aligned}\tag{4.32}$$

Moreover, if (A5) *holds, then all of the above are equivalent to*

(vi) $N_6(f):=\|f\|_{L_p}+\|\sup_{Q\ni x}2^{k_Q(\alpha+d/q)}\|T_{k_Q}f-T_{k_Q-1}f\|_{L_q(Q)}\|_{L_p}$,

where the constants of equivalence depend on q,T and d.

Proof. First we consider the equivalence $N_1(f)\approx N_2(f)$. For $p=\infty$, as we already mentioned, the result is an immediate consequence of Theorem 4.19. For $1<p<\infty$, the result follows trivially from Theorem 4.24 and the definitions of the norms.

Next we prove that $N_2(f)\lesssim N_3(f)$. For any $x\in\mathbb{R}^d$ we have

$$f_\alpha^T(x)=\sup_{Q\ni x}\frac{1}{|Q|^{1+\alpha/d}}\int_Q|f-T_{k_Q}f|$$
$$\le\sup_{Q\ni x}\frac{1}{|Q|}\int_Q\sup_{k\in\mathbb{Z}}2^{k\alpha}|f-T_kf|\le M(\sup_{k\in\mathbb{Z}}2^{k\alpha}|f-T_kf|)(x).$$

Using now the boundedness of the maximal operator on L_p, $1<p\le\infty$, we get $N_2(f)\lesssim N_3(f)$.

In order to prove that

$$\|\sup_{k\in\mathbb{Z}}2^{k\alpha}|f-T_kf|\|_{L_p}\lesssim\|\sup_{k\in\mathbb{Z}}2^{k\alpha}|T_{k+1}-T_kf|\|_{L_p},\tag{4.33}$$

we note that

$$|f-T_kf|\le\sum_{\nu\ge k+1}|T_\nu f-T_{\nu-1}f|\quad\text{ a.e.}$$

It follows that

$$2^{k\alpha}|f-T_kf|\le2^{k\alpha}\Big(\sum_{\nu\ge k+1}2^{-\nu\alpha}\Big)\sup_{\nu\in\mathbb{Z}}2^{\nu\alpha}|T_\nu f-T_{\nu-1}f|\tag{4.34}$$
$$\lesssim\sup_{\nu\in\mathbb{Z}}2^{\nu\alpha}|T_\nu f-T_{\nu-1}f|.$$

Taking the supremum with respect to $k\in\mathbb{Z}$ and applying the L_p norm on both sides of (4.34) we get (4.33).

The inequality $N_4(f)\le N_5(f)$ follows trivially. Thus, we only need to establish

$$\|\sup_{Q\ni x}2^{k_Q\alpha}\|T_{k_Q}f-T_{k_Q-1}f\|_{L_\infty(Q)}\|_{L_p}\lesssim\|f_\alpha^T\|_{L_p}\tag{4.35}$$

in order to conclude the proof of (4.32). From (3.1) we know that if $Q\ni x$ then

$$2^{k_Q\alpha}\|T_{k_Q}f-T_{k_Q-1}f\|_{L_\infty(Q)}\tag{4.36}$$
$$\lesssim\frac{2^{(k_Q-1)\alpha}}{|Q_L^{k_Q-1}(x)|}\int_{Q_L^{k_Q-1}(x)}|f-T_{k_Q-1}f|\lesssim f_\alpha^T(x).$$

Taking now the sup with respect to all cubes containing x and then applying the L_p norm gives (4.35).

Finally, to finish the proof of the theorem we note that if ϕ satisfies (A5) then for any $0<q<\infty$,

$$N_5(f)\lesssim N_4(f)\lesssim\|f\|_{L_p}+\|\sup_{D\ni Q\ni x}2^{k_Q\alpha}\|T_{k_Q}f-T_{k_Q-1}f\|_{L_\infty(Q)}\|_{L_p}$$
$$\lesssim\|f\|_{L_p}+\|\sup_{D\ni Q\ni x}2^{k_Q(\alpha+d/q)}\|T_{k_Q}f-T_{k_Q-1}f\|_{L_q(Q)}\|_{L_p}$$
$$\lesssim\|f\|_{L_p}+\|\sup_{Q\ni x}2^{k_Q(\alpha+d/q)}\|T_{k_Q}f-T_{k_Q-1}f\|_{L_q(Q)}\|_{L_p}$$
$$\lesssim N_5(f).$$

In the third inequality we used (2.2) and in the fifth Hölder's inequality. ∎

Finally, employing (2.12) in the previous theorem we derive (1.7).

THEOREM 4.37. *Let $1<p\le\infty$ and $0<q\le\infty$. Let also $\Psi=\{\psi^e:e\in E\}$ be the family of compactly supported orthonormal wavelets given in* (2.8). *If the corresponding sequence $(P_k)_{k\in\mathbb{Z}}$ of operators satisfies assumptions* (A2, 4) *for some $\alpha>0$, then for every $f\in C_p^\alpha$,*

$$\|f\|_{C_p^\alpha}\approx\|f\|_{L_p}+\Big\|\sup_{D\ni Q\ni x}2^{k_Q(\alpha+d/2)}\sum_{e\in E}\|(a^e_{k_Q,j})_j\|_{l_q(\Lambda^e(Q))}\Big\|_{L_p},\tag{4.38}$$

where $\Lambda^e(Q)$ denotes the set of $j\in\mathbb{Z}^d$ such that $\psi^e(2^{k_Q}\cdot-j)$ is not identically zero on Q and $a^e_{k_Q,j}$ are the wavelet coefficients defined in (2.11).

Proof. We recall that the sequence $(P_k)_{k\in\mathbb{Z}}$ satisfies, by construction, the assumptions (A1, 3, 5). Thus, by using (2.12) in (4.32)(vi) we get the result. ∎

5. Smoothness spaces in L_p, $0<p\le1$. Next we want to extend Theorem 4.31 to $0<p\le1$. The only difficulty appears to be that the maximal operator used in the proof of $N_2(f)\lesssim N_3(f)$ is not bounded on L_p, $0<p\le1$. To overcome this obstacle we will establish that for any $\alpha>d(1/p-1)$, $0<q,s\le1$,

$$\|f^\sharp_{\alpha,s}\|_{L_p}\approx\|f^\sharp_{\alpha,q}\|_{L_p},\tag{5.1}$$

and that

$$\|f^T_{\alpha,s}\|_{L_p}\approx\|f^T_{\alpha,q}\|_{L_p},\tag{5.2}$$

where for any $0<q<\infty$,

$$f^T_{\alpha,q}(x):=\sup_{Q\ni x}\frac{1}{|Q|^{\alpha/d}}\Big(\frac{1}{|Q|}\int_Q|f-T_{k_Q}f|^q\Big)^{1/q}.\tag{5.3}$$

Another, rather minor, problem is that T_kf, $k\in\mathbb{Z}$, may not be defined on L_p, $0<p<1$. However, if we restrict our attention to $\alpha>d(1/p-1)$ it is not hard to show that the space C_p^α is continuously embedded in L_1. To see this we recall the following embeddings for Besov spaces (see [DS] and [DP]):

$$\begin{aligned}&B_p^{\alpha,p}\hookrightarrow C_p^\alpha\hookrightarrow B_p^{\alpha,\infty}, && \alpha>0,\\ &B_p^{\alpha,q}\hookrightarrow B_p^{\beta,r}, && \alpha>\beta,\ 0<q,\ r\le\infty,\\ &B_p^{\alpha,p}\hookrightarrow L_1, && \alpha>d(1/p-1).\end{aligned}\tag{5.4}$$

Let now ε be small enough so that $\alpha - 2\varepsilon > d(1/p-1)$. From (5.4) we have

$$C_p^\alpha \hookrightarrow B_p^{\alpha,\infty} \hookrightarrow B_p^{\alpha-\varepsilon,\infty} \hookrightarrow B_p^{\alpha-2\varepsilon,p} \hookrightarrow L_1. \tag{5.5}$$

We start with the proof of (5.1). For this we employ the following inequality from [DS]:

LEMMA 5.6. *If $0 < s < r$ and $f \in L_s(\mathrm{loc})$, then*

$$f_{\alpha,r}^\sharp(x) \lesssim M_\sigma(f_{\alpha,s}^\sharp)(x), \tag{5.7}$$

with $1/\sigma = 1/r + \alpha/d$ and $M_\sigma(g) := [M(|g|^\sigma)]^{1/\sigma}$.

THEOREM 5.8. *Let $0 < p \le 1$ and $\alpha > d(1/p-1)$. For every $0 < l \le q \le 1$ and $f \in L_1(\mathrm{loc})$,*

$$\|f_{\alpha,l}^\sharp\|_{L_p} \approx \|f_{\alpha,q}^\sharp\|_{L_p}. \tag{5.9}$$

Proof. From Hölder's inequality it is easily seen that

$$f_{\alpha,l}^\sharp \le f_{\alpha,q}^\sharp.$$

For the other direction it is sufficient to consider only the case $q = 1$. Applying the previous lemma with $r = 1$ and $s = l$ we get

$$f_\alpha^\sharp \lesssim M_\sigma(f_{\alpha,l}^\sharp). \tag{5.10}$$

Since $1/\sigma = 1 + \alpha/d$ it follows that $p/\sigma = p + \alpha p/d > p + p(1/p-1) > 1$. Applying the L_p norm on both sides of (5.10) and the boundedness of M on $L_{p/\sigma}$ yields the result. ∎

Next we prove (5.2). In what follows for every cube $I \in \mathcal{Q}^k$, $k \in \mathbb{Z}$, we denote by $\widetilde{I}$ the smallest cube that contains the union of all dyadic cubes in D_k that intersect I. It is readily seen that $|\widetilde{I}| \lesssim |I|$ for some constant independent of $k \in \mathbb{Z}$.

LEMMA 5.11. *Let $f \in L_1(\mathrm{loc})$ and $0 < q \le 1$. Then for every cube $I \subset \mathbb{R}^d$,*

$$\begin{aligned}[(f - T_{k_I} f)\chi_I]^*(t) \\ \lesssim \Big(\int_t^{|\widetilde{I}|} F^*(s) s^{\alpha/d} \frac{ds}{s} + t^{\alpha/d} F^*(t)\Big), \quad 0 < t \le |I|/2^d,\end{aligned} \tag{5.12}$$

where, for $x \in \widetilde{I}$,

$$F(x) := \sup_{\widetilde{I} \supset Q \ni x} \frac{1}{|Q|^{\alpha/d}} \Big(\frac{1}{|Q|} \int_Q |f - T_{k_Q} f|^q\Big)^{1/q}, \tag{5.13}$$

and g^* denotes the decreasing rearrangement of g (see [BS] for the definition).

Proof. We let $I \in \mathcal{Q}_\nu, \nu \in \mathbb{Z}$, and define $E := \{x \in I : F(x) > F^*(t)\}$. From the definition of F^* we know that $|E| \le t$.

Let $x \in I \setminus E$ be such that $\lim_{k\to\infty} T_k f(x) = f(x)$. We choose dyadic cubes $I_j \in D_j$, $j \ge \nu$, containing x and such that $\widetilde{I} \supset I_\nu \supset I_{\nu+1} \supset \ldots$ Since $t \le |I|/2^d$ we let $m > \nu$ be the integer with $2^{-(m+1)d} \le t < 2^{-md}$. The triangle inequality gives

$$|f(x) - T_\nu f(x)| \le |f(x) - T_m f(x)| + |T_m f(x) - T_\nu f(x)| =: A + B. \tag{5.14}$$

For A we have

$$\begin{aligned} A &\le \sum_{k \ge m} \|T_{k+1} f - T_k f\|_{L_\infty(I_{k+1})} \\ &\lesssim \sum_{k\ge m} \Big(\frac{1}{|I_{k+1}|} \int_{I_{k+1}} |T_{k+1} f - T_k f|^q\Big)^{1/q} \\ &\lesssim \sum_{k\ge m} \Big(\Big(\frac{1}{|I_{k+1}|} \int_{I_{k+1}} |T_{k+1} f - f|^q\Big)^{1/q} \\ &\qquad + \Big(\frac{1}{|I_k|} \int_{I_k} |T_k f - f|^q\Big)^{1/q}\Big) \\ &\lesssim \sum_{k\ge m} |I_k|^{\alpha/d} F(x) \lesssim |I_m|^{\alpha/d} F^*(t) \lesssim t^{\alpha/d} F^*(t), \end{aligned} \tag{5.15}$$

where in the second inequality we used the fact that

$$\begin{aligned}\|T_{k+1} f - T_k f\|_{L_\infty(I_{k+1})} \\ \approx \Big(\frac{1}{|I_{k+1}|} \int_{I_{k+1}} |T_{k+1} f - T_k f|^q\Big)^{1/q}, \quad k \ge \nu,\end{aligned} \tag{5.16}$$

which follows from (2.2).

As far as B is concerned, using a similar argument we get

$$\begin{aligned} B &\le \sum_{m > k \ge \nu} |T_{k+1} f(x) - T_k f(x)| \\ &\le \sum_{m\ge k\ge \nu} |I_k|^{\alpha/d} \inf_{u \in I_k} F(u) \le \sum_{m \ge k \ge \nu} |I_k|^{\alpha/d} F^*(|I_k|) \\ &\lesssim \int_t^{|\widetilde{I}|} s^{\alpha/d - 1} F^*(s)\, ds. \end{aligned} \tag{5.17}$$

Finally, taking into account (5.17) and (5.15) in (5.14) we get for every $x \in I \setminus E$,

$$|f(x) - T_\nu f(x)| \lesssim \Big(\int_t^{|\widetilde{I}|} F^*(s) s^{\alpha/d-1}\, ds + t^{\alpha/d} F^*(t) \Big), \quad 0 < t \le |I|/2^d.$$

From this and the definition of $[(f - T_\nu f)\chi_I]^*$ the result follows. ■

We are now ready to prove (5.2).

LEMMA 5.18. *If $0 < s < r$ and $f \in L_s(\mathrm{loc}) \cap L_1(\mathrm{loc})$, then*

$$f^T_{\alpha,r}(x) \lesssim M_\sigma(f^T_{\alpha,s})(x), \tag{5.19}$$

with $1/\sigma = 1/r + \alpha/d$.

Proof. Let I be an arbitrary cube and define $\eta_I := [(f - T_{k_I})\chi_I]^*(t)$. Then

$$f^T_{\alpha,r}(x) = \sup_{I \ni x} \frac{1}{|I|^{1/\sigma}} \|f - T_{k_I} f\|_{L_r(I)} = \sup_{I \ni x} \frac{1}{|I|^{1/\sigma}} \|\eta_I\|_{L_r}. \tag{5.20}$$

Employing (5.12) and using the fact that η_I is a decreasing function it is easily seen that

$$\begin{aligned} \int_0^{|I|} \eta_I^r\, dt &\le 2^d \int_0^{|I|/2^d} \eta_I^r\, dt \\ &\lesssim \int_0^{|I|/2^d} \Big[\int_t^{|\widetilde{I}|} F^*(s) s^{\alpha/d} \frac{ds}{s} + t^{\alpha/d} F^*(t) \Big]^r dt \\ &\lesssim \int_0^{|I|/2^d} \Big[\int_t^{|\widetilde{I}|} F^*(s) s^{\alpha/d} \frac{ds}{s} \Big]^r dt + \int_0^{|I|/2^d} [t^{1/\sigma} F^*(t)]^r \frac{dt}{t}. \end{aligned}$$

Using Hardy's inequality (see [BS]) we get

$$\int_0^{|I|/2^d} \Big[\int_t^{|\widetilde{I}|} F^*(s) s^{\alpha/d} \frac{ds}{s} \Big]^r dt \lesssim \int_0^{|\widetilde{I}|} [t^{1/\sigma} F^*(t)]^r \frac{dt}{t},$$

which implies that

$$\|\eta_I\|_{L_r}^r \lesssim \int_0^{|\widetilde{I}|} [t^{1/\sigma} F^*(t)]^r \frac{dt}{t}. \tag{5.21}$$

Employing (5.21) in (5.20) we get

$$f^T_{\alpha,r}(x) \lesssim \sup_{I \ni x} \frac{1}{|I|^{1/\sigma}} \Big(\int_0^{|\widetilde{I}|} [t^{1/\sigma} F^*(t)]^r \frac{dt}{t} \Big)^{1/r} = \sup_{I \ni x} \frac{1}{|I|^{1/\sigma}} \|F\|_{L_{\sigma,r}}, \tag{5.22}$$

where $\|\cdot\|_{L_{\sigma,r}}$ is the norm of the corresponding Lorentz space. We refer the reader to [BS] for the definitions of the Lorentz spaces as well as for some of their properties. In particular, it is known that for $\sigma < r$,

$$\|\cdot\|_{L_{\sigma,r}} \lesssim \|\cdot\|_{L_{\sigma,\sigma}} \approx \|\cdot\|_{L_\sigma}.$$

Using this last inequality in (5.22) we get

$$\begin{aligned} f^T_{\alpha,r}(x) &\lesssim \sup_{I \ni x} \Big(\frac{1}{|\widetilde{I}|} \int_{\widetilde{I}} F^\sigma \Big)^{1/\sigma} \lesssim \sup_{I \ni x} \Big(\frac{1}{|\widetilde{I}|} \int_{\widetilde{I}} (f^T_{\alpha,q})^\sigma \Big)^{1/\sigma} \\ &\lesssim M_\sigma(f^T_{\alpha,q})(x). \ ■ \end{aligned}$$

Following now verbatim the proof of Theorem 5.8 we derive

THEOREM 5.23. *Let $0 < p, q, l \le 1$ and $\alpha > d(1/p - 1)$. If $T = (T_k)_{k \in \mathbb{Z}}$ satisfies (A1–5) then, for any $f \in L_1(\mathrm{loc})$,*

$$\|f^T_{\alpha,l}\|_{L_p} \approx \|f^T_{\alpha,q}\|_{L_p}. \tag{5.24}$$

THEOREM 5.25. *Let $0 < p \le 1$, and $0 < q \le \infty$. If $T = (T_k)_{k \in \mathbb{Z}}$ satisfies (A1–4) for some $\alpha > 0$ then, for every $f \in C^\alpha_p$, the following are equivalent:*

(i) $N_1(f) := \|f\|_{C^\alpha_p}$,

(ii) $N_2(f) := \|f\|_{L_p} + \|f^T_\alpha\|_{L_p}$.

Moreover, if ϕ has linearly independent shifts, i.e., if (A5) holds, then the above are equivalent to

$$\begin{aligned} &\text{(iii)} \quad N_3(f) := \|f\|_{L_p} + \|\sup_{k \in \mathbb{Z}} 2^{k\alpha} |f - T_k f|\|_{L_p}, \\ &\text{(iv)} \quad N_4(f) := \|f\|_{L_p} + \|\sup_{k \in \mathbb{Z}} 2^{k\alpha} |T_k f - T_{k-1} f|\|_{L_p}, \\ &\text{(v)} \quad N_5(f) := \|f\|_{L_p} + \|\sup_{Q \ni x} 2^{k_Q \alpha} \|T_{k_Q} f - T_{k_Q - 1} f\|_{L_\infty(Q)}\|_{L_p}, \\ &\text{(vi)} \quad N_6(f) := \|f\|_{L_p} + \|\sup_{Q \ni x} 2^{k_Q(\alpha + d/q)} \|T_{k_Q} f - T_{k_Q - 1} f\|_{L_q(Q)}\|_{L_p}, \end{aligned} \tag{5.26}$$

where the constants of equivalence depend on q, T and d.

Proof. First we prove the equivalence between $N_1(f)$ and $N_2(f)$. In view of (5.5) we know from Theorem 4.24 that

$$f^T_\alpha \approx f^\sharp_\alpha. \tag{5.27}$$

Therefore, applying the L_p norm on both sides of (5.27) and taking into account (5.9) with $q = 1$ and $l = p$ completes the proof.

Assuming now that (A5) holds, we will prove that $N_2(f) \lesssim N_3(f)$. Let $0<q<p$. For any $x\in\mathbb{R}^d$, we have

$$f^T_{\alpha,q}(x)=\sup_{Q\ni x}\frac{1}{|Q|^{\alpha/d}}\left(\frac{1}{|Q|}\int_Q |f-T_{k_Q}f|^q\right)^{1/q}$$
$$\le \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q(\sup_{k\in\mathbb{Z}}2^{k\alpha}|f-T_kf|)^q\right)^{1/q}=M_q(\sup_{k\in\mathbb{Z}}2^{k\alpha}|f-T_kf|)(x).$$

Using now the boundedness of the maximal operator on $L_{p/q}$, $p/q>1$, and (5.24) we get

$$\|f^T_\alpha\|_{L_p}\lesssim \|f^T_{\alpha,q}\|_{L_p}\le \|M_q(\sup_{k\in\mathbb{Z}}2^{k\alpha}|f-T_kf|)\|_{L_p}$$
$$\lesssim \|\sup_{k\in\mathbb{Z}}2^{k\alpha}|f-T_kf|\|_{L_p}.$$

Finally, the inequalities $N_3(f)\lesssim N_4(f)\approx N_5(f)\lesssim N_6(f)\lesssim N_2(f)$ follow in the same manner as in Theorem 4.31. ■

At last, similarly to Theorem 4.37 we have

THEOREM 5.28. *Let $0<p\le 1$ and $0<q\le\infty$. Let also $\Psi=\{\psi^e : e\in E\}$ be the family of compactly supported orthonormal wavelets given in* (2.8). *If the corresponding sequence $(P_k)_{k\in\mathbb{Z}}$ of operators satisfies assumptions* (A2, 4) *for some $\alpha>0$ then, for every $f\in C^\alpha_p$,*

$$\|f\|_{C^\alpha_p}\approx\|f\|_{L_p}+\|\sup_{D\ni Q\ni x}2^{k_Q(\alpha+d/2)}\sum_{e\in E}\|(a^e_{k_Q,j})_j\|_{l_q(\Lambda^e(Q))}\|_{L_p}, \tag{5.29}$$

where $\Lambda^e(Q)$ denotes the set of $j\in\mathbb{Z}^d$ such that $\psi^e(2^{k_Q}\cdot-j)$ is not identically zero on Q and $a^e_{k_Q,j}$ are the wavelet coefficients defined in (2.11).

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