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# Finite rank elements in semisimple Banach algebras

by

MATEJ BREŠAR and PETER ŠEMRL (Maribor)

**Abstract.** Let $\mathcal{A}$ be a semisimple Banach algebra. We define the rank of a nonzero element $a$ in the socle of $\mathcal{A}$ to be the minimum of the number of minimal left ideals whose sum contains $a$. Several characterizations of rank are proved.

**1. Introduction and statement of the main result.** Throughout this paper, $\mathcal{A}$ will be a unital semisimple complex Banach algebra. Recall that the sum of all minimal left ideals of $\mathcal{A}$ coincides with the sum of all minimal right ideals of $\mathcal{A}$, and is called the *socle* of $\mathcal{A}$. It will be denoted by $\operatorname{soc}(\mathcal{A})$. If $\mathcal{A}$ does not have minimal one-sided ideals, we define $\operatorname{soc}(\mathcal{A}) = \{0\}$.

According to the definition, for every nonzero element $a$ in $\operatorname{soc}(\mathcal{A})$ there exist finitely many minimal left ideals such that $a$ belongs to their sum. Of course, the choice of these minimal left ideals is not unique. We are interested in the minimum of the number of minimal left ideals whose sum contains $a$.

DEFINITION. The element $0 \in \operatorname{soc}(\mathcal{A})$ has *rank* 0. An element $a \in \operatorname{soc}(\mathcal{A})$ has *rank one* if $a$ belongs to some minimal left ideal of $\mathcal{A}$ and $a \neq 0$. An element $b \in \operatorname{soc}(\mathcal{A})$ has *rank* $n > 1$ if $b$ belongs to a sum of $n$ minimal left ideals, but does not belong to any sum of less than $n$ minimal left ideals.

Recall that every minimal left ideal $\mathcal{L}$ of $\mathcal{A}$ is of the form $\mathcal{L} = \mathcal{A}e$ where $e \in \mathcal{A}$ is a minimal idempotent, that is, a nonzero idempotent such that $e\mathcal{A}e = \mathbb{C}e$. Using this fact it is easy to see that a bounded linear operator $A$ on a Banach space $X$ has rank $n$ as an element of the Banach algebra $\mathcal{B}(X)$, the algebra of all bounded linear operators on $X$, if and only if the range of $A$ is an $n$-dimensional subspace of $X$. This justifies the choice of the word rank.

Note that $b \in \mathcal{A}$ has rank $n > 1$ if and only if $b$ can be written as a sum of $n$ elements of rank one, but cannot be written as a sum of less than $n$ elements of rank one.

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We claim that $a \in \mathcal{A}$ has rank one if and only if $a \neq 0$ and $a\mathcal{A}a = \mathbb{C}a$. Suppose that $a$ has rank one. Then $a \neq 0$ and $a$ belongs to some minimal left ideal $\mathcal{A}e$ where $e$ is a minimal idempotent. But then it follows at once that $a\mathcal{A}a = \mathbb{C}a$. Conversely, $a \neq 0$ and $a\mathcal{A}a = \mathbb{C}a$ imply that $\mathcal{A}a$ is a minimal left ideal of $\mathcal{A}$—this can be, for instance, shown by adapting the proof of [10, Lemma 2.1.8]—which clearly implies that $a$ has rank one. We have thereby shown that our definition of rank coincides with the one given in [9] and [8].

The object of this paper is to prove several characterizations of rank. In order to state the main result we need some more terminology.

We shall say that an element $a$ of an algebra $\mathcal{B}$ is *indecomposable* if it cannot be written as a sum $a = b + c$ where $b, c \in \mathcal{B}$ are nonzero elements satisfying $b\mathcal{B}c = 0$. Note that an algebra $\mathcal{B}$ is prime if and only if every element in $\mathcal{B}$ is indecomposable, and $\mathcal{B}$ is semiprime if and only if $0 \in \mathcal{B}$ is an indecomposable element. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be prime algebras and let $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$ be their direct sum. Then every element in $\mathcal{B}$ of the form $(b_1, 0)$ or $(0, b_2)$ is indecomposable while the element $(b_1, b_2)$ with $b_1 \neq 0$, $b_2 \neq 0$ is not. Indecomposable elements in $\operatorname{soc}(\mathcal{A})$ will be characterized in Proposition 2.4.

By $M_{r,n}$, where $r \leq n \leq 2r$, we denote the algebra of all $n \times n$ matrices $A = [a_{ij}]$ satisfying $a_{ij} = 0$ whenever $i > r$ or $j \leq n - r$. We will show in Lemma 2.7 that an operator $T \in \mathcal{B}(X)$ has rank $r$ if and only if the algebra $T\mathcal{B}(X)T$ is isomorphic to $M_{r,n}$ for some $n$. Note also that if $a \in \mathcal{A}$ is a rank one element, then, as already noted, $a\mathcal{A}a = \mathbb{C}a$, and so $a\mathcal{A}a$ is either isomorphic to $M_{1,2}$ (when $a$ is nilpotent) or to $\mathbb{C} \cong M_{1,1}$ (when $a$ is not nilpotent).

As usual, $\sigma(x)$ denotes the spectrum of $x$.

We are now in a position to state the

MAIN THEOREM. *Let $\mathcal{A}$ be a semisimple unital complex Banach algebra and let $n \in \mathbb{N}$. For $a \in \mathcal{A}$ the following assertions are equivalent:*

(A) *$a$ has rank $n$;*

(B) *the left ideal $\mathcal{A}a$ is a sum of $n$ minimal left ideals, but is not a sum of less than $n$ minimal left ideals;*

(C) *$\sigma(xa)$ contains at most $n$ nonzero points for every $x \in \mathcal{A}$ and there exists $x_0 \in \mathcal{A}$ such that $\sigma(x_0 a)$ contains $n$ nonzero points;*

(D) *$a$ satisfies*

 (D.1) *$\bigcap_{\lambda \in F} \sigma(x + \lambda a) \subset \sigma(x)$ for every $x \in \mathcal{A}$ and every subset $F \subset \mathbb{C} \setminus \{0\}$ having $n + 1$ elements,*

 (D.2) *there exists $x_1 \in \mathcal{A}$ and a subset $F_1 \subset \mathbb{C} \setminus \{0\}$ with $n$ elements such that $\bigcap_{\lambda \in F_1} \sigma(x_1 + \lambda a) \not\subset \sigma(x_1)$;*



(E) *$a$ satisfies*

 (E.1) *there exist finitely many distinct primitive ideals $P_1, \ldots, P_k$ of $\mathcal{A}$ such that $a \in P$ for every primitive ideal $P \neq P_i$, $i = 1, \ldots, k$,*

 (E.2) *if $\pi_i$, $i = 1, \ldots, k$, are continuous irreducible representations of $\mathcal{A}$ on Banach spaces such that $\operatorname{Ker} \pi_i = P_i$, then $\pi_i(a)$ are finite rank operators and $n = \operatorname{rank} \pi_1(a) + \ldots + \operatorname{rank} \pi_k(a)$;*

(F) *there exist $a_1, \ldots, a_k \in \mathcal{A}$ such that*

 (F.1) *$a = a_1 + \ldots + a_k$,*

 (F.2) *each $a_i$ is indecomposable,*

 (F.3) *$a_i \mathcal{A} a_j = 0$ whenever $i \neq j$,*

 (F.4) *$a_i \mathcal{A} a_i$ is isomorphic to $M_{r_i, n_i}$ for some $r_i, n_i \in \mathbb{N}$, $r_i \leq n_i \leq 2r_i$;*

 (F.5) *$n = r_1 + \ldots + r_k$.*

*Moreover, $a_1, \ldots, a_k$ are unique nonzero elements in $\operatorname{soc}(\mathcal{A})$ satisfying (F.1)–(F.3).*

When $n = 1$, the condition (B) should be understood as stating that $\mathcal{A}a$ is a minimal left ideal.

Not all the equivalences in this theorem are new. In a recent paper [3], Aupetit and Mouton defined the rank via the condition (C). At the beginning of the paper they show that (C) and (D) are equivalent [3, Theorem 2.1] (see also [6, 7, 2, 8] for the background concerning the conditions (C) and (D)). Using the subadditivity of the rank they also show that these two conditions are equivalent to (A) [3, Corollary 2.18]. Let us also point out [3, Corollary 2.17] which indicates that an element can be written as a sum of elements of lower rank (cf. the condition (F)).

The assertion (F) is probably the most illuminating one. It could be considered as a structure theorem for elements having rank $n$. Let $a \in \mathcal{A}$ be an indecomposable element. Obviously, $k$ in (F) must then be 1. Therefore, we have the following characterization of rank of indecomposable elements $a$: the rank of $a$ is $r$ if and only if $a\mathcal{A}a \cong M_{r,n}$ for some $n$, $r \leq n \leq 2r$. Let us show by a simple example that this is not true for all elements in $\mathcal{A}$.

EXAMPLE. Let $X$ be a Banach space, and let $\mathcal{A} = \mathcal{B}(X) \oplus \mathcal{B}(X) \oplus \mathcal{B}(X) \oplus \mathcal{B}(X)$. Pick $\xi \in X$ and $\phi$ in the dual $X^*$ of $X$ such that $\phi(\xi) = 0$. Let $\xi \otimes \phi \in \mathcal{B}(X)$ be a rank one nilpotent defined by $(\xi \otimes \phi)(\eta) = \phi(\eta)\xi$. Set $a = (\xi \otimes \phi, \xi \otimes \phi, \xi \otimes \phi, \xi \otimes \phi) \in \mathcal{A}$. The rank of $a$ is 4. However, the algebra $a\mathcal{A}a$ is isomorphic to $M_{1,2} \oplus M_{1,2} \oplus M_{1,2} \oplus M_{1,2}$, which in turn is isomorphic to $M_{2,4}$ for they both are 4-dimensional algebras with trivial multiplication.

In particular, this example shows that the rank of an element $a \in \mathcal{A}$ is not necessarily determined by the algebra $a\mathcal{A}a$. This is the reason why we have to decompose $a$ into a sum of indecomposable elements.

**2. Preliminaries.** In this section we gather together several auxiliary results needed for the proof of the main theorem. We begin with some more or less well-known theorems.

The first one is an extension of the Jacobson density theorem. It can be deduced from [5, p. 283].

THEOREM 2.1. *Let $\pi_1, \ldots, \pi_k$ be irreducible representations of $\mathcal{A}$ on Banach spaces $X_1, \ldots, X_k$, respectively. Assume that* $\operatorname{Ker} \pi_i \neq \operatorname{Ker} \pi_j$ *whenever $i \neq j$. Let $V_1, \ldots, V_k$ be finite-dimensional subspaces of $X_1, \ldots, X_k$, respectively, and let $A_i : X_i \to X_i$, $i = 1, \ldots, k$, be any linear operators. Then there exists $x \in \mathcal{A}$ such that*

$$\pi_i(x)|V_i = A_i|V_i, \quad i = 1, \ldots, k.$$

The set of all primitive ideals of $\mathcal{A}$ will be denoted by $\Pi(\mathcal{A})$. The following theorem follows from [10, Theorem 2.2.9(v)].

THEOREM 2.2. $\sigma(a) = \bigcup_{P \in \Pi(\mathcal{A})} \sigma(a + P)$ *for every $a \in \mathcal{A}$.*

The next theorem characterizes elements lying in the socle (see [1, Theorem 7.2], [2, Theorem 2.1], and [4, Proposition 2.2]).

THEOREM 2.3. *For $a \in \mathcal{A}$ the following assertions are equivalent*:

(i) $a \in \operatorname{soc}(\mathcal{A})$;

(ii) *the algebra $a\mathcal{A}a$ is finite-dimensional*;

(iii) $\sigma(xa)$ *is finite for every $x \in \mathcal{A}$*;

(iv) *there exist finitely many primitive ideals $P_1, \ldots, P_k$ of $\mathcal{A}$ such that $a \in P$ for every primitive ideal $P \neq P_i$, $i = 1, \ldots, k$, and $a + P_i \in \operatorname{soc}(\mathcal{A}/P_i)$, $i = 1, \ldots, k$.*

The condition (iv) shows, in particular, that $a + P \in \operatorname{soc}(\mathcal{A}/P)$ for every $a \in \operatorname{soc}(\mathcal{A})$ and every $P \in \Pi(\mathcal{A})$. Another useful observation is the following: $a + P \in \operatorname{soc}(\mathcal{A}/P)$ if and only if $\pi(a)$ is a finite rank operator where $\pi$ is an irreducible representation of $\mathcal{A}$ with $\operatorname{Ker} \pi = P$. We will use these two facts without making explicit reference.

In the next result we show how one can recognize indecomposable elements among all elements satisfying (iv) of Theorem 2.3.

PROPOSITION 2.4. *For a nonzero element $a \in \mathcal{A}$ the following assertions are equivalent*:



(i) *$a$ is indecomposable and $a \in \operatorname{soc}(\mathcal{A})$*;

(ii) *there exists a primitive ideal $P_1$ of $\mathcal{A}$ such that $a \in P$ for every primitive ideal $P \neq P_1$, and $a + P_1 \in \operatorname{soc}(\mathcal{A}/P_1)$.*

Proof. Suppose first that (i) holds. By Theorem 2.3 there exist $P_1, \ldots$ $\ldots, P_k \in \Pi(\mathcal{A})$, $P_i \neq P_j$ when $i \neq j$, such that $a \in P$ whenever $P \in \Pi(\mathcal{A})$ and $P \neq P_i$, $i = 1, \ldots, k$, and $a + P_i \in \operatorname{soc}(\mathcal{A}/P_i)$. Since $a \neq 0$, we can of course assume that $a \notin P_i$. All we need to show is that $k = 1$.

Suppose that $k \geq 2$. Let $\pi_1, \ldots, \pi_k$ be irreducible representations of $\mathcal{A}$ on Banach spaces $X_1, \ldots, X_k$, respectively, such that $\operatorname{Ker} \pi_i = P_i$. Since $a + P_i \in \operatorname{soc}(\mathcal{A}/P_i)$, $\pi_i(a)$ are finite rank operators. Therefore, Theorem 2.1 tells us that there is $x \in \mathcal{A}$ such that $\pi_1(x)|\operatorname{Im}\pi_1(a) = 0$ and $\pi_i(x)|\operatorname{Im}\pi_i(a) = I|\operatorname{Im}\pi_i(a)$, $i = 2, \ldots, k$. That is, $\pi_1(x)\pi_1(a) = 0$ and $\pi_i(x)\pi_i(a) = \pi_i(a)$, $i = 2, \ldots, k$. Similarly, there is $y \in \mathcal{A}$ such that $\pi_1(y)\pi_1(a) = \pi_1(a)$ and $\pi_i(y)\pi_i(a) = 0$, $i = 2, \ldots, k$. Thus, the elements $b = xa$ and $c = ya$ satisfy

$$\pi_1(b) = 0, \quad \pi_i(b) = \pi_i(a), \quad i = 2, \ldots, k,$$
$$\pi_1(c) = \pi_1(a), \quad \pi_i(c) = 0, \quad i = 2, \ldots, k.$$

That is,

$$b \in P_1, \quad a - b \in P_2 \cap \ldots \cap P_k,$$
$$a - c \in P_1, \quad c \in P_2 \cap \ldots \cap P_k.$$

Using these relations one concludes that $a - b - c \in P_1 \cap \ldots \cap P_k$. Now pick $P \in \Pi(\mathcal{A})$ such that $P \neq P_i$, $i = 1, \ldots, k$. As $a \in P$ and $b, c \in \mathcal{A}a$, we also have $b, c \in P$, and in particular, $a - b - c \in P$. Thus we proved that $a - b - c$ lies in every primitive ideal of $\mathcal{A}$. But this means that $a = b + c$ for $\mathcal{A}$ is semisimple. Similarly we see that $b\mathcal{A}c$ is contained in every primitive ideal of $\mathcal{A}$, so that $b\mathcal{A}c = 0$. However, $b \neq 0$ and $c \neq 0$ for $\pi_i(b) = \pi_i(a) \neq 0$, $i = 2, \ldots, k$, and $\pi_1(c) = \pi_1(a) \neq 0$. This contradicts the indecomposability of $a$. Thus, we proved that (i) implies (ii).

Now suppose that (ii) holds true. Then $a \in \operatorname{soc}(\mathcal{A})$ by Theorem 2.3. Assume that there are nonzero $b, c \in \mathcal{A}$ such that $a = b + c$ and $b\mathcal{A}c = 0$. As $b, c \neq 0$, there exist $P, Q \in \Pi(\mathcal{A})$ such that $b \notin P$, $c \notin Q$. Every primitive ideal is also prime, and so $b\mathcal{A}c = 0 \subseteq P$ implies $c \in P$. Similarly, $b \in Q$. But then $a = b + c$ lies neither in $P$ nor in $Q$. Of course, $P \neq Q$ for $b \notin P$ and $b \in Q$. This contradicts the assumption that $a$ lies in every primitive ideal but one. The proof of the proposition is complete.

LEMMA 2.5. *If $a \in \mathcal{A}$ has rank one, then $a$ is indecomposable.*

Proof. Every element of rank one lies in a left ideal $\mathcal{A}e$ where $e$ is a minimal idempotent. Therefore, using Proposition 2.4 we see that it suffices to show that every minimal idempotent is indecomposable.

We will prove slightly more, namely, that every minimal idempotent $e$ in a semiprime algebra $\mathcal{B}$ is indecomposable. Assume that $e = b + c$ and $b\mathcal{B}c = 0$. We must show that $b = 0$ or $c = 0$. Since $e$ is a minimal idempotent, there exist scalars $\lambda, \mu$ such that $ebe = \lambda e$ and $ece = \mu e$. Hence $(\lambda + \mu)e = ebe + ece = e^3 = e$. In particular, this shows that at least one of $\lambda, \mu$, say $\lambda$, is not zero. Consequently, $\lambda ec = ebec \in eb\mathcal{B}c = 0$ yields $ec = 0$. Therefore, $b + c = e = e^2 = eb + ec = eb$. That is, $c = (e-1)b$. Hence $c\mathcal{B}c = (e-1)b\mathcal{B}c = 0$ and so $c = 0$ by the semiprimeness of $\mathcal{B}$.

LEMMA 2.6. *Let $s \in \operatorname{soc}(\mathcal{A})$, $s \neq 0$. Then there exist $s_1, \ldots, s_k \in \operatorname{soc}(\mathcal{A})$ and distinct primitive ideals $P_1, \ldots, P_k$ of $\mathcal{A}$ such that*

(i) $s = s_1 + \ldots + s_k$;

(ii) $s_i \notin P_i$ *and* $s_i \in P$ *for every primitive ideal* $P \neq P_i$;

(iii) *each* $s_i$ *is indecomposable*;

(iv) *the rank of* $s$ *is* $r_1 + \ldots + r_k$ *where* $r_i$ *is the rank of* $s_i$;

(v) *each* $s_i$ *is a sum of* $r_i$ *elements of rank one none of which lies in* $P_i$ *and each of which lies in every primitive ideal* $P \neq P_i$;

(vi) *if* $\pi_i$ *is a continuous irreducible representation of* $\mathcal{A}$ *on a Banach space* $X_i$ *such that* $\operatorname{Ker}\pi_i = P_i$, *then* $\pi_i(s) = \pi_i(s_i)$ *is an operator of rank* $r_i$.

Proof. We denote the rank of $s$ by $n$. Let $u_1, \ldots, u_n$ be elements of rank one whose sum is $s$. According to Lemma 2.5 and Proposition 2.4 for every $u_j$ there exists exactly one primitive ideal not containing it. Of course, some of these $n$ primitive ideals may coincide. Adding together all the $u_j$'s which do not belong to the same primitive ideal $P_i$ and denoting their sum by $s_i$, we can then easily verify that the assertions (i)–(v) are true. It remains to prove (vi). In order to simplify the notation we write $t$ instead of $s_i$, and $r, \pi, X, P$ instead of $r_i, \pi_i, X_i, P_i$, respectively. Without loss of generality we may also assume that $t = u_1 + \ldots + u_r$, $u_i \notin P$, $i = 1, \ldots, r$. We must show that $\pi(t) = \pi(u_1) + \ldots + \pi(u_r)$ is an operator of rank $r$.

Let $e$ be any minimal idempotent in $\mathcal{A}$. Thus, $e\mathcal{A}e = \mathbb{C}e$, and hence $\pi(e)\pi(\mathcal{A})\pi(e) = \mathbb{C}\pi(e)$. As $\pi(\mathcal{A})$ is a subalgebra of $\mathcal{B}(X)$ acting densely on $X$, it follows easily that $\pi(e)$ is either 0 or a rank one projection. Using this we see that $\pi(u_1), \ldots, \pi(u_r)$ are rank one operators. Thus, there are nonzero vectors $\xi_1, \ldots, \xi_r$ in $X$ and nonzero functionals $\phi_1, \ldots, \phi_r$ in the dual $X^*$ such that $\pi(u_i) = \xi_i \otimes \phi_i$, $i = 1, \ldots, r$. Now

$$\pi(t) = \xi_1 \otimes \phi_1 + \ldots + \xi_r \otimes \phi_r$$

has rank $r$, unless the $\xi_i$'s or the $\phi_i$'s are linearly dependent.

Suppose that the $\xi_i$'s are dependent. Assume, for instance, that $\xi_r = \lambda_1\xi_1 + \ldots + \lambda_{r-1}\xi_{r-1}$ for some $\lambda_i \in \mathbb{C}$. Then

$$\pi(t) = \xi_1 \otimes (\phi_1 + \lambda_1\phi_r) + \ldots + \xi_{r-1} \otimes (\phi_{r-1} + \lambda_{r-1}\phi_r).$$



As $\pi$ is irreducible and $\xi_r \neq 0$, there exist $x_1, \ldots, x_{r-1} \in \mathcal{A}$ such that $\pi(x_i)\xi_r = \xi_i$, $i = 1, \ldots, r-1$. Set $v_i = u_i + \lambda_i x_i u_r$, $i = 1, \ldots, r-1$. As the $u_i$'s belong to every primitive ideal except $P$, the same is true for the $v_i$'s. Note that

$$\pi(v_i) = \xi_i \otimes \phi_i + \lambda_i \pi(x_i)(\xi_r \otimes \phi_r) = \xi_i \otimes (\phi_i + \lambda_i\phi_r).$$

Hence

$$\pi(t) = \pi(v_1) + \ldots + \pi(v_{r-1}),$$

that is, $t - v_1 - \ldots - v_{r-1} \in P$. As $t$ as well as the $v_i$'s lie in every primitive ideal different from $P$, it follows that $t = v_1 + \ldots + v_{r-1}$.

We claim that the $v_i$'s are elements of rank at most one. We know that $\pi(v_i)$ is an operator of rank at most one. Therefore, $\pi(v_i)\pi(\mathcal{A})\pi(v_i) = \mathbb{C}\pi(v_i)$. From this relation and the fact that $v_i$ lies in every primitive ideal except possibly $P$, it follows that $v_i\mathcal{A}v_i = \mathbb{C}v_i$. As already observed in the introduction, this relation yields that $v_i$ is either 0 or an element of rank one. Thus, $t$ can be written as a sum of less than $r$ elements of rank one. However, $r$ is the rank of $t$. This contradiction tells us that the $\xi_i$'s are linearly independent. Similarly we see that the $\phi_i$'s are linearly independent. Therefore, $\pi(t)$ has rank $r$. The lemma is thereby proved.

LEMMA 2.7. *Let $X$ be a Banach space and $\mathcal{D}$ be a subalgebra of $\mathcal{B}(X)$ acting densely on $X$. Let $T \in \mathcal{B}(X)$ and $r \in \mathbb{N}$. Then $T$ has rank $r$ if and only if the subalgebra $T\mathcal{D}T$ is isomorphic to $M_{r,n}$ for some $n$, $r \leq n \leq 2r$.*

Proof. Assume that $\operatorname{rank} T = r$. Set $X_1 = \operatorname{Ker} T \cap \operatorname{Im} T$. As $X_1$ is finite-dimensional, there exists a closed subspace $X_4 \subseteq X$ such that $\operatorname{Ker} T = X_1 \oplus X_4$. Choose a subspace $X_2$ satisfying $X_1 \oplus X_2 = \operatorname{Im} T$. A direct sum of a closed subspace and a finite-dimensional subspace is closed. Therefore, $X_1 \oplus X_2 \oplus X_4$ is closed. Moreover, it contains $\operatorname{Ker} T$, so that it is of finite codimension in $X$. It follows that we can find $X_3 \subseteq X$ such that $X$ has the direct sum decomposition into closed subspaces $X = X_1 \oplus X_2 \oplus X_3 \oplus X_4$. Then

$$T = \begin{bmatrix} 0 & T_1 & T_2 & 0 \\ 0 & T_3 & T_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with respect to this direct sum decomposition. Let $A \in \mathcal{B}(X)$. Then

$$TAT = \begin{bmatrix} 0 & A_1 & A_2 & 0 \\ 0 & A_3 & A_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, $\dim(X_1 \oplus X_2) = r$. As $X_2 \oplus X_3$ is complementary to $\operatorname{Ker} T$ we have also $\dim(X_2 \oplus X_3) = r$. It follows that $n = \dim(X_1 \oplus X_2 \oplus X_3) \leq 2r$

and $n \geq r$. It is now easy to verify that the mapping $\phi : T\mathcal{D}T \to M_{r,n}$ defined by

$$\phi\left(\begin{bmatrix} 0 & A_1 & A_2 & 0 \\ 0 & A_3 & A_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & A_1 & A_2 \\ 0 & A_3 & A_4 \\ 0 & 0 & 0 \end{bmatrix}$$

is an isomorphism.

Conversely, assume that $T\mathcal{D}T$ is isomorphic to $M_{r,n}$. In particular, $T\mathcal{D}T$ is finite-dimensional. As $\mathcal{D}$ acts densely on $X$, it is easy to see that then the operator $T$ must have finite rank. But then we already know that $T\mathcal{D}T$ is isomorphic to $M_{r_1,n_1}$ where $\operatorname{rank} T = r_1$. Since $M_{r,n}$ and $M_{r_1,n_1}$ are isomorphic and $\dim M_{r,n} = r^2$ we finally get the desired equality $r_1 = r$.

## 3. Proof of the main theorem

(A)$\Rightarrow$(B). We first settle the easier part. If the left ideal $\mathcal{A}a$ were a sum of less than $n$ minimal left ideals, then $a$ would lie in this sum, so that the rank of $a$ would be less than $n$.

Now, let us prove that the left ideal $\mathcal{A}a$, where $a$ is an element of rank $n$, is a sum of $n$ minimal left ideals. Apply Lemma 2.6 with $a$ playing the role of $s$. We keep the notation from that lemma with $a_i$ instead of $s_i$. We will show that

(a) $\mathcal{A}a = \mathcal{A}a_1 + \ldots + \mathcal{A}a_k$; and
(b) $\mathcal{A}a_i$ is a sum of $r_i$ minimal left ideals.

Of course, (a) and (b) together imply that $\mathcal{A}a$ is a sum of $n$ minimal left ideals.

It is trivial that $\mathcal{A}a = \mathcal{A}(a_1 + \ldots + a_k) \subseteq \mathcal{A}a_1 + \ldots + \mathcal{A}a_k$. Now pick an arbitrary element $x_1a_1 + \ldots + x_ka_k$ in $\mathcal{A}a_1 + \ldots + \mathcal{A}a_k$. Let $\pi_i$ be a continuous irreducible representation of $\mathcal{A}$ on a Banach space $X_i$ such that $\operatorname{Ker}\pi_i = P_i$, $i = 1, \ldots, k$. As each $\pi_i(a_i)$ is a finite rank operator, Theorem 2.1 tells us that there exists $x \in \mathcal{A}$ such that

$$\pi_i(x)|\operatorname{Im}\pi_i(a_i) = \pi_i(x_i)|\operatorname{Im}\pi_i(a_i), \quad i = 1, \ldots, k.$$

That is, $\pi_i(x)\pi_i(a_i) = \pi_i(x_i)\pi_i(a_i)$, for all $i$, which means that $(x - x_i)a_i \in P_i$. As $a_i$ belongs to any primitive ideal different from $P_i$, it follows that $xa_i = x_ia_i$, $i = 1, \ldots, k$. Hence $x_1a_1 + \ldots + x_ka_k = x(a_1 + \ldots + a_k) = xa \in \mathcal{A}a$. We have thereby proved (a).

In order to prove (b), we simplify the notation as in the proof of Lemma 2.6. Let $t \in \mathcal{A}$ be a sum of rank one elements $u_1, \ldots, u_r$ such that each $u_i$ lies in every primitive ideal different from $P \in \Pi(\mathcal{A})$. Note that (b) will be proved by showing that $\mathcal{A}t = \mathcal{A}u_1 + \ldots + \mathcal{A}u_r$.



Now, as in the proof of Lemma 2.6 we have $\pi(u_i) = \xi_i \otimes \phi_i$, $i = 1, \ldots, r$, where $\pi$ is a continuous irreducible representation of $\mathcal{A}$ on a Banach space $X$ with $\operatorname{Ker}\pi = P$, $\xi_1, \ldots, \xi_r$ are linearly independent vectors in $X$, and $\phi_1, \ldots, \phi_r$ are linearly independent functionals in $X^*$. Pick arbitrary $x_1, \ldots, x_r \in \mathcal{A}$. By the Jacobson density theorem there is $x \in \mathcal{A}$ satisfying $\pi(x)\xi_i = \pi(x_i)\xi_i$, $i = 1, \ldots, r$. Consequently,

$$\pi(x)(\xi_1 \otimes \phi_1) + \ldots + \pi(x)(\xi_r \otimes \phi_r) = \pi(x_1)(\xi_1 \otimes \phi_1) + \ldots + \pi(x_r)(\xi_r \otimes \phi_r).$$

That is,

$$\pi(xu_1 + \ldots + xu_r) = \pi(x_1u_1 + \ldots + x_ru_r),$$

which means that $x(u_1 + \ldots + u_r) - (x_1u_1 + \ldots + x_ru_r) \in P$. Since the $u_i$'s lie in every primitive ideal different from $P$, we conclude that $xt = x(u_1 + \ldots + u_r) = x_1u_1 + \ldots + x_ru_r$. Thus we have proved that $\mathcal{A}u_1 + \ldots + \mathcal{A}u_r \subseteq \mathcal{A}t$. The reverse inclusion, $\mathcal{A}t \subseteq \mathcal{A}u_1 + \ldots + \mathcal{A}u_r$, is trivial.

(B)$\Rightarrow$(C). Assuming (B) we see that every element in $\mathcal{A}a$ has rank at most $n$. Therefore, $\sharp\sigma(xa) \setminus \{0\} \leq n$, $x \in \mathcal{A}$, will be proved by showing that the number of nonzero points in the spectrum of every element in $\operatorname{soc}(\mathcal{A})$ cannot exceed its rank (we use $\sharp F$ to denote the cardinality of the set $F$).

Pick $s \in \operatorname{soc}(\mathcal{A})$ and denote its rank by $m$. Let $s_1, \ldots, s_k$ and $P_1, \ldots, P_k \in \Pi(\mathcal{A})$ satisfy the assertions of Lemma 2.6. By Theorem 2.2 we have

$$\begin{aligned} \sigma(s) \setminus \{0\} &= \Big(\bigcup_{P \in \Pi(\mathcal{A})} \sigma(s + P)\Big) \setminus \{0\} = \bigcup_{P \in \Pi(\mathcal{A})} \sigma(s + P) \setminus \{0\} \\ &= \sigma(s + P_1) \setminus \{0\} \cup \ldots \cup \sigma(s + P_k) \setminus \{0\} \\ &= \sigma(\pi_1(s)) \setminus \{0\} \cup \ldots \cup \sigma(\pi_k(s)) \setminus \{0\}, \end{aligned}$$

and hence

$$\sharp\sigma(s) \setminus \{0\} \leq \sharp\sigma(\pi_1(s)) \setminus \{0\} + \ldots + \sharp\sigma(\pi_k(s)) \setminus \{0\}.$$

Since $\pi_i(s)$ is an operator of rank $r_i$, it has at most $r_i$ nonzero points in the spectrum. Therefore,

$$\sharp\sigma(s) \setminus \{0\} \leq r_1 + \ldots + r_k = m,$$

which is our desired conclusion.

Next, we have to find an element $x_0 \in \mathcal{A}$ such that $\sharp\sigma(x_0a) \setminus \{0\} = n$. Again we apply Lemma 2.6 to $s = a$, and, as above, we keep the notation of the lemma with $a_i$ instead of $s_i$, and, to avoid confusion, $r_i'$ instead of $r_i$. Using Theorem 2.1 it is easy to see that there is $x_0 \in \mathcal{A}$ such that $\pi_1(x_0)\pi_1(a) = \pi_1(x_0a)$ has eigenvalues $1, \ldots, r_1'$, $\pi_2(x_0)\pi_2(a) = \pi_2(x_0a)$ has eigenvalues $r_1' + 1, \ldots, r_1' + r_2'$ etc. Finally, $\pi_k(x_0a)$ has eigenvalues $r_1' + \ldots + r_{k-1}' + 1, \ldots, r_1' + \ldots + r_k'$. Therefore, $\{1, \ldots, r_1' + \ldots + r_k'\} \subseteq \sigma(x_0a)$. This means that the number of nonzero points in $\sigma(x_0a)$ is not smaller than the rank of $a$. As $\mathcal{A}a$ is a sum of $n$ minimal left ideals, and $a \in \mathcal{A}a$, the rank of $a$ is at most $n$. However, it cannot be less than $n$. Namely, we already know that (A) implies (B); therefore, if the rank of $a$ were smaller than $n$, then the left ideal $\mathcal{A}a$ would be equal to a sum of less than $n$ minimal left ideals, contrary to assumption. Thus, the rank of $a$ is $n$, and so $\sharp\sigma(x_0a)\setminus\{0\} \geq n$. On the other hand, $\sharp\sigma(xa)\setminus\{0\} \leq n$ holds for every $x \in \mathcal{A}$. Hence $\sharp\sigma(x_0a)\setminus\{0\} = n$.

(C)$\Leftrightarrow$(D). As already mentioned, this equivalence was proved in [3].

(C)$\Rightarrow$(E). Let $a$ satisfy (C). We first prove that there is at most $n$ primitive ideals not containing $a$. Suppose that, on the contrary, $P_1, \ldots, P_{n+1}$ are primitive ideals none of which contains $a$. For each $i = 1, \ldots, n+1$, let $\pi_i$ be an irreducible representation on a Banach space $X_i$ with $\operatorname{Ker}\pi_i = P_i$. As $\pi_i(a) \neq 0$, there exists $\xi_i \in X_i$ such that $\pi_i(a)\xi_i \neq 0$. According to Theorem 2.1 there is $x \in \mathcal{A}$ satisfying $\pi_i(x)\pi_i(a)\xi_i = i\xi_i$, $i = 1, \ldots, n+1$. Thus, $i \in \sigma(\pi_i(xa)) \subseteq \sigma(xa)$. But then $\{1, \ldots, n+1\} \subseteq \sigma(xa)$, contrary to assumption.

Thus, we have proved (E.1), that is, there exist $k \leq n$ primitive ideals $P_1, \ldots, P_k$ such that $a$ lies in every primitive ideal different from them. Let $\pi_i$ be a continuous irreducible representation such that $\operatorname{Ker}\pi_i = P_i$, $i = 1, \ldots, k$. As $a \in \operatorname{soc}(\mathcal{A})$ by Theorem 2.3, $\pi_i(a)$ are finite rank operators. Set $r_i = \operatorname{rank}\pi_i(a)$. Arguing as in the proof of implication (B)$\Rightarrow$(C) we see that there is $y \in \mathcal{A}$ such that $\{1, \ldots, r_1 + \ldots + r_k\} \subseteq \sigma(ya)$. Therefore, $r_1 + \ldots + r_k \leq \sharp\sigma(ya)\setminus\{0\} \leq n$. We have to prove that this inequality is actually an equality.

Using Theorem 2.2 we have

$$\begin{aligned}\sigma(x_0a)\setminus\{0\} &= \Big(\bigcup_{P\in\Pi(\mathcal{A})} \sigma(x_0a+P)\Big)\setminus\{0\} = \bigcup_{P\in\Pi(\mathcal{A})} \sigma(x_0a+P)\setminus\{0\}\\ &= \sigma(x_0a+P_1)\setminus\{0\} \cup \ldots \cup \sigma(x_0a+P_k)\setminus\{0\}\\ &= \sigma(\pi_1(x_0a))\setminus\{0\} \cup \ldots \cup \sigma(\pi_k(x_0a))\setminus\{0\},\end{aligned}$$

and hence

$$n = \sharp\sigma(x_0a)\setminus\{0\} \leq \sharp\sigma(\pi_1(x_0a))\setminus\{0\} + \ldots + \sharp\sigma(\pi_k(x_0a))\setminus\{0\}.$$

Since $\pi_i(a)$ is an operator of rank $r_i$, $\pi_i(x_0a)$ has rank at most $r_i$. Consequently, $\sharp\sigma(\pi_i(x_0a))\setminus\{0\} \leq r_i$, and so $n \leq r_1 + \ldots + r_k$. Hence $n = r_1 + \ldots + r_k$.

(E)$\Rightarrow$(F). Suppose $a$ satisfies (E). Using Theorem 2.1 we see that for each $i = 1, \ldots, k$ there exists $x_i \in \mathcal{A}$ such that

$$\pi_i(x_i)\pi_i(a) = \pi_i(a), \quad \pi_j(x_i)\pi_j(a) = 0, \quad i \neq j.$$

Set $a_i = x_ia$, $i = 1, \ldots, k$. Thus, $\pi_i(a_i) = \pi_i(a)$ and $\pi_j(a_i) = 0$, $i \neq j$, so that $a - a_i \in P_i$ and $a_i$ lies in every primitive ideal except $P_i$. It follows at once that $a - a_1 - \ldots - a_k$ lies in every primitive ideal of $\mathcal{A}$, and so (F.1) holds. Theorem 2.3 tells us that $a$ belongs to $\operatorname{soc}(\mathcal{A})$. Therefore, Proposition 2.4 implies that (F.2) is true. If $i \neq j$ then $a_i \in P_j$ and $a_j \in P_i$, which further implies that $a_i\mathcal{A}a_j$ is contained in every primitive ideal of $\mathcal{A}$. This proves (F.3).

Set $r_i = \operatorname{rank}\pi_i(a)$. Lemma 2.7 states that the algebra $\pi_i(a)\pi_i(\mathcal{A})\pi_i(a)$ is isomorphic to $M_{r_i,n_i}$ for some $n_i$ such that $r_i \leq n_i \leq 2r_i$. On the other hand, $\pi_i(a)\pi_i(\mathcal{A})\pi_i(a)$ is isomorphic to $a_i\mathcal{A}a_i$. An isomorphism is exactly the restriction of $\pi_i$ to $a_i\mathcal{A}a_i$—it is certainly a homomorphism from one algebra onto the other, but it is also one-to-one. Namely, if $\pi_i(a_ixa_i) = 0$ for some $x \in \mathcal{A}$, then $a_ixa_i$ lies in $P_i$, which clearly yields $a_ixa_i = 0$. Thus, (F.4) is true. Of course, (F.5) holds by assumption.

It remains to show that $a_1, \ldots, a_k$ are unique nonzero elements in $\operatorname{soc}(\mathcal{A})$ satisfying (F.1)–(F.3). First of all, they are indeed nonzero elements lying in $\operatorname{soc}(\mathcal{A})$ as can be seen using (F.4) and Theorem 2.3. Now suppose that $a = b_1 + \ldots + b_l$ where the $b_i$'s are nonzero indecomposable elements in $\operatorname{soc}(\mathcal{A})$ satisfying $b_i\mathcal{A}b_j = 0$, $i \neq j$. According to Proposition 2.4, for every $b_i$ there exists exactly one $Q_i \in \Pi(\mathcal{A})$ not containing $b_i$. Observe that $Q_i \neq Q_j$ whenever $i \neq j$ for $b_i\mathcal{A}b_j = 0 \subseteq Q_i$ yields $b_j \in Q_i$. Note also that $a - b_i \in Q_i$, $i = 1, \ldots, l$.

Since $b_1 \notin Q_1$, we also have $a_1 + \ldots + a_k = a \notin Q_1$. But then at least one of the $a_i$'s, say $a_{i_1}$, does not belong to $Q_1$. For any $j \neq i_1$ we have $a_{i_1}\mathcal{A}a_j = 0 \subseteq Q_1$, which gives $a_j \in Q_1$. Hence $b_1 + Q_1 = a + Q_1 = a_{i_1} + Q_1$, that is, $b_1 - a_{i_1} \in Q_1$. As both $b_1$ and $a_{i_1}$ are indecomposable, they lie in every primitive ideal except $Q_1$. But then $b_1 = a_{i_1}$. Similarly, for each $j = 1, \ldots, l$ there is $i_j \leq k$ such that $b_j = a_{i_j}$. Of course, $j_1 \neq j_2$ yields $i_{j_1} \neq i_{j_2}$ for otherwise $b_{j_1}$ would be equal to $b_{j_2}$—this is impossible for $b_{j_1} \neq 0$ and $b_{j_1}\mathcal{A}b_{j_2} = 0$. It follows that $l \leq k$. However, as the $b_i$'s and the $a_i$'s appear symmetrically, we must have $k = l$. Therefore, $\{b_1, \ldots, b_l\} = \{a_1, \ldots, a_k\}$.

(F)$\Rightarrow$(A). As already observed, (F) implies that $a_i \in \operatorname{soc}(\mathcal{A})$ and so $a \in \operatorname{soc}(\mathcal{A})$. In other words, $a$ has a finite rank, say $m$. We already know that (A) implies (F). Therefore, there exist indecomposable elements $b_1, \ldots, b_l \in \mathcal{A}$ whose sum is $a$ and such that $b_i\mathcal{A}b_j = 0$, $i \neq j$, $b_i\mathcal{A}b_i \cong M_{p_i,m_i}$ for some $p_i, m_i \in \mathbb{N}$, $m = p_1 + \ldots + p_k$. However, as the elements satisfying (F.1)–(F.3) are unique, it follows that $\{b_1, \ldots, b_l\} = \{a_1, \ldots, a_k\}$. In particular, this yields $m = p_1 + \ldots + p_l = r_1 + \ldots + r_k = n$. The proof of the theorem is complete.

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Matej Brešar
University of Maribor
PF, Koroška 160
2000 Maribor, Slovenia
E-mail: bresar@uni-mb.si

Peter Šemrl
University of Maribor
SF, Smetanova 17
2000 Maribor, Slovenia
E-mail: peter.semrl@uni-mb.si

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