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Analyticity for some degenerate one-dimensional evolution equations

by

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Abstract. We study the analyticity of the semigroups generated by some degenerate second order differential operators in the space $C([\alpha,\beta])$, where $[\alpha,\beta]$ is a bounded real interval. The asymptotic behaviour and regularity with respect to the space variable are also investigated.

1. Introduction. In this paper we prove the analyticity of the semigroups generated by differential operators of the form

$$A_1 = m(x)[(x-\alpha)(\beta-x)D^2 + b(x)D]$$

or

$$A_2 = m(x)\left[D^2 + \frac{b(x)}{(x-\alpha)(\beta-x)}D\right],$$

where $D = d/dx$, in the space $C([\alpha,\beta])$, with suitable boundary conditions. The functions m and b are real-valued, continuous on the compact interval $[\alpha,\beta]$ and m is strictly positive; moreover, we assume that b satisfies a Hölder condition at the endpoints α and β.

The study of degenerate parabolic problems like

$$\begin{cases} du/dt = Bu, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where

$$B = a(x)D^2 + b(x)D, \quad x \in I,$$

and I is a real interval, already started in the fifties with the papers by Feller [10] and [11], motivated by some one-dimensional diffusion problems; the subsequent work of Clément and Timmermans (see [7] and [15]) clarified which (necessary and sufficient) conditions on the coefficients a and b guarantee the existence of the semigroup generated by $(B, D(B))$ if $D(B)$

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is defined by

$$D(B) = \{u \in C(\bar I) \cap C^2(I) : \lim_{x\to\partial \bar I} Bu(x) = 0\}$$

or by

$$D(B) = \{u \in C(\bar I) \cap C^2(I) : \lim_{x\to\partial \bar I} Bu(x) \text{ exists}\}.$$

Specializing the results of [7] and [15] to our case and using the fact that b is Hölder continuous at α and β we see that A_i, $i = 1,2$, generates a C_0-semigroup on $C([\alpha,\beta])$ if $D(A_i)$ is defined as follows: $u \in D(A_i) \Leftrightarrow u \in C([\alpha,\beta]) \cap C^2(]\alpha,\beta[)$ and u satisfies the following boundary conditions:

$$(T_\alpha) \qquad \lim_{x\to\alpha} A_i u(x) \in \mathbb{C} \text{ exists} \quad \text{if } \frac{b(\alpha)}{\beta-\alpha} \ge 1,$$

$$(V_\alpha) \qquad \lim_{x\to\alpha} A_i u(x) = 0 \quad \text{if } \frac{b(\alpha)}{\beta-\alpha} < 1,$$

$$(T_\beta) \qquad \lim_{x\to\beta} A_i u(x) \in \mathbb{C} \text{ exists} \quad \text{if } \frac{b(\beta)}{\beta-\alpha} \le -1,$$

$$(V_\beta) \qquad \lim_{x\to\beta} A_i u(x) = 0 \quad \text{if } \frac{b(\beta)}{\beta-\alpha} > -1.$$

Conditions (V_α) and (V_β) are the classical Ventcel's boundary conditions at α and β respectively. In the terminology of Feller, we can impose (V_α) if α is not an entrance point while we can impose (T_α) if α is neither a regular nor an exit point, and similarly for the point β.

We shall not deal with degenerate evolution problems in several variables for which we refer to [14] and to the references quoted therein.

The regularity of the semigroup $(T(t))_{t\ge0}$ generated by $(B, D(B))$ is considered neither in Feller's original papers nor in the more recent work of Clément and Timmermans. The first results in this direction are contained in a paper by Brézis, Rosenkrantz and Singer (see [4]) where some differentiability properties of the semigroup generated by (1.1) on the half-line are stated in the special case $a(x) \equiv 1$ and $b(x) = c$ with $c > -1$. The analyticity of the above semigroup if $a(x)$ is bounded away from zero, $b(x) = c(x)$, where c is a bounded continuous function on $[0,\infty[$ satisfying $c(0) > -1$, has been proved by Angenent in [2] assuming Neumann boundary conditions at $x = 0$; his methods are the starting point for this paper. Other results have been obtained in [12], [3], [9], [8] and [6] with different techniques; in particular, the analyticity of $(T(t))_{t\ge0}$ is established in $C([\alpha,\beta])$ if a vanishes at α, β and $\sqrt{a} \in C^1([\alpha,\beta])$ (with the additional assumption that $b/\sqrt{a}$ is bounded, see [9] and also [8]). This fact forces a to have (at least) double

zeros at the endpoints and excludes the operator A_1 which is the most important in applications and, above all, is the most natural in the class of one-dimensional degenerate differential operators.

Here we completely solve the problem of the analyticity of the solutions to the degenerate evolution problem (1.1) in the case of simple zeros. Our methods apply to the case of the half-line as well as to that of bounded interval. In particular, we prove analyticity results for Ventcel's boundary conditions and, as a special case, we obtain the analyticity of the semigroup generated by $x(1-x)D^2$ on $C([0,1])$, a problem which has been left open for a long time. Applications to higher order of degeneracy are given in [5].

The paper is organized as follows.

In Section 2 we consider the operator

$$L = m(x)\left[D^2 + \frac{b(x)}{x}D\right], \quad x \in]0,\infty[,$$

and we prove that it generates (with an appropriate domain) an analytic semigroup in $C([0,\infty])$. Angenent proved in [2] that if $b(0) > -1$ the operator L generates an analytic semigroup on $C([0,\infty])$ if its domain is defined by

$$D_1(L) = \{u \in C^2([0,\infty]) : u'(0) = 0\}. \tag{1.2}$$

In order to study the operators A_1 and A_2 we need results for L for all values of $b(0)$. In particular, the case $b(0) = -1$ is necessary to investigate $A_1 = x(1-x)D^2$.

If $b(0) \le -1$ we put

$$D_2(L) = \{u \in C([0,\infty]) \cap C^2(]0,\infty]) : \lim_{x\to0} Lu(x) = 0\}, \tag{1.3}$$

i.e. we impose the Ventcel boundary condition at $x = 0$, and we use Angenent's techniques to show that $(L, D_2(L))$ generates an analytic semigroup in $C([0,\infty])$. Observe that for $-1 < b(0) < 1$ the point $x = 0$ is regular and hence many other boundary conditions at $x = 0$ make L the generator of a semigroup (see [11]). We consider here only the case of Neumann conditions; other possible choices are considered in [5].

The analyticity of the semigroups generated by $(L, D_i(L))$, $i = 1,2$, will imply the same property for the semigroup generated by A_2 provided $D(A_2)$ is defined in a "natural" way using the boundary condition for L.

In Section 3 we reduce the operator A_1 to A_2 by a change of variable and deduce the analyticity of the generated semigroup. We shall see that our proof also works for operators like

$$G = m(x)[(x-\alpha)^r(\beta-x)^s D^2 + b(x)D]$$

where $0 < r, s \le 1$. The compactness of the semigroups, as well as their asymptotic behaviour, are studied in Section 4. In Section 5 we study the regularity of the solutions with respect to the space variable and use these results to show that the spectrum of A_i, $i = 1, 2$, is contained in $]-\infty, 0]$ and to deduce that the semigroups are bounded analytic of angle $\pi/2$.

Notation. All the function spaces considered in this paper consist of complex-valued functions.

$C^m(]\alpha,\infty])$ denotes the space of all m-times continuously differentiable functions u on $]\alpha,\infty[$ such that $\lim_{x\to\infty} u^{(k)}(x)$ exists and is finite for all $0 \le k \le m$. Of course $\lim_{x\to\infty} u^{(k)}(x) = 0$ if $k \ge 1$. $C^m([-\infty, b[)$ is defined in a similar way.

The symbol $\mathbf{1}$ denotes the constant function of value 1.

A *bounded analytic semigroup of angle* Θ $(0 < \Theta \le \pi/2)$ is an analytic semigroup defined in the sector

$$S_\Theta = \{z \in \mathbb{C} : |\arg z| < \Theta\}$$

that is bounded on closed subsectors of S_Θ.

An operator $(A, D(A))$ defined on a $C(K)$-space (K compact) satisfies the *positive minimum principle* if for every $0 \le f \in D(A)$ and $x \in K$ the equality $f(x) = 0$ implies $(Af)(x) \ge 0$. If $(A, D(A))$ is the generator of a semigroup $(T(t))_{t\ge0}$ on $C(K)$, then the semigroup is positive, i.e. each operator $T(t)$ is a positive operator on $C(K)$, if and only if $(A, D(A))$ satisfies the positive minimum principle (see [13, B-II, Theorem 1.6]).

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2. The operators L and A_2. We first consider

$$L = m(x)\left[D^2 + \frac{b(x)}{x}D\right]$$

and we suppose that b is continuous and bounded on $[0,\infty[$, and m is uniformly continuous and bounded on $[0,\infty[$ with $\inf_{x>0} m(x) > 0$; moreover, we assume that the estimate

$$|b(x) - b(0)| \le Cx^\sigma$$

holds in a neighbourhood of 0, for some positive constants σ and C. We also suppose b to be real-valued even though most of the results of this section are valid for complex functions. If $b(0) > -1$ Angenent proved in [2] that $(L, D_1(L))$ generates an analytic semigroup on $C([0,\infty])$ where $D_1(L)$ is defined by (1.2). His proof also shows that the generated semigroup is analytic of angle $\pi/2$. Actually, Angenent uses spaces of C^m functions bounded and uniformly continuous on $]0,\infty[$ but his methods work also in our case.

If $b(0) \le -1$ we define $D_2(L)$ by (1.3); in particular, a function $u \in D_2(L)$ satisfies condition (V_0).

We say that a function $u \in C([\alpha,\beta])$ satisfies *condition* (N_α) if

$$(N_\alpha) \qquad u \in C^2([\alpha,\alpha+\delta]) \quad \text{and} \quad u'(\alpha) = 0$$

for some positive δ.

Clearly every function in $D_1(L)$ satisfies (N_0).

By [15] and [7] we know that condition (T_0) can be imposed if and only if $b(0) \ge 1$ while condition (V_0) can be imposed if and only if $b(0) < 1$.

We put

$$f(x) = u''(x) + \frac{b(x)}{x}u'(x) \tag{2.1}$$

and

$$\gamma(x) = \exp\int_c^x \frac{b(t)}{t}\,dt, \tag{2.2}$$

where c is a fixed positive number, so that

$$Lu(x) = \frac{m(x)}{\gamma(x)}\frac{d}{dx}[\gamma(x)u'(x)] = m(x)f(x).$$

In the following propositions we discuss the boundary conditions (N_0), (V_0) and (T_0) for the operator L; because of their local character these results will also be valid for A_2. Proposition 2.1 will not be used in this paper and appears here for the sake of completeness.

PROPOSITION 2.1. *Let $b(0) \ge 1$ and $u \in C([0,\infty]) \cap C^2(]0,\infty[)$; then conditions (N_0) and (T_0) are equivalent for u.*

Proof. Suppose (T_0) holds. Then the function f defined by (2.1) is continuous in a (right) neighbourhood of 0. Integrating we obtain from (2.1) and (2.2),

$$u'(x) = \frac{1}{\gamma(x)}\int_0^x \gamma(t)f(t)\,dt + \frac{C}{\gamma(x)}. \tag{2.3}$$

Observe that $\gamma(x)$ behaves like $Kx^{b(0)}$, $K \ne 0$, for x near 0 and that

$$\lim_{x\to0^+}\frac{1}{\gamma(x)}\int_0^x \gamma(t)f(t)\,dt = 0.$$

Since u is continuous and $b(0) \ge 1$, the constant C must be 0 whence u' vanishes at $x = 0$ and (2.3) gives

$$u'(x) = \frac{f(0)}{b(0)+1}x + o(x), \qquad x \to 0.$$

Then $u'(x)/x$ is continuous at $x = 0$ and so is u'', i.e. u satisfies (N_0).

The converse is obvious and always true. ∎

If $-1 < b(0) < 1$, then (N_0) and (V_0) are independent. In fact, any C^2-function with $u'(0) = 0$, $u''(0) \neq 0$ satisfies (N_0) but not (V_0). Conversely, if $-1 < \lambda < 1$ and $L = D^2 + (\lambda/x)D$ then $u(x) = x^{1-\lambda}$ satisfies (V_0) but not (N_0). In the limit case $b(0) = -1$, clearly (N_0) implies (V_0) but not conversely. Consider, for example, u such that

$$u''(x) + \frac{b(x)}{x}u'(x) = \frac{1}{\log x}$$

for x near 0.

PROPOSITION 2.2. *Let $b(0) < -1$ and $u \in C([0,\infty]) \cap C^2(]0,\infty[)$; then u satisfies condition (V_0) if and only if $u \in C^2([0,\delta])$ for some positive δ and $u'(0) = u''(0) = 0$.*

Proof. Suppose that f vanishes at $x = 0$; then

$$u'(x) = \frac{1}{\gamma(x)}\int_c^x \gamma(t)f(t)\,dt + \frac{u'(c)}{\gamma(x)}.$$

Since $\gamma(x) \approx Cx^{b(0)}$ as $x \to 0$, for a suitable $C > 0$, it is not difficult to see that

$$\lim_{x\to 0}\frac{u'(x)}{x} = \frac{f(0)}{b(0)+1} = 0$$

from which it easily follows that $u \in C^2([0,\delta])$ for some positive δ and $u'(0) = u''(0) = 0$.

The converse implication is obvious. ∎

The case $b(0) = -1$ is slightly different.

PROPOSITION 2.3. *Let $b(0) = -1$ and $u \in C([0,\infty]) \cap C^2(]0,\infty[)$; then u satisfies condition (V_0) if and only if $u \in C^1([0,\delta])$ for some positive δ, $u'(0) = 0$ and*

$$\lim_{x\to 0}\frac{1}{x}\int_0^x [u''(x) - u''(t)]\,dt = 0.$$

Moreover, if u satisfies condition (V_0) then $u''(x) = o(\log x)$ and $u'(x)/x = o(\log x)$ as $x \to 0$.

Proof. As in the above proposition we obtain

$$u'(x) = \frac{1}{\gamma(x)}\int_c^x \gamma(t)f(t)\,dt + \frac{u'(c)}{\gamma(x)}$$

with γ defined by (2.2) and satisfying $\gamma(x) \approx C/x$ as $x \to 0$ for a suitable $C > 0$. Since $\lim_{x\to 0} f(x) = 0$ it follows easily that $u'(x)/x = o(\log x)$ and $u''(x) = o(\log x)$ as $x \to 0$. Writing $b(x) = -1 + O(x^\sigma)$ for $x \to 0$ with $\sigma > 0$,

we see that $u''(x) - (1/x)u'(x)$ tends to 0 as $x \to 0$. This immediately implies

$$\lim_{x\to 0}\frac{1}{x}\int_0^x [u''(x) - u''(t)]\,dt = 0.$$

Conversely, if $u'(0) = 0$ and this last condition is satisfied, then

$$u''(x) - \frac{1}{x}u'(x) \to 0 \quad \text{as } x \to 0,$$

that is, u satisfies Ventcel's condition at $x = 0$ with respect to the operator $D^2 - (1/x)D$. The argument above then shows that $u'(x)/x = o(\log x)$ as $x \to 0$, whence

$$f(x) = u''(x) - \frac{1}{x}u'(x) + O(x^\sigma \log x) \quad \text{as } x \to 0$$

and u satisfies (V_0). ∎

Observe that the function b enters the definition of $D_2(L)$ only through the value $b(0)$.

LEMMA 2.4. *The operator $(L, D_2(L))$ is closed, dissipative and satisfies the positive minimum principle.*

Proof. Let $u \in D_2(L)$ and $x \ge \delta > 0$; then

$$\left|\frac{b(x)}{x}u'(x)\right| \le \delta^{-1}\|b\|_\infty \sup_{x\ge\delta}|u'(x)|.$$

Hence for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for $x \ge \delta$,

$$\left|\frac{b(x)}{x}u'(x)\right| \le \varepsilon \sup_{x\ge\delta}|u''(x)| + C_\varepsilon \sup_{x\ge\delta}|u(x)|.$$

Since $\inf_{x>0} m(x) > 0$ it follows that

$$\sup_{x\ge\delta}|u''(x)| \le K_\delta[\sup_{x\ge\delta}|Lu(x)| + \sup_{x\ge\delta}|u(x)|].$$

If $(u_n) \subset D_2(L)$, $u_n \to u$, $L(u_n) \to v$ in $C([0,\infty])$, then the inequality above implies that u_n', u_n'' converge uniformly on $[\delta,\infty[$ for all $\delta > 0$, whence $u \in C([0,\infty]) \cap C^2(]0,\infty])$ and $Lu(x) = v(x)$ for $x > 0$. Since $v(0) = 0$ it follows that $u \in D_2(L)$ and $Lu = v$.

Let us show that L is dissipative, i.e. that

$$\lambda\|u\|_\infty \le \|\lambda u - Lu\|_\infty \tag{2.4}$$

for $\lambda > 0$.

If $\|u\|_\infty = u(x_0)$ with $0 \le x_0 < \infty$, then $Lu(x_0) \le 0$ and (2.4) holds, and similarly for $\|u\|_\infty = -u(x_0)$. If $\|u\|_\infty = \pm\lim_{x\to\infty} u(x)$ one concludes similarly since $\lim_{x\to\infty} Lu(x) = 0$.

The proof of the fact that L satisfies the positive minimum principle is similar. ∎

Remark 2.5. Arguing as in Propositions 2.2, 2.3 and Lemma 2.4 it follows that the map

$$u \mapsto \frac{f(x)}{x} u'(x)$$

is continuous from $D_2(L)$ to $C([0,\infty])$ if

(i) f is continuous and bounded, in the case $b(0) < -1$;

(ii) f is continuous, bounded and Hölder continuous at $x = 0$, with $f(0) = 0$, in the case $b(0) = -1$.

Our aim is now to prove that $(L, D_2(L))$ generates an analytic semigroup; we start with the case $b(x) \equiv b$ constant, with $b \le -1$, and $m(x) \equiv 1$.

For $\lambda \notin]-\infty, 0]$, $\lambda = \mu^2$ with $\operatorname{Re}\mu > 0$, consider the singular differential equation in $]0,\infty[$,

$$u'' + \frac{b}{x}u' = \mu^2 u.$$

It has in $]0,\infty[$ two linearly independent solutions $p(\mu x)$, $q(\mu x)$ given by

$$p(x) = x^\nu K_\nu(x), \qquad q(x) = x^\nu I_\nu(x),$$

where $\nu = (1-b)/2$ and $K_\nu(x)$, $I_\nu(x)$ are the Bessel functions of imaginary argument. Since $\operatorname{Re}\mu > 0$ it follows that

$$|q(\mu x)| \to \infty, \qquad p(\mu x) \to 0, \tag{2.5}$$

exponentially as $x \to \infty$.

For the function q the following expansion holds:

$$q(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+2\nu},$$

whence

$$q(\mu x) \approx C x^{1-b} \quad \text{as } x \to 0 \tag{2.6}$$

for a suitable $C \neq 0$. Since

$$K_\nu(x) \approx \frac{1}{2}\Gamma(\nu)\left(\frac{x}{2}\right)^{-\nu} \quad \text{as } x \to 0$$

for $\operatorname{Re}\mu > 0$, we obtain

$$\lim_{x\to 0} p(\mu x) = 2^{\nu-1}\Gamma(\nu), \tag{2.7}$$

i.e. $p(\mu x)$ is regular at $x = 0$ with a non-zero value (see [1] for all these properties of Bessel functions).

Lemma 2.6. *If b is a constant, $b \le -1$ and $m \equiv 1$ then the spectrum of $(L, D_2(L))$ is contained in $]-\infty, 0]$.*

Proof. Let $\lambda = \mu^2$, $\operatorname{Re}\mu > 0$ and consider the operator $L - \mu^2$ from $D_2(L)$ to $C([0,\infty])$. Clearly $q(\mu x)$ does not belong to $D_2(L)$ and the same holds for $p(\mu x)$ since $(L-\mu^2)p(\mu x) = 0$ and, by (2.7), $p(\mu x)$ does not vanish for $x = 0$. By (2.5) no linear combination of $p(\mu x)$ and $q(\mu x)$ belongs to $D_2(L)$, whence $L - \mu^2$ is injective.

Now we prove the surjectivity. Let $f \in C([0,\infty])$ and consider the equation

$$u'' + \frac{b}{x}u' - \mu^2 u = f. \tag{2.8}$$

If f is a constant c then $u = -c/\mu^2$ is a solution of (2.8) and clearly is in $D_2(L)$, so that we may suppose $f(0) = 0$. Let $v(x) = q(\mu x)$ and write $u = wv$. Inserting this in (2.8) we find that the new unknown w satisfies the differential equation

$$w'' + \left(2\frac{v'}{v} + \frac{b}{x}\right)w' = \frac{f}{v}$$

in a suitable interval $]0,\delta[$ where $v(x) \neq 0$, from which we get

$$w'(x) = \frac{1}{x^b v(x)^2}\int_0^x f(t)v(t)t^b\,dt,$$

the integral being convergent by (2.6). For the same reason we deduce $w'(x) = O(x^b)$ as $x \to 0$, whence

$$w(x) = \begin{cases} O(x^{b+1}) & \text{if } b < -1, \\ O(\log x) & \text{if } b = -1. \end{cases}$$

It follows that $u = wv$ is a solution of (2.8) in $]0,\delta[$ and that $\lim_{x\to0} u(x) = 0$. Since $Lu = f + \mu^2 u$ and $f(0) = 0$, u satisfies condition (V_0). Extending u to a maximal solution defined in the whole of $]0,\infty[$ we obtain a global solution u_1 of (2.8) satisfying (V_0) (but perhaps unbounded at ∞).

Consider now the operator

$$M = D^2 - \mu^2$$

with

$$D_r(M) = \{u \in C^2([r,\infty]) : u'(r) = 0\}.$$

Then M is invertible from $D_r(M)$ to $C([r,\infty])$ for every positive r and the norm of M^{-1} is independent of r. Since

$$L - \mu^2 = M + \frac{b}{x}D$$

and

$$\sup_{x\ge r}|u'(x)| \le 2[\sup_{x\ge r}|u''(x)| + \sup_{x\ge r}|u(x)|]$$

we obtain, for every $u \in D_r(M)$ and sufficiently large r,

$$\sup_{x\geq r}\left|\frac{b}{x}\frac{d}{dx}M^{-1}u(x)\right| \leq \frac{1}{2}\sup_{x\geq r}|u(x)|.$$

Then $L-\mu^2$ is invertible from $D_r(M)$ to $C([r,\infty])$ and we can find $u \in D_r(M)$ satisfying (2.8) in $[r,\infty]$, whence (by extending it to a maximal solution) a global solution u_2 which belongs to $C^2(]0,\infty[)$. Clearly $u_2(x) = u_1(x)+c_1p(\mu x)+c_2q(\mu x)$ for suitable constants c_1, c_2.

Consider the solution $u_0(x) = u_1(x)+c_2q(\mu x)$; then $u_0 \in C^2(]0,\infty])$ and u_0 satisfies (V_0) since u_1 does by construction and $q(\mu x)$ by (2.6) (recall that $(L-\mu^2)q(\mu x) = 0$).

Thus $u_0 \in D_2(L)$ and $(L-\mu^2)u_0 = f$. ∎

PROPOSITION 2.7. *If b is a constant, $b \leq -1$ and $m \equiv 1$, then $(L, D_2(L))$ generates a bounded analytic semigroup of angle $\pi/2$. Moreover, the semigroup is positive and contractive.*

Proof. Consider the group $(I_t)_{t>0}$ of isometries of $C([0,\infty])$ defined by

$$I_tf(x) = f(t^{1/2}x).$$

It is easily verified that $LI_t = tI_tL$, whence $I_t(D_2(L)) = D_2(L)$. If $\lambda = t\omega$ with $|\omega| = 1$ and $\omega \neq -1$, then

$$(\lambda-L)^{-1} = t^{-1}I_t(\omega-L)^{-1}I_{t^{-1}},$$

whence

$$\|(\lambda-L)^{-1}\| \leq |\lambda|^{-1}C(\omega)$$

where $C(\omega)$ depends continuously on ω. This estimate, together with the preceding lemma, shows that $(L, D_2(L))$ generates a bounded analytic semigroup of angle $\pi/2$. Positivity and contractivity are consequences of Lemma 2.4. ∎

Now we turn to the general case

$$L = m(x)\left[D^2+\frac{b(x)}{x}D\right], \quad b(0) \leq -1.$$

THEOREM 2.8. *The operator $(L, D_2(L))$, with $D_2(L)$ defined in* (1.3), *generates an analytic semigroup of angle $\pi/2$. The semigroup is positive and contractive.*

Proof. The proof can be achieved through the artifice of Korn, i.e. the constant coefficient case (dealt with in Proposition 2.7) and a partition of unity argument suitably adapted in order to take care of the degeneracy at $x=0$. In fact, observe that the continuity of the map

$$u \mapsto \frac{b(x)-b(0)}{x}u'(x)$$

from the domain of L into $C([0,\infty])$, guaranteed by Propositions 2.2 and 2.3 (see also Remark 2.5), allows us to approximate the operator L in a neighbourhood of the singularity $x=0$ with the constant coefficients operator

$$L_0 = m(0)\left[D^2+\frac{b(0)}{x}D\right].$$

We omit further details and refer to [2] where similar arguments are discussed more extensively. An alternative approach is presented in [5]. ∎

Remark 2.9. Observe that the hypothesis of b being Hölder continuous at $x=0$ is necessary only for the case $b(0) = -1$ (see Remark 2.5) so that the above theorem holds for $b(0) < -1$ if the coefficients are merely continuous.

We use the result of this section and of [2] about the operator L to study the operator

$$A_2 = m(x)\left[D^2+\frac{b(x)}{(x-\alpha)(\beta-x)}D\right]$$

in the bounded interval $[\alpha,\beta]$. We suppose m to be strictly positive and continuous, b continuous on $[\alpha,\beta]$ and Hölder continuous at the endpoints.

We define the domain of A_2 in the following way: $u \in D(A_2) \Leftrightarrow u \in C([\alpha,\beta]) \cap C^2(]\alpha,\beta[)$ and u satisfies the following boundary conditions:

$$\begin{aligned} &(N_\alpha) \quad \text{if } \frac{b(\alpha)}{\beta-\alpha} > -1, \qquad (V_\alpha) \quad \text{if } \frac{b(\alpha)}{\beta-\alpha} \leq -1,\\ &(N_\beta) \quad \text{if } \frac{b(\beta)}{\beta-\alpha} < 1, \qquad (V_\beta) \quad \text{if } \frac{b(\beta)}{\beta-\alpha} \geq 1. \end{aligned} \tag{2.9}$$

Observe that Propositions 2.1–2.3 are of local character and apply to this situation with 0 replaced by α or β. In particular, we see that (N_α) is equivalent to (T_α) if $b(\alpha)/(\beta-\alpha) \geq 1$ and that (N_β) is equivalent to (T_β) if $b(\beta)/(\beta-\alpha) \leq -1$.

By Propositions 2.2 and 2.3,

$$u \in D(A_2) \Rightarrow u \in C^1([\alpha,\beta]) \text{ and } u'(\alpha) = u'(\beta) = 0;$$

moreover, provided we exclude the critical cases $b(\alpha)/(\beta-\alpha) = -1$ and $b(\beta)/(\beta-\alpha) = 1$, if u is in $D(A_2)$ then $u \in C^2([\alpha,\beta])$. In the critical case the integral condition of Proposition 2.3 holds instead of the continuity of u''. We do not specialize Propositions 2.2 and 2.3 to the operator A_2 but just point out that, once $b(\alpha)$ and $b(\beta)$ are known, the boundary conditions are completely explicit.

LEMMA 2.10. *The operator $(A_2, D(A_2))$ is dissipative and satisfies the positive minimum principle.*

Proof. Identical to a part of the proof of Lemma 2.4. ∎

We may now prove the main theorem of this section.

THEOREM 2.11. *The operator $(A_2, D(A_2))$ with $D(A_2)$ defined in* (2.9) *generates a bounded analytic semigroup of angle $\pi/2$ in $C([\alpha,\beta])$. The semigroup is positive and contractive for $t > 0$.*

Proof. Positivity and contractivity come from Lemma 2.10 once the existence of the semigroup has been established.

Let ϕ_0, ϕ_1 be cut-off functions such that $\phi_0^2+\phi_1^2 = 1$ on $[\alpha,\beta]$ and $\phi_0 = 1$, $\phi_0 = 0$ and $\phi_1 = 0$, $\phi_1 = 1$ in neighbourhoods of α and β respectively.

Let ψ_0, ψ_1 be other cut-off functions such that $\psi_i = 1$ on the support of ϕ_i, $i = 0,1$, $\psi_0 = 0$ in a neighbourhood of β, $\psi_1 = 0$ in a neighbourhood of α.

Extend m putting $m(x) = m(\alpha)$ for $x \le \alpha$ and $m(x) = m(\beta)$ for $x \ge \beta$ and consider, for $i = 0,1$, the differential operators

$$L_i = m(x)\left[D^2 + \frac{b\psi_i}{(x-\alpha)(\beta-x)}D\right]$$

where $D(L_0) \subset C([\alpha,\infty])$ and $D(L_1) \subset C([-\infty,\beta])$ are defined as follows:

$$u \in D(L_0) \Leftrightarrow u \in C([\alpha,\infty]) \cap C^2(]\alpha,\infty]) \text{ and } u\phi_0 \in D(A_2),$$
$$u \in D(L_1) \Leftrightarrow u \in C([-\infty,\beta]) \cap C^2([-\infty,\beta[) \text{ and } u\phi_1 \in D(A_2).$$

By [2, Theorem 5.1] and Theorem 2.8 above, L_0 and L_1 generate analytic semigroups of angle $\pi/2$. Hence if $0 < \Theta < \pi$ is a fixed angle, we can find positive constants C and R such that for $|\lambda| \ge R$ and $|\arg\lambda| < \Theta$ the resolvents $(\lambda - L_i)^{-1}$, $i = 0,1$, exist and satisfy the estimate

$$\|(\lambda - L_i)^{-1}\| \le C/|\lambda|.$$

Put, for $|\lambda| \ge R$ and $|\arg\lambda| < \Theta$,

$$S(\lambda) = \phi_0(\lambda - L_0)^{-1}\phi_0 + \phi_1(\lambda - L_1)^{-1}\phi_1.$$

Clearly, for these λ's,

$$S(\lambda) : C([\alpha,\beta]) \to D(A_2)$$

and satisfies $\|S(\lambda)\| \le K|\lambda|^{-1}$. Let us prove the equality

$$(\lambda - A_2)S(\lambda) = I - [A_2,\phi_0](\lambda - L_0)^{-1}\phi_0 - [A_2,\phi_1](\lambda - L_1)^{-1}\phi_1$$

where $[A_2,\phi_i]u = A_2(\phi_i u) - \phi_i(A_2 u)$. In fact, the choice of the cut-off functions and the definitions of the operators L_0, L_1 give $\phi_0 A_2 v = \phi_0 L_0 v$ for every $v \in C^2(]\alpha,\infty[)$ and $A_2(\phi_0 u) = L_0(\phi_0 u)$ for every $u \in D(A_2)$ whose support is contained in the support of ϕ_0. Similarly, $\phi_1 A_2 v = \phi_1 L_1 v$ for every $v \in C^2(]-\infty,\beta[)$ and $A_2(\phi_1 u) = L_1(\phi_1 u)$ for every $u \in D(A_2)$ with

support contained in the support of ϕ_1. Then for $v \in C([\alpha,\beta])$ we obtain

$$\begin{aligned}(\lambda - A_2)S(\lambda)v &= (\lambda - L_0)(\phi_0(\lambda - L_0)^{-1}\phi_0 v)\\ &\quad + (\lambda - L_1)(\phi_1(\lambda - L_1)^{-1}\phi_1 v)\\ &= v - [L_0,\phi_0](\lambda - L_0)^{-1}\phi_0 v - [L_1,\phi_1](\lambda - L_1)^{-1}\phi_1 v\\ &= v - [A_2,\phi_0](\lambda - L_0)^{-1}\phi_0 v - [A_2,\phi_1](\lambda - L_1)^{-1}\phi_1 v.\end{aligned}$$

Observe now that $[A_2,\phi_0]$ is a first order differential operator supported on $[\alpha+\delta,\beta-\delta]$ for some $\delta > 0$, hence without singularities. Since L_0 has no singularities in $[\alpha+\delta,\infty]$, it follows that

$$\sup_{x\ge\alpha+\delta}|u''(x)| \le C_1\big[\sup_{x\ge\alpha+\delta}|L_0 u(x)| + \sup_{x\ge\alpha+\delta}|u(x)|\big]$$

for every $u \in C^2([\alpha+\delta,\infty])$.

The above considerations (and the analogous ones for L_1) imply that for $|\lambda| \ge R$ and $|\arg\lambda| < \Theta$,

$$\|[A_2,\phi_i](\lambda - L_i)^{-1}\| \le C_3/|\lambda|^{1/2},$$

whence for $|\lambda| \ge R_1 > R$ and $|\arg\lambda| < \Theta$,

$$\|(\lambda - A_2)S(\lambda) - I\| < 1/2$$

and $(\lambda - A_2)S(\lambda)$ has a bounded inverse Z of norm less than 2.

It follows that for $|\lambda| \ge R_1$ and $|\arg\lambda| < \Theta$, $S(\lambda)Z$ is a right inverse of $\lambda - A_2$ and the above argument shows that for $|\lambda| \ge R_1$ and $|\arg\lambda| < \Theta$,

$$\|(\lambda - A_2)^{-1}\| \le C_4/|\lambda| \tag{2.10}$$

provided that $\lambda - A_2$ is injective and, in particular, for $\lambda > 0$ since A_2 is dissipative.

Remembering that if λ belongs to the resolvent set of A_2 and

$$|\omega - \lambda| < \|(\lambda - A_2)^{-1}\|^{-1}$$

then ω belongs to the resolvent set of A_2, it is not difficult to deduce, using (2.10) and a simple argument based on connectedness, that

$$\varrho(A_2) \supset \{\lambda \in \mathbb{C} : |\arg\lambda| < \Theta \text{ and } |\lambda| > R_1\}.$$

This last fact, together with (2.10), shows that A_2 generates an analytic semigroup of angle $\pi/2$.

The proof that the semigroup is bounded analytic of the same angle will be given at the end of Section 5. ∎

3. The operator A_1. By a linear change of variable we may reduce the study of the operator A_1 to the interval $[0,1]$, i.e. we may suppose $x \in [0,1]$ and

$$A_1 = m(x)[x(1-x)D^2 + b(x)D].$$

As in the preceding section we suppose that $m, b \in C([0,1])$, m is strictly positive and b is Hölder continuous at $x = 0, 1$.

Put

$$x = \frac{1-\cos t}{2}, \quad 0 \le t \le \pi,$$

whence $t = \arccos(1-2x)$.

With this change of variable the operator A_1 transforms into

$$A_2 = m\left(\frac{1-\cos t}{2}\right)\left[D^2 + \frac{g(t)}{t(\pi - t)}D\right]$$

where

$$g(t) = [2b((1-\cos t)/2) - \cos t]\frac{t(\pi-t)}{\sin t}.$$

Observe that

$$(3.1) \qquad g(0) = \pi[2b(0)-1], \quad g(\pi) = \pi[2b(1)+1].$$

We define $D(A_2)$ according to (2.9) with $\alpha = 0$, $\beta = \pi$ and $b = g$. Consequently, we define $D(A_1)$ by

$$u \in D(A_1) \Leftrightarrow v \in D(A_2)$$

where v is defined by

$$v(t) = u((1-\cos t)/2).$$

Of course the reduction of A_1 to A_2 will be of some interest when we will clarify the boundary condition implicitly contained in $D(A_1)$. Clearly, if u belongs to $D(A_1)$ then u belongs to $C([0,1]) \cap C^2(]0,1[)$.

As in Section 2, we say that u satisfies (V_0), (V_1) if $\lim_{x\to 0} A_1u(x) = 0$ or $\lim_{x\to 1} A_1u(x) = 0$ respectively.

PROPOSITION 3.1. *Let $u \in C([0,1]) \cap C^2(]0,1[)$. Then $u \in D(A_1)$ if and only if the following conditions are satisfied:*

(i) *u satisfies (V_0)* *if $b(0) \le 0$,*

$u \in C^1([0,\delta])$ *and* $\lim_{x\to 0^+} xu''(x) = 0$ *if $b(0) > 0$,*

(ii) *u satisfies (V_1)* *if $b(1) \ge 0$,*

$u \in C^1([1-\delta,1])$ *and* $\lim_{x\to 1^-}(1-x)u''(x) = 0$ *if $b(1) < 0$.*

Proof. We prove only point (i). The proof of (ii) is similar.

If $b(0) \le 0$, then by (3.1) and (2.9), v must satisfy (V_0), i.e. u satisfies (V_0).

Similarly, if $b(0) > 0$ then v satisfies (N_0), that is, $v \in C^2([0,\eta])$ for some $\eta > 0$ and $v'(0) = 0$. Note that

$$(3.2) \qquad \frac{dv}{dt} = \frac{1}{2}\frac{du}{dx}\sin t, \qquad \frac{d^2v}{dt^2} = x(1-x)\frac{d^2u}{dx^2} + \left(\frac{1}{2} - x\right)\frac{du}{dx}.$$

If $u \in C^1([0,\delta])$ for some $\delta > 0$ and $\lim_{x\to 0^+} xu''(x) = 0$, then by (3.2), $v \in C^2([0,\eta])$ for some positive η and $v'(0) = 0$.

Conversely, suppose this last fact is true. Then by (3.2) again, we obtain

$$\lim_{x\to 0^+} \sqrt{x}\frac{du}{dx} = 0$$

and the function

$$f(x) = x\frac{d^2u}{dx^2} + \frac{1}{2}\frac{du}{dx}$$

is continuous on $[0,\delta]$ for some positive δ. Then

$$\frac{du}{dx} = \frac{1}{\sqrt{x}}\int_0^x \frac{f(s)}{\sqrt{s}}\,ds,$$

whence $\lim_{x\to 0^+} du/dx = 2f(0)$ and $u \in C^1([0,\delta])$.

By taking differences $\lim_{x\to 0^+} xu''(x)$ exists and must be 0 since u' is continuous at $x = 0$. ∎

PROPOSITION 3.2. *Let $u \in C([0,1]) \cap C^2(]0,1[)$. Then*

(i) *If $b(0) < 0$, then u satisfies (V_0) if and only if $u \in C^1([0,\delta])$, $u'(0) = 0$ and $\lim_{x\to 0^+} xu''(x) = 0$. If $b(1) > 0$, then u satisfies (V_1) if and only if $u \in C^1([1-\delta,1])$, $u'(1) = 0$ and $\lim_{x\to 1^-}(1-x)u''(x) = 0$.*

(ii) *If $b(0) = 0$, then u satisfies (V_0) if and only if $\lim_{x\to 0^+} xu''(x) = 0$. If $b(1) = 0$, then u satisfies (V_1) if and only if $\lim_{x\to 1^-}(1-x)u''(x) = 0$.*

Proof. The proof of (i) is identical to that of Proposition 3.1, with the use of Proposition 2.2.

For (ii) we use Proposition 2.3 and note that v satisfies (V_0) if and only if

$$\lim_{t\to 0^+}\left[\frac{d^2v}{dt^2} - (\cot t)\frac{dv}{dt}\right] = 0.$$

Since

$$\left[\frac{d^2v}{dt^2} - (\cot t)\frac{dv}{dt}\right] = x(1-x)\frac{d^2u}{dx^2}$$

we obtain (ii). ∎

We now state the main result of this section which is an immediate consequence of Theorem 2.11.

THEOREM 3.3. *The operator $(A_1, D(A_1))$ with $D(A_1)$ defined according to Proposition* 3.1 *generates a bounded analytic semigroup of angle $\pi/2$ on $C([0,1])$. The semigroup is positive and contractive.*

In particular, we obtain the analyticity of the semigroups generated by $m(x)[x(1-x)D^2]$ with Ventcel boundary conditions at 0 and 1, and by $D[m(x)x(1-x)D]$ with the degenerate boundary Neumann conditions $u \in C^1([0,1])$ and $\lim_{x\to 0,1} x(1-x)u''(x) = 0$ (see also [6]).

We examine briefly the case of low order zeros, i.e. we consider

$$G = m(x)[x^r(1-x)^s D^2 + b(x)D]$$

where, for simplicity, we suppose both $0 < r, s < 1$.

In this case we change variables putting

$$t = \int_0^x \frac{1}{\sqrt{p(z)}}\,dz$$

where $p(x) = x^r(1-x)^s$, $x \in [0,1]$, $t \in [0,\ell]$, $\ell = B(1-r/2, 1-s/2)$ (B denotes the Euler beta function).

In the new function $v(t) = u(x(t))$ the operator G has the form

$$A_2 = m(t)\left[D^2 + \frac{g(t)}{t(\ell-t)}D\right]$$

where

$$\frac{g(t)}{t(\ell-t)} = \frac{b(x) - (1/2)p'(x)}{\sqrt{p(x)}}.$$

It is not difficult to see that

$$\lim_{x\to 0+} g(t) = -\frac{r\ell}{2-r}, \qquad \lim_{x\to \ell-} g(t) = \frac{s\ell}{2-s}.$$

Since $-1 < -r(2-r)$ and $s(2-s) < 1$ the boundary conditions for A_2 are (N_0) and (N_ℓ). If we define

$$D(G) = \{u \in C([0,1]) : v \in D(A_2)\}$$

then we have the following proposition whose proof is similar to those of Propositions 3.1 and 3.2.

PROPOSITION 3.4. *Let $u \in C([0,1]) \cap C^2(]0,1[)$. Then $u \in D(G)$ if and only if $u \in C^1([0,1]) \cap C^2(]0,1[)$, $u'(0) = u'(1) = 0$ and both $\lim_{x\to 0^+} x^r u''(x)$ and $\lim_{x\to 1^-}(1-x)^s u''(x)$ exist and are finite.*

From Theorem 2.11 we obtain

THEOREM 3.5. *The operator $(G, D(G))$ generates a bounded analytic semigroup of angle $\pi/2$ on $C([0,1])$. The semigroup is positive and contractive.*

4. Compactness and asymptotic behaviour. In this section we show that the semigroups generated by A_1, A_2 and G are compact and we use this result to investigate their asymptotic behaviour. We recall that $D(A_1)$ is defined in Proposition 3.1, $D(A_2)$ is defined in (2.9) and $D(G)$ is defined in Proposition 3.4.

THEOREM 4.1. *The semigroups generated by $(A_1, D(A_1))$, $(A_2, D(A_2))$ and $(G, D(G))$ are compact.*

Proof. It is sufficient to prove the theorem only for $(A_2, D(A_2))$ since the other operators reduce to it via a change of variable (which is a linear isometry for the corresponding spaces of continuous functions).

Since the semigroup is analytic, hence norm-continuous, we only have to show the compactness of the resolvent of A_2.

But $D(A_2)$ embeds into $C^1([\alpha,\beta])$ (see Propositions 2.2 and 2.3 in the case of Ventcel conditions; for conditions (N_α) and (N_β) this is obvious) and $D(A_2)$, endowed with the graph norm, continuously embeds into $C([\alpha,\beta])$; by the closed graph theorem the inclusion of $D(A_2)$ into $C^1([\alpha,\beta])$ is continuous.

The compactness of the inclusion of $D(A_2)$ into $C([\alpha,\beta])$ and of the resolvent of A_2 then follow by the Ascoli–Arzelà Theorem. ■

We write (V_α, V_β), (V_α, V_β) and (N_α, N_β) to specify which boundary conditions define the domain of A_2; of course we tacitly assume in each case that the values $b(\alpha)$ and $b(\beta)$ are chosen according to (2.9).

We denote by $(T_i(t))_{t\geq 0}$, $i = 1,2,3$, the semigroups generated by $(A_2, D(A_2))$ in the following cases:

$$\begin{aligned} &(T_1(t))_{t\geq 0} && \text{in the case } (V_\alpha, V_\beta),\\ &(T_2(t))_{t\geq 0} && \text{in the case } (N_\alpha, V_\beta),\\ &(T_3(t))_{t\geq 0} && \text{in the case } (N_\alpha, N_\beta). \end{aligned}$$

Since, for each $i = 1,2,3$, $t > 0$, $T_i(t)$ is compact, positive and $T_i(t)\mathbf{1} = \mathbf{1}$, it follows from [13, B-IV, Theorem 2.5] that $(T_i(t))_{t\geq 0}$ converges in norm as $t \to \infty$ to a projection P_i satisfying $\mathrm{Im}(P_i) = \mathrm{Ker}(A_2)$.

The differential equation $A_2 u = 0$ has two linearly independent solutions u_1 and u_2 in $]\alpha,\beta[$ given by $u_1(x) = \mathbf{1}$ and $u_2(x) = \psi(x)$ where

$$\psi'(x) = W(x) = \exp\left[-\int_c^x \frac{b(t)}{(t-\alpha)(\beta-t)}\,dt\right]$$

and c is an arbitrary (but fixed) point in $]\alpha,\beta[$.

Observe that W is the function introduced in [7].

Clearly, W is summable near α (that is, u_2 is continuous at $x = \alpha$) if and only if $b(\alpha)/(\beta-\alpha) < 1$, and W is summable near β (that is, u_2 is continuous at $x = \beta$) if and only if $b(\beta)/(\beta-\alpha) > -1$.

In the following propositions we give explicit formulas for the limit projections.

PROPOSITION 4.2. *We have*

$$P_1 u = u(\alpha) + \frac{u(\beta)-u(\alpha)}{\psi(\beta)}\psi(x) \quad \text{for } u \in C([\alpha,\beta]),$$

with $\psi(x) = \int_\alpha^x W(t)\,dt$.

Proof. The functions $\mathbf{1}$ and ψ are in $D(A_2)$, whence

$$P_1 u = c_1 \mathbf{1} + c_2 \psi$$

for suitable constants c_1 and c_2.

Let $U(x,t) = T_1(t)u(x)$. Then the equality

$$\frac{\partial}{\partial t}U(x,t) = A_2 U(x,t)$$

holds pointwise for $t > 0$ and $x \in [\alpha,\beta]$. In particular, $(\partial/\partial t)U(x,t) = 0$ for $t > 0$, $x = \alpha$ and $x = \beta$. It follows that $U(\alpha,t) = u(\alpha)$ and $U(\beta,t) = u(\beta)$ from which we deduce $P_1 u(\alpha) = u(\alpha)$ and $P_1 u(\beta) = u(\beta)$, and the assertion follows. ∎

PROPOSITION 4.3. $P_2 u = u(\beta)$ *for* $u \in C([\alpha,\beta])$.

Proof. $\mathbf{1} \in D(A_2)$ but ψ does not. This is clear if $b(\alpha)/(\beta-\alpha) \ge 1$ since in that case ψ is not continuous at $x = \alpha$. If $-1 < b(\alpha)/(\beta-\alpha) < 1$ then ψ is continuous but does not satisfy (N_α). In fact, suppose by contradiction $\psi \in C^2([\alpha,\alpha+\delta])$ and $\psi'(\alpha) = 0$. Then we would have

$$\int_\alpha^c \frac{b(t)}{(t-\alpha)(\beta-t)}\,dt = \infty. \tag{4.1}$$

Since

$$W'(x) = -W(x)\frac{b(x)}{(x-\alpha)(\beta-x)}$$

and $W' = \psi''$ is bounded near α, we deduce

$$\left|\frac{b(x)}{(x-\alpha)(\beta-x)}\right| \le \frac{C}{|W(x)|}$$

near α and so (4.1) would be false.

Thus $\operatorname{Ker}(A_2)$ consists of the constant functions and the same argument of Proposition 4.2 proves the formula for P_2. ∎

PROPOSITION 4.4. *For* $u \in C([\alpha,\beta])$ *we have*

$$P_3 u = \frac{\int_\alpha^\beta u(x)\gamma(x)m(x)^{-1}\,dx}{\int_\alpha^\beta \gamma(x)m(x)^{-1}\,dx}$$

where $\gamma(x) = W(x)^{-1}$.

Proof. Since

$$\frac{b(\alpha)}{\beta-\alpha} > -1 \quad \text{and} \quad \frac{b(\beta)}{\beta-\alpha} < 1$$

it is easily seen that $\gamma \in L^1(\alpha,\beta)$.

Let $U(x,t) = T_3(t)u(x)$ and write

$$A_2 u(x) = \frac{m(x)}{\gamma(x)}\left[\frac{d}{dx}\gamma(x)u'(x)\right].$$

Since $\gamma(x) = o(x-\alpha)^{-1}$ as $x \to \alpha$ and $\gamma(x) = o(\beta-x)^{-1}$ as $x \to \beta$, it follows that

$$\lim_{x\to\alpha,\beta}\gamma(x)\frac{\partial}{\partial x}U(x,t) = 0$$

for $t \ge 0$. Then

$$\begin{aligned}\frac{d}{dt}\int_\alpha^\beta U(x,t)\frac{\gamma(x)}{m(x)}\,dx &= \int_\alpha^\beta A_2U(x,t)\frac{\gamma(x)}{m(x)}\,dx \\ &= \int_\alpha^\beta \frac{d}{dx}\left[\gamma(x)\frac{\partial}{\partial x}U(x,t)\right]dx = 0,\end{aligned}$$

whence

$$\int_\alpha^\beta U(x,t)\gamma(x)m(x)^{-1}\,dx = \int_\alpha^\beta u(x)\gamma(x)m(x)^{-1}\,dx$$

and, letting $t \to \infty$,

$$\int_\alpha^\beta P_3u(x)\gamma(x)m(x)^{-1}\,dx = \int_\alpha^\beta u(x)\gamma(x)m(x)^{-1}\,dx.$$

Since $\operatorname{Ker}(A_2)$ consists only of constant functions (as in the preceding proposition) the result follows. ∎

In the case of the operators A_1 and G we denote by $(S_i(t))_{t\ge0}$, $i = 1,2,3$, the generated semigroups according to the following list:

$$\begin{aligned}&(S_1(t))_{t\ge0} \quad \text{in the case } (V_0,V_1),\\ &(S_2(t))_{t\ge0} \quad \text{in the case } (N_0,V_1),\\ &(S_3(t))_{t\ge0} \quad \text{in the case } (N_0,N_1),\end{aligned}$$

and by Q_i the limit projections. Considering $0 < r, s \le 1$ we include A_1 in the definition of G.

Defining

$$W(x) = \exp\left[-\int_{1/2}^{x} \frac{b(t)}{p(t)}\,dt\right]$$

where $p(x) = x^r(1-x)^s$ and $\psi(x) = \int_0^x W(t)\,dt$ we deduce from Propositions 4.2–4.4, by a change of variable, the following formulas for the projections Q_i.

PROPOSITION 4.5. *Let $u \in C([0,1])$. Then*

(a) $Q_1u(x) = u(0) + \dfrac{u(1)-u(0)}{\psi(1)}\psi(x)$;

(b) $Q_2u(x) = u(1)$;

(c) $Q_3u(x) = \dfrac{\int_0^1 u(x)\tau(x)m(x)^{-1}\,dx}{\int_0^1 \tau(x)m(x)^{-1}\,dx}$,

where $\tau(x) = [p(x)W(x)]^{-1}$.

In particular, $Q_1u(x) = u(0) + [u(1)-u(0)]x$ for the operator

$$A_1 = m(x)[x(1-x)D^2]$$

with Ventcel boundary conditions at 0 and 1 (see [12]), and

$$Q_3u(x) = \int_0^1 u(x)\,dx \quad \text{for } A_1 = D[m(x)x(1-x)D]$$

with the degenerate homogeneous Neumann boundary conditions (see [6])

$$\lim_{x\to 0,1} x(1-x)u''(x) = 0 \quad \text{for } u \in C^1([0,1]) \cap C^2(]0,1[).$$

5. Regularity in the space variable. Let $(T_i(t))_{t\ge0}$, $i = 1,2$, be the semigroup generated by A_i; in this section we investigate, for fixed t, the regularity of the solution $T_i(t)u(x)$ in the variable x and use these results to show that the spectrum of A_i is contained in $]-\infty, 0]$ and that the generated semigroup is bounded analytic.

Observe that (keeping the notation of Section 3), since $v(t) = u((1-\cos t)/2)$ ($u \in D(A_1)$, $v \in D(A_2)$), every regularity result with respect to the space variable for A_1 will imply the analogous result for A_2. For this reason we formulate and prove the following results in the case of A_1. We also recall that $D(A_1)$ is defined according to Proposition 3.1.

In the following three propositions we suppose $m, b \in C^\infty([0,1])$ and, only for simplicity, we assume "similar" boundary conditions at $0, 1$.

PROPOSITION 5.1. *Let $b(0) > 0$, $b(1) < 0$ and $u \in D(A_1)$. If*

$$A_1u - \lambda u \in C^n([0,1])$$

then $u \in C^{n+1}([0,1])$. Consequently,

$$\bigcap_{n=1}^{\infty} D(A_1^n) = C^\infty([0,1])$$

and $U(x,t) = T_1(t)u(x) \in C^\infty([0,1]\times]0,\infty[)$ for every $u \in C([0,1])$.

Proof. Let $u \in D(A_1)$. By Proposition 3.1, $u \in C^1([0,1])$.

Suppose first $A_1u \in C^n([0,1])$. Then the function

$$f(x) = x(1-x)u''(x) + b(x)u'(x) \tag{5.1}$$

belongs to $C^n([0,1])$.

Consider only the interval $[0,1/2]$ and write

$$f(x) = \frac{x(1-x)}{\gamma(x)}\left[\frac{d}{dx}\gamma(x)u'(x)\right]$$

where

$$\gamma(x) = \exp\left[\int_{1/2}^{x} \frac{b(t)}{t(1-t)}\,dt\right] = x^{b(0)}\phi(x) \tag{5.2}$$

with $\phi \in C^\infty([0,1/2])$ and $\phi > 0$.

Then we find

$$u'(x) = \frac{1}{\phi(x)}\int_0^1 s^{b(0)-1}\eta(sx)f(sx)\,ds \tag{5.3}$$

with $\eta(x) = \phi(x)/(1-x) \in C^\infty([0,1/2])$.

Differentiating under the integral gives $u \in C^{n+1}([0,1/2])$ and the same argument in $[1/2,1]$ shows that u is $(n+1)$-times continuously differentiable on $[0,1]$.

If $A_1u - \lambda u \in C^n([0,1])$, one obtains immediately $A_1u \in C^1([0,1])$, whence $u, A_1u \in C^2([0,1])$ and, in n steps, $u \in C^{n+1}([0,1])$.

It follows, by induction on n, that $D(A_1^n) \subset C^n([0,1])$, whence

$$\bigcap_{n=1}^{\infty} D(A_1^n) = C^\infty([0,1]).$$

The C^∞-regularity of the solution $U(x,t)$ then follows from the analyticity of the semigroup. ∎

PROPOSITION 5.2. *Let $b(0) = b(1) = 0$ and $u \in D(A_1)$ be such that*

$$A_1u - \lambda u \in C^{n,\sigma}([0,1]), \quad 0 < \sigma < 1.$$

Then $u \in C^{n+1,\sigma}([0,1])$. *Consequently,*

$$\bigcap_{n=1}^{\infty} D(A_1^n) = C^\infty([0,1])$$

and $U(x,t) = T_1(t)u(x) \in C^\infty([0,1] \times]0,\infty[)$ *for every* $u \in C([0,1])$.

Proof. Let first $\lambda = 0$ and suppose that $A_1 u \in C^{n,\sigma}([0,1])$. Then the function f defined by (5.1) belongs to $C^{n,\sigma}([0,1])$ and vanishes at $0,1$.

Therefore the function $f(x)/(x(1-x))$ is in $C^{n-1,\sigma}([0,1])$ for $n \geq 1$, and is dominated by $K[x(1-x)]^{\sigma-1}$ if $n = 0$. Moreover, the function γ defined by (5.2) is infinitely differentiable and strictly positive on $[0,1]$. Since

$$u'(x) = \frac{1}{\gamma(x)}\left[\int_{1/2}^{x} \frac{\gamma(t)f(t)}{t(1-t)}\,dt + u'(1/2)\right]$$

we obtain $u' \in C^{n,\sigma}([0,1])$ and $u \in C^{n+1,\sigma}([0,1])$.

In the case $\lambda \neq 0$, let

$$g = A_1 u - \lambda u \in C^{n,\sigma}([0,1]).$$

Since A_1 annihilates the functions $\mathbf{1}$ and $w(x) = \int_0^x \gamma(t)^{-1}\,dt$ which are infinitely differentiable we may subtract from g the term

$$g(0) + \frac{g(1)-g(0)}{w(1)}w(x)$$

and assume that $g(0) = g(1) = 0$. Then $u(0) = u(1) = 0$ and $g(x)/(x(1-x))$ has the same properties as above.

Since $\lim_{x\to 0,1} x(1-x)u''(x) = 0$, u is Hölder continuous (of any exponent $\sigma < 1$) and the formula

$$u'(x) = \frac{1}{\gamma(x)}\left[\int_{1/2}^{x} \gamma(t)\frac{g(t)+\lambda u(t)}{m(t)t(1-t)}\,dt + u'(1/2)\right]$$

gives immediately $u' \in C^\sigma([0,1])$ and, by induction, $u \in C^{n,\sigma}([0,1])$.

It follows by induction that $D(A_1^n) \subset C^{n,\sigma}([0,1])$ for all n and the proof of the second statement follows exactly as in the previous proposition. ■

In the case $b(0) < 0$ and $b(1) > 0$ the above propositions are (in general) no longer valid. For example the function w defined in the proof of the proposition above is annihilated by A_1 but is not C^∞ near 0 unless $b(0)$ is a negative integer. Similar considerations hold for the point $x = 1$.

We denote by n_0 the smallest integer greater than $-b(0)$ and $b(1)$.

PROPOSITION 5.3. *Let* $b(0) < 0$ *and* $b(1) > 0$. *If* $u \in D(A_1^{n_0+k})$ *then*

$$u = h + g_0 + g_1$$

where

$$h \in C^{k+1}([0,1]), \quad h^{(m)}(x) = O[x^{1-b(0)-m}(1-x)^{1+b(1)-m}]$$

for $0 \leq m \leq k+1$ *and*

$$g_0(x) = x^{1-b(0)}\phi_1(x), \quad g_1(x) = (1-x)^{1+b(1)}\phi_2(x)$$

with ϕ_1 *and* ϕ_2 C^∞*-functions on* $[0,1]$.

Proof. Since $\mathrm{Ker}(A_1)$ is generated by the functions $\mathbf{1}$ and $\int_0^x \gamma(t)^{-1}\,dt$ which are of the form indicated in the statement of the proposition, we can assume $u(0) = u(1) = 0$.

First we show that if $u \in D(A_1^{n_0})$ then

$$u(x) = O[x^{1-b(0)}(1-x)^{1+b(1)}]. \tag{5.4}$$

Consider only the interval $[0,1/2]$ and choose $0 < \sigma < 1$ such that $n - r + \sigma$ is never an integer for $n \in \mathbb{N}$, where $r = -b(0)$.

Since $u \in D(A_1)$, we have $u \in C^1([0,1/2])$, $u'(0) = 0$ and $\lim_{x\to 0} xu''(x) = 0$, whence $u(x) = O(x^{1+\sigma})$.

Suppose, by induction, that

$$u \in D(A_1^n) \Rightarrow u(x) = O(x^{r+1} + x^{n+\sigma+1}).$$

If $u \in D(A_1^{n+1})$ let $f = A_1 u$. With the notation of Proposition 5.1, formula (5.3), we obtain

$$u'(x) = \frac{x^r}{\phi(x)}\left[\int_{1/2}^{x} \frac{\phi(t)f(t)}{t^{r+1}(1-t)}\,dt + u'(1/2)\right] \tag{5.5}$$

for $0 < x < 1/2$.

Since $f(x) = O(x^{r+1} + x^{n+\sigma+1})$, by the inductive hypothesis, we obtain by (5.5),

$$u'(x) = O(x^r + x^{n+\sigma+1}), \tag{5.6}$$

whence

$$u(x) = O(x^{r+1} + x^{n+\sigma+2}). \tag{5.7}$$

Using a similar argument near 1 and putting $n = n_0$ we obtain (5.4).

Now we prove the stated representation of u in $[0,1/2]$ (i.e. only with the functions h and g_0); similar computations give it in $[1/2,1]$ and a partition of unity argument yields the final formula in $[0,1]$.

The statement is true for $k = 0$ since $u \in C^1([0,1/2])$ and (5.6) and (5.7) hold.

Suppose $u \in D(A_1^{n_0+k+1})$. Then $f = A_1 u \in D(A_1^{n_0+k})$ and $f(x)/x^{r+1}$ is bounded, whence

$$(5.8)\qquad u'(x) = \frac{x^r}{\phi(x)}\left[\int_0^x \frac{\eta(t)f(t)}{t^{r+1}}\,dt + C\right]$$

for $0 < x < 1/2$, with $\eta(x) = \phi(x)/(1-x)$ and C a suitable positive constant.

Observe that

$$(5.9)\qquad \frac{x^r}{\phi(x)}\left[\int_0^x \frac{\eta(t)f(t)}{t^{r+1}}\,dt\right] = \frac{1}{\phi(x)}\left[\int_0^1 \frac{\eta(sx)f(sx)}{s^{r+1}}\,ds\right].$$

Hence, if $f(x) = h(x) + x^{r+1}\phi_1(x)$ with $h \in C^k([0,1/2])$, $h^{(m)}(x) = O(x^{r+1-m})$ for $0 \le m \le k$, and $\phi_1 \in C^\infty([0,1/2])$ by the inductive hypothesis, then we obtain the assertion from (5.8) and (5.9). ∎

We use the results and methods of this section to show that the spectrum of A_1 (hence of A_2) is contained in $]-\infty, 0]$.

Remark 5.4. We point out that, even without assuming b and m to be infinitely differentiable, any eigenfunction u of A_1 in Propositions 5.1 and 5.2 is in $C^1([0,1])$ and any eigenfunction u of A_1 in Proposition 5.3 satisfies the estimate $u'(x) = O[x^{-b(0)}(1-x)^{b(1)}]$; this is readily seen by inspection of the proofs.

THEOREM 5.5. *The spectrum of A_1 (hence of A_2) is contained in $]-\infty, 0]$.*

Proof. We distinguish three cases and, only for simplicity, we assume similar boundary conditions at the endpoints.

Suppose that $\lambda \ne 0$ and $u \in D(A_1)$ satisfies $A_1 u = \lambda u$. Using the function γ defined in (5.2) we may write

$$(5.10)\qquad \frac{d}{dx}[\gamma(x)u'(x)] = \lambda u(x)\frac{\gamma(x)}{m(x)x(1-x)}.$$

(i) $b(0) > 0$ and $b(1) < 0$. In this case $\gamma(x) \to 0$ as $x \to 0,1$ and $\gamma(x)/(x(1-x)) \in L^1(0,1)$. Hence, if we multiply both sides of (5.10) by $\overline{u}$ and integrate by parts, the boundary terms vanish and we get

$$(5.11)\qquad \int_0^1 \gamma(x)|u'(x)|^2\,dx = -\lambda\int_0^1 |u(x)|^2\frac{\gamma(x)}{m(x)x(1-x)}\,dx$$

and so $\lambda \in]-\infty, 0]$.

(ii) $b(0) = b(1) = 0$. The function γ is continuous and u vanishes at the endpoints. Since $u \in C^1([0,1])$, the function $u(x)/(x(1-x))$ is continuous and we may repeat the argument used in (i) to get (5.11) and deduce $\lambda \in]-\infty, 0]$.

(iii) $b(0) < 0$ and $b(1) > 0$. Since $u(0) = u(1) = 0$, we deduce from Remark 5.4 that $u(x) = O[x^{1-b(0)}(1-x)^{1+b(1)}]$, whence $\gamma(x)|u'(x)|^2$ and $|u(x)|^2\gamma(x)/(x(1-x))$ are integrable over $[0,1]$. Also, $\lim_{x\to 0,1}\gamma(x)u'(x)\overline{u}(x) = 0$, so that one gets (5.11) and concludes as before. ∎

We use the results of Section 4 and of this section to complete the proof of Theorem 2.11, showing that the generated semigroup is bounded analytic of angle $\pi/2$.

This will be a consequence of Theorems 4.1 and 5.5 and of the following proposition.

PROPOSITION 5.6. *Let $(A, D(A))$ be the generator of an analytic semigroup $T(z)$ of angle $\pi/2$; suppose that the spectrum of A is contained in $]-\infty, 0]$, that A has compact resolvent and that $T(t)$ is uniformly bounded for positive real t. Then $T(z)$ is bounded analytic of angle $\pi/2$.*

Proof. We only have to show that $T(z)$ is uniformly bounded on any closed subsector of the right half-plane. By the semigroup property this is equivalent to showing that $T(z)$ is bounded on every ray $\{z = re^{i\theta}\}$ with $r \ge 0$, and $-\pi/2 < \theta < \pi/2$ fixed.

Fix $-\pi/2 < \theta < \pi/2$ and set $S_\theta(t) = T(e^{i\theta}t)$. Then $(S_\theta(t))_{t\ge 0}$ is the C_0-semigroup generated by $e^{i\theta}A$ (with the same domain as A), whence $\|S_\theta(t)\|$ is bounded on bounded intervals and it is sufficient to show boundedness as $t \to \infty$.

Observe that $e^{i\theta}A$ has compact resolvent and spectrum contained in $\{z = re^{i\theta} : r \le 0\}$.

Let λ_0 be the eigenvalue of $e^{i\theta}A$ of maximal real part. Since $(S_\theta(t))_{t\ge 0}$ is norm-continuous, its growth bound coincides with $\operatorname{Re}\lambda_0$ (see [13, Remark 1.7], so that, if $\operatorname{Re}\lambda_0 < 0$ then

$$\|S_\theta(t)\| \to 0 \quad \text{as } t \to \infty.$$

Suppose now $\operatorname{Re}\lambda_0 = 0$, i.e. $\lambda_0 = 0$. Then $0 \in \sigma(A)$ and 0 is a simple pole of the resolvent of A since the semigroup $(T(t))_{t\ge 0}$ is uniformly bounded on the positive real line. Then 0 is also a simple pole of the resolvent of $e^{i\theta}A$ and, by [13, B-IV, Theorem 2.1], we conclude that $S_\theta(t)$ converges in norm to a projection as $t \to \infty$.

This concludes the proof. ∎

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Intrinsic characterizations of distribution spaces on domains

by

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Abstract. We give characterizations of Besov and Triebel–Lizorkin spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ in smooth domains $\Omega \subset \mathbb{R}^n$ via convolutions with compactly supported smooth kernels satisfying some moment conditions. The results for $s \in \mathbb{R}$, $0 < p, q \le \infty$ are stated in terms of the mixed norm of a certain maximal function of a distribution. For $s \in \mathbb{R}$, $1 \le p \le \infty$, $0 < q \le \infty$ characterizations without use of maximal functions are also obtained.

1. Introduction. The Besov and Triebel–Lizorkin spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p, q \le \infty$, are well-known scales of spaces of tempered distributions on $\mathbb{R}^n$, covering classical Hölder–Zygmund spaces, fractional Sobolev spaces, local Hardy spaces and their duals.

After being introduced in the 60s–70s in the pioneering papers by

O. V. Besov [Bes1,2] (B^s_{pq} spaces, $s > 0$, $1 \le p, q \le \infty$),
M. H. Taibleson [Tai] (B^s_{pq} spaces, $s \in \mathbb{R}$, $1 \le p, q \le \infty$),
P. I. Lizorkin [Liz1,2] (F^s_{pq} spaces, $s > 0$, $1 < p, q < \infty$),
H. Triebel [Tri1] (F^s_{pq} spaces, $s \in \mathbb{R}$, $1 < p, q < \infty$),
J. Peetre [P1,2] (extensions of B^s_{pq} and F^s_{pq} to all $0 < p, q \le \infty$),

these spaces were studied in detail. General references for the theory of B^s_{pq} and F^s_{pq} spaces are two monographs by H. Triebel [Tri2,3], and the fundamental paper by M. Frazier and B. Jawerth [FrJ].

In this paper B^s_{pq} and F^s_{pq} spaces on domains are studied. Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary. The natural way (also used here) to introduce distribution spaces $B^s_{pq}(\Omega), F^s_{pq}(\Omega) \subset \mathcal{D}'(\Omega)$ is to define them as restrictions of corresponding spaces from $\mathbb{R}^n$ to Ω. Then the problem of finding intrinsic characterizations of these spaces arises.

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