STUDIA MATHEMATICA 127 (1) (1998)

A universal modulus for normed spaces

by

CARLOS BENÍTEZ (Badajoz),
KRZYSZTOF PRZESŁAWSKI (Zielona Góra) and
DAVID YOST (Lyon)

Abstract. We define a handy new modulus for normed spaces. More precisely, given any normed space X, we define in a canonical way a function $\xi : [0,1) \to \mathbb{R}$ which depends only on the two-dimensional subspaces of X. We show that this function is strictly increasing and convex, and that its behaviour is intimately connected with the geometry of X. In particular, ξ tells us whether or not X is uniformly smooth, uniformly convex, uniformly non-square or an inner product space.

Introduction. In short, we define a new concept for normed spaces and prove some basic results concerning it. Why bother? This definition did not just come out of the blue, but arose naturally from studying Lipschitz continuous set-valued functions [24], a topic with ramifications in diverse areas of mathematics. Moreover, this modulus behaves quite well, compared with the moduli of convexity and smoothness. (For example, the modulus of convexity need not be convex [21], may well be constant in a neighbourhood of the origin [easy examples], and its value at a single point does not always characterize inner product spaces [1].) This motivates our belief that these ideas are worth pursuing in their own right.

It is high time to recall the modulus defined in [24]. Given a normed space X, one observes that for any $x, y \in X$ with $\|y\| < 1 < \|x\|$, there is a unique $z = z(x,y)$ in the line segment $[x,y]$ with $\|z\| = 1$. We put

$$\omega(x,y) = \frac{\|x - z(x,y)\|}{\|x\| - 1}$$

and define $\xi = \xi_X : [0,1) \to [1,\infty]$ by

$$\xi(\beta) = \sup\{\omega(x,y) : \|y\| \le \beta < 1 < \|x\|\}.$$

If $\xi(\beta) = 1$, for some non-zero value of β, it is easy to show that X must

1991 *Mathematics Subject Classification*: Primary 46B10, 46B20; Secondary 46B03, 46B08.

be the real line. Henceforth we assume that X has dimension at least two. Throughout we consider only real scalars; results for complex normed spaces can be obtained by applying the forgetful functor. In [24] we called ξ the *modulus of squareness*, because its extreme values characterize uniform non-squareness. It is easily shown [24, pp. 557–558] that for an inner product space, $\xi(\beta)=\xi_2(\beta)=1/\sqrt{1-\beta^2}$, and that for any normed space containing $\ell_1(2)$, $\xi(\beta)=\xi_1(\beta)=(1+\beta)/(1-\beta)$. It is not hard to show that $\xi\le\xi_1$ in any normed space; thus ξ always takes finite values. One proof of this inequality appears in [24, Proposition 2]; three more proofs crop up here.

We state our main results now, in the following omnibus theorem.

THEOREM O. *Let X be any normed space, ξ its modulus of squareness. Then*

(a) $\xi(\beta)=\sup\{\xi_M(\beta): M\subset X,\ \dim M=2\}$,
(b) *ξ is strictly increasing and convex*,
(c) *$\xi<\xi_1$ everywhere on $(0,1)$, unless X contains arbitrarily close copies of $\ell_1(2)$*,
(d) *$\xi'\le\xi_1'$ almost everywhere on $(0,1)$*,
(e) *$\xi>\xi_2$ everywhere on $(0,1)$, unless X is an inner product space*,
(f) *X is uniformly convex if and only if $\lim_{\beta\to1}(1-\beta)\xi(\beta)=0$*,
(g) *X is uniformly smooth if and only if $\xi'(0)=0$*,
(h) *the modulus of squareness of X^*, at β, is $1/\xi^{-1}(1/\beta)$*,
(i) *if $\xi(\beta)<1/(1-\beta)$ for some β, then X has uniformly normal structure.*

Statement (a) is fairly obvious. Statements (b) and (e) were announced in [24]. Statements (b) and (d) will be proved in §1, while (c), (f), (g), (h) and (i) will be proved in §2. The proof of (e) is rather long; it occupies all of §3.

We note in particular that the behaviour of ξ near one is related to convexity, and that the behaviour of ξ near zero is related to smoothness.

1. Essential properties of ξ.

We will need some familiar concepts from the geometry of normed spaces. For any $u,v\in X$, the one-sided directional derivatives of the norm (at u in the direction v) are defined by

$$N_+(u,v)=\lim_{\lambda\downarrow0}\frac{\|u+\lambda v\|-\|u\|}{\lambda},\qquad N_-(u,v)=\lim_{\lambda\uparrow0}\frac{\|u+\lambda v\|-\|u\|}{\lambda}.$$

By convexity, the quotient in these expressions is a monotonic function of λ, which guarantees that both limits exist, and that $N_-(u,v)\le N_+(u,v)$. Furthermore, for fixed u, $N_+(u,v)$ and $N_-(u,v)$ are sublinear functions of v. The collection of support functionals for u is defined as

$$D(u)=\{f\in X^*:\|f\|=1,\ f(u)=\|u\|\}.$$

Easy calculations show that $D(u)(v)=\{f(v): f\in D(u)\}$ coincides with the interval $[N_-(u,v),N_+(u,v)]$.

We say that u is *orthogonal* to v (in the sense of Birkhoff) provided $\|u\|\le\|u+\lambda v\|$ for all real scalars λ. This concept was first defined in [6] and studied later in [16]. We will say that u is *B-orthogonal* to v, in order to distinguish this relation from other orthogonality relations which will be defined later, but we write simply $u\perp v$. We denote by S the unit sphere of our normed space. The next lemma is elementary but useful.

LEMMA 1.1 [16]. *For any $\alpha\in\mathbb{R}$, the relationship $u\perp v-\alpha u$ is equivalent to the inequality $N_-(u,v)\le\alpha\|u\|\le N_+(u,v)$.*

The following alternative definition of the modulus of squareness will be used many times in the sequel.

LEMMA 1.2. *For any $\beta\in[0,1)$, we have*

$$\xi(\beta)=\sup_{w,z\in S}\frac{\|z-\beta w\|}{1-\beta N_-(z,w)}=\sup_{w,z\in S,\,N_-(z,w)\ge0}\frac{\|z-\beta w\|}{1-\beta N_-(z,w)}$$

and

$$\xi(\beta)=\sup_{w,z\in S}\frac{\|z-\beta w\|}{1-\beta N_+(z,w)}=\sup_{w,z\in S,\,f\in D(z)}\frac{\|z-\beta w\|}{1-\beta f(w)}.$$

Proof. Given any y,z with $\|y\|\le\beta$ and $\|z\|=1$, put $x_\lambda=z+\lambda(z-y)$. A moment's reflection shows that

$$\xi(\beta)=\sup_{\|y\|\le\beta,\,\|z\|=1,\,\lambda>0}\frac{\|x_\lambda-z\|}{\|x_\lambda\|-1}.$$

However,

$$\sup_{\lambda>0}\frac{\|x_\lambda-z\|}{\|x_\lambda\|-1}=\lim_{\lambda\downarrow0}\frac{\|x_\lambda-z\|}{\|z+\lambda(z-y)\|-\|z\|}=\frac{\|z-y\|}{N_+(z,z-y)}=\frac{\|z-y\|}{1-N_-(z,y)}.$$

This establishes the first equality.

To establish the second, suppose that $N_-(z,w)<0$, and put $y=\beta w$. Then there is a unique $\lambda>0$ with $\|z-\lambda y\|=\|z\|=1$ and so, by convexity, $\|z-(\lambda+1)y\|>\|z-y\|$ and $N_-(z-\lambda y,y)>0>N_-(z,y)$. Thus

$$\frac{\|(z-\lambda y)-y\|}{1-N_-(z-\lambda y,y)}>\frac{\|z-y\|}{1-N_-(z,y)}.$$

To deduce the second pair of equalities from the first, it suffices to show that, given any norm one vectors z,w, we can find $x\in S$ arbitrarily close to z with $N_-(x,w)\ge N_+(z,w)$. Put $x=\|z+\delta w\|^{-1}(z+\delta w)$, where δ is positive, but may be chosen arbitrarily small. For any $f\in D(x)$ and any $g\in D(z)$ we have

$$1+\delta f(w)\ge f(z+\delta w)=\|z+\delta w\|\ge g(z+\delta w)=1+\delta g(w).$$

Thus $f(w)\ge g(w)$, as required. ■

Lemma 1.2 clearly implies that $\xi \le \xi_1$ for any normed space. It also implies that the modulus of squareness of any finite-dimensional polyhedral space can be calculated exactly. See Example C at the end of §2.

LEMMA 1.3. *For $a, b, c, d > 0$ and $0 < t \le 1$, the inequality*

$$\frac{ta + (1-t)c}{tb + (1-t)d} \le t\frac{a}{b} + (1-t)\frac{c}{d}$$

is equivalent to $(bc - ad)(b - d)(1 - t) \ge 0$. In particular, it is true if $b \ge d$ and $a/b \le c/d$.

THEOREM 1.4. *The modulus of squarenesss ξ of any normed space is strictly increasing and convex, hence absolutely continuous and differentiable except perhaps at countably many points.*

Proof. Obviously, $0 < \beta < \gamma < 1 \Rightarrow \xi(\beta) \le \xi(\gamma)$. Since ξ is not constant in any neighborhood of 0, convexity will imply that ξ is strictly increasing.

To prove convexity, write $z_\beta = \|z - \beta w\|^{-1}(z - \beta w)$. First we show (for fixed unit vectors z and w) that the function $\beta \mapsto N_+(z, z_\beta)$ is decreasing. Given $0 < \beta_1 < \beta_2$, there are $\lambda \in (0,1)$ and $\gamma \ge 1$ such that $z_{\beta_1} = \gamma((1-\lambda)z_0 + \lambda z_{\beta_2})$. For any functional $f \in D(z)$, we have

$$f(z_{\beta_1}) = \gamma((1-\lambda) + \lambda f(z_{\beta_2})) \ge \gamma(1 - \lambda + \lambda) f(z_{\beta_2}) \ge f(z_{\beta_2}).$$

The inequality $N_+(z, z_{\beta_1}) \ge N_+(z, z_{\beta_2})$ follows by taking the supremum over all such f.

Now fix $0 < \beta_1 < \beta_2 < 1$, $0 < t < 1$ and $\varepsilon > 0$. It clearly suffices to show that $\xi(t\beta_1 + (1-t)\beta_2) - \varepsilon < t\xi(\beta_1) + (1-t)\xi(\beta_2)$. Choose w and z of norm one so that $N_-(z, w) \ge 0$ and

$$\xi(t\beta_1 + (1-t)\beta_2) - \varepsilon < \frac{\|z - (t\beta_1 + (1-t)\beta_2)w\|}{1 - (t\beta_1 + (1-t)\beta_2)N_-(z, w)}.$$

It is fairly clear that

$$\frac{\|z - \beta_1 w\|}{1 - \beta_1 N_-(z, w)} = \frac{1}{N_+(z, z_{\beta_1})} \le \frac{1}{N_+(z, z_{\beta_2})} = \frac{\|z - \beta_2 w\|}{1 - \beta_2 N_-(z, w)}$$

and that $1 - \beta_2 N_-(z, w) < 1 - \beta_1 N_-(z, w)$. Applying Lemma 1.3, we then have

$$\begin{aligned}\xi(t\beta_1 + (1-t)\beta_2) - \varepsilon &< \frac{t\|z - \beta_1 w\| + (1-t)\|z - \beta_2 w\|}{t(1 - \beta_1 N_-(z, w)) + (1-t)(1 - \beta_2 N_-(z, w))}\\ &\le \frac{t\|z - \beta_1 w\|}{1 - \beta_1 N_-(z, w)} + \frac{(1-t)\|z - \beta_2 w\|}{1 - \beta_2 N_-(z, w)}\\ &\le t\xi(\beta_1) + (1-t)\xi(\beta_2). \ \blacksquare\end{aligned}$$

We recall that the modulus of smoothness of a normed space is easily seen to be convex, hence absolutely continuous. A surprisingly simple example ([21] or [12, Example 5.8]) shows that the modulus of convexity need not be a convex function. Nevertheless it is continuous [13]; see also [28, Proposition 3.4] for a generalization of this. Absolute continuity of ξ also follows from the following precise estimate.

THEOREM 1.5. *The modulus of squareness ξ of any normed space satisfies the inequality $\xi(\gamma) - \xi(\beta) \le \xi_1(\gamma) - \xi_1(\beta)$ whenever $0 \le \beta < \gamma < 1$. Thus $\xi \le \xi_1$ and, at all points of differentiability, $\xi' \le \xi_1'$.*

Proof. It clearly suffices to show, for fixed x and y with $\|x\| > 1 = \|y\|$, that the function $\eta \mapsto \omega(x, \eta y)$ satisfies the same estimate. So let us consider $z = \alpha\gamma y + (1-\alpha)x$ and $z' = \alpha'\beta y + (1-\alpha')x$, where $\|z\| = \|z'\| = 1$ and $\alpha, \alpha' \in (0,1)$. Then

$$(\clubsuit) \quad \omega(x, \gamma y) - \omega(x, \beta y) = \alpha\frac{\|x - \gamma y\| - \|x - \beta y\|}{\|x\| - 1} + \frac{(\alpha - \alpha')\|x - \beta y\|}{\|x\| - 1}.$$

The first item on the right of ($\clubsuit$) does not exceed

$$\frac{\alpha(\gamma - \beta)}{\|x\| - 1} < \frac{\gamma - \beta}{1 - \gamma},$$

since $1 = \|z\| \le (1-\alpha)\|x\| + \alpha\gamma \le \|x\| - \alpha + \alpha\gamma$. The second item is a bit trickier to estimate. Let $f \in X^*$ be a support functional for z. Then $f(\gamma y) < 1$ and therefore $f(x) > 1$. Since $(1-\alpha)f(x) + \alpha f(\gamma y) = 1$ and $(1-\alpha')f(x) + \alpha' f(\beta y) \le 1$, we have

$$\alpha = \frac{f(x) - 1}{f(x) - f(\gamma y)} \quad \text{and} \quad \alpha' \ge \frac{f(x) - 1}{f(x) - f(\beta y)}.$$

This implies that the second term in ($\clubsuit$) is negative if $f(y) < 0$; otherwise it is dominated by

$$\begin{aligned}&\frac{\|x\| + \beta}{\|x\| - 1}(f(x) - 1)\frac{(\gamma - \beta)f(y)}{(f(x) - \beta f(y))(f(x) - \gamma f(y))}\\ &\quad\le \frac{f(x) + \beta}{f(x) - 1}(f(x) - 1)\frac{\gamma - \beta}{(f(x) - \beta)(f(x) - \gamma)}\\ &\quad= \frac{(f(x) + \beta)(\gamma - \beta)}{(f(x) - \beta)(f(x) - \gamma)} \le \frac{(1 + \beta)(\gamma - \beta)}{(1 - \beta)(1 - \gamma)}.\end{aligned}$$

Adding these two estimates completes the proof. ■

We remark that in every example we have calculated, ξ is actually analytic and logarithmically convex. It would be interesting to characterize those functions ξ which can be the modulus of squareness of some (uniformly non-square) normed space.

2. Connections with the geometry of X. For any normed space X, Theorem 1.5 ensures the existence of a unique β_0 for which $\xi(\beta) = \xi_1(\beta)$ for $\beta < \beta_0$ and $\xi(\beta) < \xi_1(\beta)$ for $\beta > \beta_0$. The next two results show that β_0 is always equal to zero or one, and provides some justification for the name "modulus of squareness": if $\xi(\beta) = \xi_1(\beta)$, even for one non-zero value of β, then the same equality holds for all values of β, and X contains subspaces arbitrarily close to $\ell_1(2)$. Recall that the *Banach–Mazur distance* between two normed spaces is the infimum, over all isomorphisms T between them, of $\|T\| \cdot \|T^{-1}\|$.

THEOREM 2.1. *Fix $\delta, \beta \in (0,1)$. If $\xi(\beta) > (1-\delta\beta)\xi_1(\beta)$, then X contains a two-dimensional subspace whose Banach–Mazur distance from $\ell_1(2)$ is less than $1/(1-(1+\beta)\delta)$.*

Proof. Obviously, there exist unit vectors z, w for which

$$\frac{\|z-\beta w\|}{1-\beta N_+(z,w)} > (1-\delta\beta)\frac{1+\beta}{1-\beta}.$$

Since $\|z-\beta w\| \le 1+\beta$ and $1/(1-\beta N_+(z,w)) \le 1/(1-\beta)$ we see at once that

$$\|z-\beta w\| \ge (1-\delta\beta)(1+\beta) \quad \text{and} \quad \frac{1}{1-\beta N_+(z,w)} \ge \frac{1-\delta\beta}{1-\beta}.$$

The latter inequality implies that $N_+(z,w) \ge 1-\delta$. This further implies that for all $\lambda, \mu > 0$, we have

$$\frac{\|z+(\mu/\lambda)w\|-1}{\mu/\lambda} \ge 1-\delta \quad \text{whence} \quad 1-\delta < \frac{\|\lambda z+\mu w\|}{\lambda+\mu} \le 1.$$

The former inequality says that $x = (z-\beta w)/((1-\delta\beta)(1+\beta))$ has norm at least one. Since x is a convex combination of z and $-w/(1-\delta-\delta\beta)$, it follows that all points between x and $-w/(1-\delta-\delta\beta)$ have norm at least one. Similarly all points between x and $z/(1-\delta\beta-\delta\beta^2)$ have norm at least one. A glance at a simple diagram implies that for all $\lambda, \mu > 0$, we have

$$1-\delta-\delta\beta \le \frac{\|\lambda z-\mu w\|}{\lambda+\mu} \le 1.$$

The estimate for the Banach–Mazur distance is now obvious. ∎

Recall that a normed space is said to be *uniformly non-square* if all of its two-dimensional subspaces have at least a certain distance from $\ell_1(2)$. Otherwise, i.e. if a normed space does contain arbitrarily close copies of $\ell_1(2)$, we will call it *nearly square*. Theorem 2.1 implies that if a normed space is uniformly non-square, then $\xi(\beta) < \xi_1(\beta)$ for each $\beta \in (0,1)$. We now show that the converse is true, i.e. if a normed space is close to $\ell_1(2)$ (with respect to the Banach–Mazur distance), then its modulus of squareness is close to ξ_1. This is equivalent to saying that, for each β, the real-valued map defined on the Banach–Mazur compactum by $X \mapsto \xi_X(\beta)$ is continuous at $\ell_1(2)$. In fact, it is continuous everywhere.

To prove this, we need the following very special case of the Bishop–Phelps Theorem. At the referee's suggestion, we include a simple proof. There is no novelty in our argument; it is only a specialization of Hiriart-Urruty's proof of Ekeland's variational principle [15], followed by routine applications of the separation theorem.

LEMMA 2.2. *Let X be a finite-dimensional normed space, $z \in X$, $f \in X^*$, $\varepsilon > 0$ with $\|z\| \le 1$, $f(z) > \|f\| - \varepsilon$. Then for any $\lambda > 0$, we can find $z' \in X$, $f' \in X^*$ with $\|z'\| \le 1$, $f'(z') = \|f'\|$, $\|z-z'\| \le \varepsilon/\lambda$ and $\|f-f'\| \le \lambda$.*

Proof. Assume that $\|f\| > \lambda$; otherwise we could take $f' = 0$ and $z' = z$. By compactness, the function $f(x) - \lambda\|x-z\|$ has a maximum on the unit ball of X, say at some point z'. Then, whenever $\|x\| \le 1$, we have

$$f(x) \le f(z') - \lambda\|z'-z\| + \lambda\|x-z\| \le f(z') + \lambda\|x-z'\|.$$

Putting $x = z$ gives us $f(z') - \lambda\|z'-z\| \ge f(z) \ge \|f\| - \varepsilon \ge f(z') - \varepsilon$ and so $\|z'-z\| \le \varepsilon/\lambda$. Writing $x = y + z'$ tells us that

$$f(y) \le \lambda\|y\| \quad \text{whenever} \quad \|y+z'\| \le 1.$$

The two open convex sets $\{y : \|z'+y\| < 1\}$ and $\{y : \lambda\|y\| < f(y)\}$ are thus disjoint, and can be separated by a linear functional $g \in X^*$. Since the latter set is closed under multiplication by positive scalars, we may as well suppose that $g(y) > 0$ whenever $\lambda\|y\| < f(y)$ and $g(y) \le 0$ whenever $\|z'+y\| \le 1$. The last condition means that $g(x) \le g(z')$ for all x with $\|x\| \le 1$, and so any positive scalar multiple of g attains its norm at z'.

The minimum value of any $y \in X = X^{**}$ on the ball $B(f,\lambda)$ is clearly $f(y) - \lambda\|y\|$. Thus, by the choice of g, there is no $y \in Y$ which is both strictly positive on $B(f,\lambda)$ and negative on $\mathbb{R}^+ g$. Again by the separation theorem, $B(f,\lambda) \cap \mathbb{R}^+ g$ is non-empty; any element f' therein will have the required properties. ∎

It may seem curious to have applied the separation theorem twice in this proof. However, this ensures that the proof in the 2-dimensional case requires only the separation theorem in two dimensions, which is much easier than the separation theorem in three dimensions. In particular, the following result does not require anything like the full strength of the Bishop–Phelps Theorem.

THEOREM 2.3. *Let X and Y be two isomorphic normed spaces whose Banach–Mazur distance is less than $1+2\delta^2$, where $\delta \le 1$. Then, for all β,*

$$|\xi_X(\beta) - \xi_Y(\beta)| \le \frac{2(\delta+\delta^2)}{(1-\beta)^2}.$$

Proof. Since the modulus of squareness of any normed space is the supremum of the moduli of squareness of its two-dimensional subspaces, we make the harmless assumption that X and Y are finite-dimensional. Our hypothesis implies that we may regard X and Y as the same vector space equipped with two equivalent norms, $\|\cdot\|$ and $|||\cdot|||$ respectively, whose ratio is between $1+\delta^2$ and $1/(1+\delta^2)$. By Lemma 1.2, we can choose $\|\cdot\|$-unit vectors z, w and $f \in D(z)$ so that $\|z-\beta w\|/(1-\beta f(w))$ is epsilonically close to $\xi_X(\beta)$. Clearly, $|||z/(1+\delta^2)||| \le 1$ and $f(z/(1+\delta^2)) > 1-\delta^2 \ge |||f||| - 2\delta^2$. Lemma 2.2 guarantees the existence of $z' \in X$ and $f' \in X^*$ with $|||z'||| = 1$, $|||f'||| = f'(z')$, $|||z' - z/(1+\delta^2)||| \le 2\delta$ and $|||f - f'||| \le \delta$. Put $w' = w/|||w|||$. Then

$$\begin{aligned}\|z-\beta w\| - |||z'-\beta w'||| &\le (1+\delta^2)|||z-\beta w||| - |||z'-\beta w'|||\\ &\le |||z-\beta w - (z'-\beta w')||| + \delta^2(1+\beta)(1+\delta^2)\\ &\le |||z-z'||| + \beta|||w-w'||| + \delta^2(1+\beta)(1+\delta^2)\\ &\le \left|\left|\left| z' - \frac{z}{1+\delta^2}\right|\right|\right| + \left|\left|\left|\left(\frac{1}{1+\delta^2}-1\right)z\right|\right|\right|\\ &\quad + \beta\big||||w|||-1\big| + \delta^2(1+\beta)(1+\delta^2)\\ &\le 2\delta + \delta^2(1+\beta)(2+\delta^2)\end{aligned}$$

and

$$\begin{aligned}|f(w)-f'(w')| &\le |f(w) - f(w/|||w|||)| + |(f-f')(w')|\\ &\le \big|1-1/|||w|||\big|\cdot|f(w)| + |||f-f'|||\cdot|||w'||| \le \delta^2+\delta.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{\|z-\beta w\|}{1-\beta f(w)} - \frac{|||z'-\beta w'|||}{1-\beta f'(w')}\\ &\quad\le \frac{\|z-\beta w\|}{1-\beta f(w)} - \frac{\|z-\beta w\| - (2\delta+\delta^2(1+\beta)(2+\delta^2))}{1-\beta f(w)+\beta(\delta^2+\delta)}\\ &\quad\le \frac{\beta(\delta^2+\delta)\|z-\beta w\| + (2\delta+\delta^2(1+\beta)(2+\delta^2))(1-\beta f(w))}{(1-\beta f(w))^2}\\ &\quad\le \frac{\beta(\delta^2+\delta)\xi_1(\beta) + (2\delta+\delta^2(1+\beta)(2+\delta^2))}{1-\beta f(w)}\\ &\quad\le \frac{(\beta+\beta^2)(\delta^2+\delta) + (2\delta+\delta^2(1+\beta)(2+\delta^2))(1-\beta)}{(1-\beta)^2}\\ &\quad= \frac{(\delta-\delta^4)\beta^2 + (\delta^2-\delta)\beta + (2\delta+\delta^2+\delta^4)}{(1-\beta)^2}\\ &\quad\le \frac{\max\{2\delta+2\delta^2, 2\delta+\delta^2+\delta^4\}}{(1-\beta)^2} = \frac{2(\delta+\delta^2)}{(1-\beta)^2}.\end{aligned}$$

It follows that

$$\xi_Y(\beta) \ge \frac{|||z'-\beta w'|||}{1-\beta f'(w')} \ge \frac{\|z-\beta w\|}{1-\beta f(w)} - \frac{2(\delta+\delta^2)}{(1-\beta)^2}$$

and hence that $\xi_Y(\beta) \ge \xi_X(\beta) - 2(\delta+\delta^2)/(1-\beta)^2$. A symmetric argument yields $\xi_X(\beta) \ge \xi_Y(\beta) - 2(\delta+\delta^2)/(1-\beta)^2$. ∎

It follows immediately that $\xi = \xi_1$ in the non-reflexive case, since uniformly non-square Banach spaces (and the completions of uniformly non-square normed spaces) are reflexive [17]. Similarly, if the *girth* [26] of a normed space equals 4, then $\xi = \xi_1$, but of course the converse need not be true.

We recall [12, Definition 7.3] that the *modulus of smoothness* of a normed space is the function $\varrho : [0,1] \to \mathbb{R}$ defined by

$$2\varrho(\beta) = \sup\{\|x+\beta y\| + \|x-\beta y\| - 2 : \|x\| = \|y\| = 1\}.$$

A normed space is said to be *uniformly smooth* [12, Definition 7.2] if and only if $\varrho(\beta)/\beta \to 0$ as $\beta \to 0$.

THEOREM 2.4. *Let ϱ be the modulus of smoothness of a normed space X. Then*

(i) *for all $\beta \in (0,1)$,*

$$1 \le \frac{\xi(\beta)-1}{\varrho(\beta)} \le \frac{2}{1-\beta},$$

(ii) *X is uniformly smooth if and only if $\xi'(0) = 0$,*

(iii) *X is nearly square if and only if $\xi'(0) = 2$,*

(iv) *$\xi'(0) = 2\varrho'(0)$.*

Proof. (i) Fix $\beta \in (0,1)$. Given two norm one vectors z and w, by definition we have

$$\|z+\beta w\| + \|z-\beta w\| \le 2 + 2\varrho(\beta), \qquad \beta N_-(z,w) \le \|z+\beta w\| - \|z\|.$$

Adding these two inequalities yields $\|z-\beta w\| \le 1 - \beta N_-(z,w) + 2\varrho(\beta)$. Since $N_-(z,w) \le 1$, Lemma 1.2 tells us that $\xi(\beta) \le 1 + 2\varrho(\beta)/(1-\beta)$.

For the other inequality, choose norm one vectors x and y such that $\|x-\beta y\| \le (1-\beta N_-(x,y))\xi(\beta)$ and $\|x+\beta y\| \le (1+\beta N_-(x,y))\xi(\beta)$. Adding these inequalities gives $\|x-\beta y\| + \|x+\beta y\| \le 2\xi(\beta) - 2$, which obviously implies $\varrho(\beta) \le \xi(\beta) - 1$.

(ii) is clear.

(iii) If X is nearly square then $\xi = \xi_1$ and so $\xi'(0) = 2$. Conversely, suppose that X is uniformly non-square. Then there is a $\delta > 0$ for which all two-dimensional subspaces of X have Banach–Mazur distance from $\ell_1(2)$ at least $1/(1-2\delta)$. By Theorem 2.1, $\xi(\beta) \le (1-\delta\beta)\xi_1(\beta)$ for all β and so $\xi(\beta) \le 1 + (2-\delta)\beta - O(\beta^2)$, whence $\xi'(0) \le 2-\delta$.

(iv) It is clear that $\xi'(0) \le 2\varrho'(0)$ and that $\xi'(0) = 0$ if $\varrho'(0) = 0$. So assume $\varrho'(0) > 0$ and choose an arbitrary $\lambda < \varrho'(0)$. We will show that $\xi'(0) \ge 2\lambda$. Note that $\lambda < 1$.

For convenience we first consider the finite-dimensional case. Given an arbitrary sequence $\beta_n \to 0$, we have $\varrho(\beta_n) > \lambda\beta_n$ and so there are unit vectors x_n and y_n with $\|x_n + \beta_n y_n\| + \|x_n - \beta_n y_n\| > 2 + 2\lambda\beta_n$. If f_n and g_n are support functionals for $x_n + \beta_n y_n$ and $x_n - \beta_n y_n$ respectively then $(f_n + g_n)(x_n) + \beta_n(f_n - g_n)(y_n) > 2 + 2\lambda\beta_n$. Since $(f_n + g_n)(x_n) \le 2$, this implies $(f_n - g_n)(y_n) > 2\lambda$ and so $\|f_n - g_n\| > 2\lambda$. Passing to subsequences if necessary, we may assume that everything converges. Write $z = \lim x_n$, $f = \lim f_n$ and $g = \lim g_n$. Then $\|z\| = 1$, $f, g \in D(z)$ and $\|f - g\| \ge 2\lambda$. Choose w with $\|w\| = 1$ and $(f - g)(w) \ge 2\lambda$. Then for all β,

$$\|z - \beta w\| \ge f(z - \beta w) + (g - f)(z - \beta w) \ge 1 - \beta f(w) + 2\lambda\beta.$$

Lemma 1.2 then forces $\xi(\beta) \ge 1 + 2\lambda\beta$ and so $\xi'(0) \ge 2\lambda$.

Perhaps the easiest way to settle the infinite-dimensional case is to permit ourselves the use of ultrapowers [12, Chapter 14]. Choose a free ultrafilter $\mathcal{U}$ over the integers and sequences x_n, y_n, f_n and g_n as before. In the ultrapowers $X_{\mathcal{U}}$ and $(X^*)_{\mathcal{U}} \subseteq (X_{\mathcal{U}})^*$ define $z = (x_n)_{\mathcal{U}}$, $w = (y_n)_{\mathcal{U}}$, $f = (f_n)_{\mathcal{U}}$ and $g = (g_n)_{\mathcal{U}}$. Then $\|z\| = 1$, $f, g \in D(z)$ and $(f - g)(w) \ge 2\lambda$. The preceding calculations then show that $\xi'_{X_{\mathcal{U}}}(0) \ge 2\lambda$. Since every finite-dimensional subspace of $X_{\mathcal{U}}$ is almost isometric to a finite-dimensional subspace of X (cf. [12, Theorem 14.2]), we have $\xi_X(\beta) = \xi_{X_{\mathcal{U}}}(\beta)$ for all β. ∎

Easy calculations with the examples $\ell_1(2)$ and $\ell_2(2)$ show that the constants 1 and 2 in part (i) cannot be improved. This does not contradict (iv); it merely exemplifies the fact that differentiation can be a discontinuous operation.

Now we relate uniform convexity of X with the behaviour of ξ. Recall [12, Definitions 5.1 and 5.2] that X is *uniformly convex* if and only if its *modulus of convexity*

$$\delta(\varepsilon) = \inf\{1 - \|\tfrac{1}{2}(x + y)\| : \|x\| = \|y\| = 1,\ \|x - y\| \ge \varepsilon\}$$

is strictly positive, for each $\varepsilon > 0$. For $\|z\| = 1$ and $0 < \beta < 1$, Kadets [19] defined the set $G(z, \beta) = \{y : [y, z] \subset B(0, 1) \setminus \mathrm{int} B(0, \beta)\}$, and noted that X is locally uniformly convex iff $\operatorname{diam} G(z, \beta) \to 0$ as $\beta \to 1$ for every z. Naturally, X is uniformly convex iff $\operatorname{diam} G(z, \beta) \to 0$ as $\beta \to 1$, uniformly with respect to $\|z\| = 1$.

One defines $D(z, \beta) = \mathrm{co}(\{z\} \cup B(0, \beta))$ as the *drop* of $B(0, \beta)$ with respect to the point z, and $R(z, \beta) = D(z, \beta) \setminus B(0, \beta)$ as the *residue*. (A well-known theorem of Abel, mentioned also in [24] because it uses the inequality $\xi \le \xi_1$ in the euclidean plane, can then be restated: if a complex power series has radius of convergence 1, and it converges at one point z_0 on the unit circle, then it converges uniformly on the drop $D(z_0, \beta)$, for each $\beta < 1$.) Rolewicz [25, Proposition 1] showed that X is uniformly convex iff $\operatorname{diam} R(z, \beta) \to 0$ as $\beta \to 1$, uniformly with respect to $\|z\| = 1$. We now offer a simpler proof of this, incorporating the behaviour of the modulus of squareness. We note in passing that X is locally uniformly convex iff $\operatorname{diam} R(z, \beta) \to 0$ as $\beta \to 1$, for all z with $\|z\| = 1$.

Recall that the *radius* of a set A relative to a point x is defined by $\mathrm{rad}(x, A) = \sup_{a \in A} \|x - a\|$. One then defines the radius (more precisely, the *self-radius*) of A by $\mathrm{rad}\, A = \inf_{x \in A} \mathrm{rad}(x, A)$. It is easily seen that $\mathrm{rad}\, A \le \operatorname{diam} A \le 2\, \mathrm{rad}\, A$.

The number $\varepsilon_0(X) = \sup\{\varepsilon : \delta(\varepsilon) = 0\}$ is called the *characteristic of convexity* of a normed space X (see [12, Definition 5.3]). Obviously, X is uniformly convex if and only if $\varepsilon_0(X) = 0$. It is well known [12, Lemma 5.1] that δ is continuous and strictly increasing on $[\varepsilon_0, 2)$, and so the right limit $\delta^{-1}(0+)$ exists and equals ε_0.

THEOREM 2.5. *For any normed space X, the following are equivalent.*

(i) *X is uniformly convex,*
(ii) $\operatorname{diam} G(z, \beta) \to 0$ *as* $\beta \to 1$, *uniformly with respect to* $\|z\| = 1$,
(iii) $\operatorname{diam} R(z, \beta) \to 0$ *as* $\beta \to 1$, *uniformly with respect to* $\|z\| = 1$,
(iv) $\limsup_{\beta \to 1}(1 - \beta)\xi(\beta) = 0$,
(v) $\liminf_{\beta \to 1}(1 - \beta)\xi(\beta) = 0$.

Furthermore, $\lim_{\beta \to 1}(1 - \beta)\xi(\beta) = \varepsilon_0(X)$. *Thus X is nearly square if and only if* $\lim_{\beta \to 1}(1 - \beta)\xi(\beta) = 2$.

Proof. We will prove, in the order given, the following four inequalities:

$$\varepsilon_0 - 1 + \beta \le (1 - \beta)\xi(\beta) \le \sup_{\|z\|=1} \operatorname{diam} R(z, \beta) \le \sup_{\|z\|=1} \operatorname{diam} G(z, \beta) \le \delta^{-1}(1 - \beta).$$

All other statements then follow by letting β go to 1.

(1) This is trivial if $\varepsilon_0 = 0$, so suppose that X is not uniformly convex. This means that we can find pairs of norm one vectors, a distance at least ε_0 apart, whose midpoints have norms arbitrarily close to one. A short calculation then shows that the norm is "almost additive" on the cone generated by these two vectors. More precisely, we can, given any $\gamma > 0$, find x_0 and y_0 of norm one for which $\|x_0 - y_0\| \ge \varepsilon_0$ and $(1 + \gamma^2)\|\lambda x_0 + \mu y_0\| \ge \lambda + \mu$ for all $\lambda, \mu \ge 0$. Now put $x = (1 + \gamma)x_0$ and $y = \beta y_0$, so that $\|x - y\| \ge \varepsilon_0 - \gamma - (1 - \beta)$. Then $z(x, y) = (1 - \alpha)x + \alpha y$ must satisfy

$$1 = \|z\| \ge \frac{1 + \gamma - \alpha(1 + \gamma - \beta)}{1 + \gamma^2} \quad \text{and so} \quad \alpha \ge \frac{\gamma - \gamma^2}{1 + \gamma - \beta}.$$

But then

$$\frac{\|x-z\|}{\|x\|-1}=\frac{\alpha\|x-y\|}{\gamma}\geq\frac{(1-\gamma)(\varepsilon_0-\gamma-(1-\beta))}{1+\gamma-\beta}.$$

Letting γ go to 0, we see that $\xi(\beta)\geq(\varepsilon_0-1+\beta)/(1-\beta)$.

(2) Consider $z=\alpha x+(1-\alpha)y$ as in the definition of ξ. Replacing y if necessary by a point on the segment $[y,z]$ with norm β, we may suppose that $y\in\overline{R(z,\beta)}$. Then $1=\|z\|\leq\alpha\|x\|+(1-\alpha)\beta$, from which it follows that $(1-\alpha)(1-\beta)\leq\alpha(\|x\|-1)$ and

$$\begin{aligned}(1-\alpha)(1-\beta)\omega(x,y)&\leq\alpha\|x-z\|=(1-\alpha)\|y-z\|\\&\leq(1-\alpha)\operatorname{rad}(z,\overline{R(z,\beta)}).\end{aligned}$$

Thus $(1-\beta)\xi(\beta)\leq\sup_z\operatorname{diam}R(z,\beta)$ in any normed space. This clearly yields the conclusion. We also note that $\operatorname{rad}(z,R(z,\beta))\leq1+\beta$; this gives another proof of the inequality $\xi\leq\xi_1$.

(3) It suffices to show that $R(z,\beta)\subseteq G(z,\beta)$. Given any $x\in R(z,\beta)$, the function $f(\alpha)=\|(1-\alpha)x+\alpha z\|$ is convex, and satisfies $f(\alpha_0)<\beta$ for some $\alpha_0<0$. Since $\beta<f(0)$, it follows that f is increasing for $\alpha\geq0$, and so $[x,z]$ does not meet $B(0,\beta)$.

(4) This is routine; let $\delta(\cdot)$ denote the modulus of convexity of X. For any $\varepsilon>\delta^{-1}(1-\beta)$, any unit vector z and any $x\in G(z,\beta)$, we have $\frac{1}{2}(x+z)\in[x,z]$ so $\|\frac{1}{2}(x+z)\|>\beta>1-\delta(\varepsilon)$ and $\|x-z\|<\varepsilon$. Thus $\operatorname{rad}(z,G(z,\beta))<\varepsilon$, independently of z. ■

We now see that X is uniformly convex whenever ξ is an integrable function. We suspect that the converse is also true. Let us make a few remarks about this problem.

LEMMA 2.6. *Let y,z,x be (in that order) three colinear points in a normed space with $\|y\|<\|z\|<\|x\|$. Then*

$$\|x-z\|\leq\int_{\|z\|}^{\|x\|}\xi\left(\frac{\|y\|}{t}\right)dt.$$

Proof. For $k=0,1,\dots,n$ put $z_k=z+\frac{k}{n}(x-z)$ and apply [24, Lemma 1] to the triple y,z_{k-1},z_k. This leads to the estimate

$$\|x-z\|\leq\sum_{k=1}^{n}\|z_k-z_{k-1}\|\leq\sum_{k=1}^{n}\xi\left(\frac{\|y\|}{\|z_{k-1}\|}\right)(\|z_k\|-\|z_{k-1}\|).$$

Taking the limit as $n\to\infty$ gives the desired inequality. ■

Now let us define $\alpha(\beta)=(1-\beta)^{-1}\sup_z\operatorname{rad}(z,G(z,\beta))$. Fix x,y with $\|x\|=1$ and $\|y\|=\beta$. Taking the limit as z approaches y in the conclusion of Lemma 2.6, we get $\|y-x\|\leq\int_{\|y\|}^{\|x\|}\xi(\|y\|/t)\,dt$, which might of course be infinite. After the change of variable $t\mapsto\beta/t$, we see that

$$\xi(\beta)\leq\alpha(\beta)\leq\frac{\beta}{1-\beta}\int_{\beta}^{1}\frac{\xi(t)}{t^2}\,dt.$$

This provides another proof that integrability of ξ implies uniform convexity of X, and leaves the converse as a tantalizing problem.

Since ξ is monotone, Lemma 2.6 is obviously an improvement of the estimate

$$\frac{\|x-z\|}{\|x\|-\|z\|}\leq\xi\left(\frac{\|y\|}{\|z\|}\right)$$

given by [24, Lemma 1]. Replacing ξ by ξ_1 yields the inequality

$$\|x-z\|\leq\|x\|-\|z\|+2\|y\|\log\frac{\|x\|-\|y\|}{\|z\|-\|y\|}.$$

For an inner product space, we have

$$\|x-z\|\leq(\|x\|^2-\|y\|^2)^{1/2}-(\|z\|^2-\|y\|^2)^{1/2}.$$

However, these seem too unwieldy to apply, in order to improve any of the estimates in [24]. Using the linear estimate $\log(1+t)\leq t$ brings us back to where we started.

Examples A and B below show that the modulus of squareness of a normed space and that of its dual need not be equal. Nevertheless, there is a very precise relationship between them.

THEOREM 2.7. *For any normed space X and any $\beta\in(0,1)$ we have*

$$\xi_{X^*}(\beta)=1/\xi_X^{-1}(1/\beta).$$

Proof. In the non-reflexive case, $\xi_X=\xi_{X^*}=\xi_1$ since X and X^* are nearly square, and the equality is easily verified. So we assume that X is reflexive. (An alternative to this argument is to deduce from the principle of local reflexivity that $\xi_{X^{**}}=\xi_X$.)

Write ξ^* instead of ξ_{X^*}. It suffices to show that $\xi^*(1/\xi(\beta))=1/\beta$. We begin by showing that $\xi^*(1/\xi(\beta))\geq1/\beta$. Fix $\tau<1$. Choose unit vectors z,w with $\delta\xi(\beta)=\|z-\beta w\|/N_-(z,z-\beta w)$ for some $\delta\in(\tau,1)$. Choose $f\in D(z)$ so that $f(z-\beta w)=N_-(z,z-\beta w)$, and choose an arbitrary $g\in D(z-\beta w)$. Then $\delta\xi(\beta)=f(z-\beta w)/g(z-\beta w)$, which implies that

$$\frac{\delta\xi(\beta)f(w)-g(w)}{\delta\xi(\beta)f(z)-g(z)}=\frac{1}{\beta}.$$

But $z\in D(f)$ so $N_-\big(f,f-\frac{1}{\delta\xi(\beta)}g\big)\leq\big(f-\frac{1}{\delta\xi(\beta)}g\big)(z)$. It follows that

$$\xi^*(1/(\delta\xi(\beta)))\geq\frac{\|f-\frac{1}{\delta\xi(\beta)}g\|}{N_-\big(f,f-\frac{1}{\delta\xi(\beta)}g\big)}\geq\frac{f(w)-\frac{1}{\delta\xi(\beta)}g(w)}{f(z)-\frac{1}{\delta\xi(\beta)}g(z)}=\frac{1}{\beta}.$$

Letting $\tau \to 1$, we conclude that $\xi^*(1/\xi(\beta)) \geq 1/\beta$.

By duality, we obtain

$$\xi\left(\frac{1}{\xi^*(1/\xi(\beta))}\right) \geq \xi(\beta),$$

which implies that $\xi^*(1/\xi(\beta)) \leq 1/\beta$. This establishes the equality. ∎

We remark that given any convex, strictly increasing (or even logarithmically convex) function $\xi : [0,1) \to [1,\infty)$, the formula appearing in Theorem 2.7 defines another function with the same properties. We sketch the proof of this in the log-convex case.

Recall that the *epigraph* of a function f is $\{(x,y) : y \geq f(x)\}$. Given any log-convex function $\xi : [0,1) \to [1,\infty)$, let $\mathcal{A}$ denote the set of all strictly increasing affine functions $A : [0,1) \to \mathbb{R}$ with $A \leq \log\xi$ everywhere. By the separation theorem, the epigraph of $\log\xi$ is the intersection of the epigraphs of all functions $A \in \mathcal{A}$. A short calculation then shows that a point (x,y) belongs to the epigraph of $\log\xi^*$ if and only if it lies in the epigraph of $(-\log)\circ A^{-1}\circ(-\log)$ for all $A \in \mathcal{A}$. Since $-\log$ is a convex function, $\log\xi^*$ also has convex epigraph.

It follows easily from Theorem 2.7 that $\xi'_{X^*}(0) = \lim_{\beta\to1}(1-\beta)\xi_X(\beta)$. This gives new proofs, albeit too complicated, of the duality between uniform smoothness and uniform convexity, and of the fact that being uniformly non-square is a self-dual property.

If Y is a subspace of X, it is obvious that $\xi_Y(\beta) \leq \xi_X(\beta)$. The previous result allows us to draw the same conclusion when Y is a quotient of X. A direct proof of this does not seem to be possible.

COROLLARY 2.8. *If Y is a quotient of X, then $\xi_Y(\beta) \leq \xi_X(\beta)$ for all $\beta \in (0,1)$.*

Proof. For any continuous strictly increasing bijection $f : [0,1) \to [1,\infty)$, let us write $f^*(\beta) = 1/f^{-1}(1/\beta)$. If $Y = X/M$ for some subspace M in X, then

$$\xi_Y^* = \xi_{Y^*} = \xi_{M^\circ} \leq \xi_{X^*} = \xi_X^*,$$

which means of course that $\xi_Y \leq \xi_X$. ∎

Now we present the fruits of some two-dimensional calculations. We use Lemma 1.2 to obtain the precise value of ξ for several spaces. Rather than including all the tedious details, we simply indicate at which points z,w the quotient $\|z-\beta w\|/(1-\beta N_+(z,w))$ from Lemma 1.2 is equal to $\xi(\beta)$. (In each of these examples, the corresponding quotient with $N_-(z,w)$ in place of $N_+(z,w)$ does not have a maximum value.) These examples can be used to settle a number of naive conjectures. For instance, Example C shows

that, even in two dimensions, the function ξ does not determine the space X uniquely.

EXAMPLE A. Let V be the two-dimensional space with the ℓ_1 norm in the second and fourth quadrants and the ℓ_2 norm in the first and third quadrants. Then

$$\xi(\beta) = \frac{\sqrt{1+\beta^2}}{1-\beta}.$$

A maximum is attained when $z = (-1,0)$ and $w = (0,1)$.

EXAMPLE B. Let W be the two-dimensional space with the ℓ_∞ norm in the second and fourth quadrants and the ℓ_2 norm in the first and third quadrants. Then

$$\xi(\beta) = \xi_{V^*}(\beta) = \frac{\sqrt{2-\beta^2}+\beta}{\sqrt{2-\beta^2}-\beta} = \frac{1+\beta\sqrt{2-\beta^2}}{1-\beta^2}.$$

A maximum is attained when $w = \frac{1}{2}\left(\sqrt{2-\beta^2}-\beta, \sqrt{2-\beta^2}+\beta\right)$ and $z = (-1,1)$.

EXAMPLE C. For $-1 < \alpha < 1$, let X_α be the two-dimensional space with hexagonal unit ball with vertices at $(-1,1)$, $(\alpha,1)$, $(1,\alpha)$ and the opposite three points. Then

$$\xi(\beta) = \frac{1+|\alpha|\beta}{1-\beta}.$$

For $\alpha \geq 0$, a maximum is attained when $z = (-1,1)$ and $w = (\alpha,1)$. For $\alpha \leq 0$, a maximum is attained when $z = (\alpha,1)$ and $w = (-1,1)$.

For a two-dimensional space, we have seen that $\xi(\beta) = \xi_1(\beta)$ for one value of β implies that the same equality holds for all values of β. We will see in the next section that the same holds for ξ_2. However, it is not true in general that ξ is completely determined by its value at a single point β; one can easily find a $\beta > 0$ for which $\xi_V(\beta) = \xi_{X_\alpha}(\beta)$ whenever $|\alpha| < \sqrt{2}-1$. More generally, let X be a uniformly smooth space which is not uniformly convex, and choose Y uniformly convex but not uniformly smooth. From the inequalities $\xi'_X(0) < \xi'_Y(0)$ and $\lim_{\beta\to1}(1-\beta)\xi_Y(\beta) < \lim_{\beta\to1}(1-\beta)\xi_X(\beta)$ one can easily see that there is a β for which $\xi_X(\beta) = \xi_Y(\beta)$.

Alternatively, denote by ξ_p the modulus of squareness of $\ell_p(2)$. (We confess that we have not been able to calculate this in general, not even for $p = 4$.) Using Theorem 2.3, we see that there are values $p < 2$ and $q > 2$ for which $\xi_p(1/2) = \xi_q(1/2) = 3/2$. It is well known that $\ell_p(2)$ and $\ell_q(2)$ are not isometric; we refer to [7] for several proofs of this.

We finish this section with an application to fixed point theory. Recall that X is said to have *uniformly normal structure* [12, Definition 4.3] if $\sup\{\operatorname{rad} A/\operatorname{diam} A : A \in H(X),\ A \text{ infinite}\} < 1$. This property is famous for its applications to fixed point problems [12]. In fact, it is a well-known easy exercise [12, Theorem 6.1] to show that $\varepsilon_0 < 1$ implies uniformly normal structure. Thus every uniformly convex space has uniformly normal structure. (A long-standing conjecture is that the converse is true, after renorming.) Slightly more difficult is the following result due independently to Sims [27], Khamsi [20] and Prus [23], drawing on earlier work by Baillon and Turett [12, Theorem 14.3]: if $\varrho'(0) < 1/2$, then X has uniformly normal structure. It follows that X has uniformly normal structure whenever $\xi'(0) < 1$. The following result includes all of these cases.

PROPOSITION 2.9. *If $\xi_X(\beta) < 1/(1-\beta)$, even for one value of β, then X has uniformly normal structure.*

Proof. A normed space is said to have *weak normal structure* if the radius of every weakly compact subset is strictly less than its diameter. If X does not have weak normal structure, then by [11, Lemma 2.3] we can find two-dimensional subspaces M_n of X, and symmetric hexagons in their unit balls, such that for each of them the vertices have norm 1, the length of each side is within $1/n$ of 1, and four of the sides are within distance $1/n$ of the unit sphere. Passing to a subsequence in the Banach–Mazur compactum, we find a two-dimensional space M which is $\lim_n M_n$ with respect to the Banach–Mazur distance. It is clear that the unit ball of M contains a regular hexagon four of whose sides lie completely in the unit sphere. Thus we may identify M with $\mathbb{R}^2$, with the ℓ_∞ norm in the first and third quadrants. Choosing $z = (1,1)$ and $w = (0,1)$ shows that $\xi_M(\beta) \geq 1/(1-\beta)$ for all non-zero β. We do not claim that M is isometric to a subspace of X. However, an application of Theorem 2.3 tells us that $\xi_X(\beta) \geq \sup_n \xi_{M_n}(\beta) = \xi_M(\beta) \geq 1/(1-\beta)$.

If $\xi_X(\beta) < 1/(1-\beta)$, the remark after Theorem 2.3 tells us that X is reflexive, and thus (by the preceding paragraph) has normal structure. As remarked at the end of the proof of Theorem 2.4, $\xi_{X_\mathcal{U}}(\beta) = \xi_X(\beta)$ for every ultrapower $X_\mathcal{U}$ of X. Thus every ultrapower of X has normal structure, whence X has uniformly normal structure. ■

3. A characterization of inner product spaces.

Dvoretzky's Theorem [22] together with Theorem 2.3 shows that $\xi_X \geq \xi_2$ for any infinite-dimensional normed space X. Here we give a two-dimensional proof of this, necessarily independent of Dvoretzky's Theorem. In fact, we show that the equality $\xi(\beta) = \xi_2(\beta)$, even for one non-zero value of β, characterizes inner product spaces.

From now on, we fix $\beta \in (0,1)$ and assume that X has dimension two. If x and y are linearly independent vectors, we will write $x \prec y$ to indicate that x precedes y in some pre-ordained orientation of the plane. We make a fuss of the next definition because it will be used so much later.

DEFINITION 3.1. We say that (y,z) is a *β-pair* if

$$\|y\| = \beta, \quad \|z\| = 1, \quad z \prec y \quad \text{and} \quad z \perp y - \beta^2 z.$$

Here $\perp$ denotes of course Birkhoff orthogonality.

Imagine two infinitesimal ants, y and z, crawling around the spheres with radii β and 1, in unison, so that at each instant (y,z) is a β-pair. It is possible that y could turn around and march backwards, with z remaining fixed, but only when z is at a non-smooth point of the unit sphere. It is also possible that z could have the option of marching forwards or backwards, with y remaining fixed, but only when z is in the interior of some line segment in the unit sphere. Since an interior point of such a segment is obviously a smooth point of the unit sphere (in two dimensions), these two possibilities cannot arise simultaneously. Now fix the initial positions y_0 and z_0 of y and z; we will write $y = y_\lambda$ and $z = z_\mu$ to indicate that y makes an angle of λ with y_0 and that z makes an angle of μ with z_0. This argument shows that for each $\theta \in [0, 4\pi)$, there exists a unique $\lambda \in [0, 2\pi)$ and a unique $\mu \in [0, 2\pi)$ so that (y_λ, z_μ) is a β-pair and $\lambda + \mu = \theta$. Now write $T = \{(\lambda,\mu) \in [0,2\pi) \times [0,2\pi) : (y_\lambda, z_\mu) \text{ is a } \beta\text{-pair}\}$. Since our ants can move around their spheres without changing direction, and without any jumps, we see that the function $(\lambda,\mu) \mapsto \lambda+\mu$ is a homeomorphism between T and $[0, 4\pi)$.

Our immediate task is to calculate the area enclosed by the curve traced out by $y - \beta^2 z$. We need the following simple result, which is implicit in [18] but perhaps not completely obvious. The hypothesis $\|f(t)\| = 1$ is stronger than necessary but convenient for our purposes.

As usual, we write $u \wedge v$ for the signed area of the parallelogram determined by two vectors u and v. That is, $u \wedge v = u_1 v_2 - u_2 v_1$, which is invariant under rotations of the coordinate system.

LEMMA 3.2. *Let $f, g : [a,b] \to X$ be two rectifiable curves such that $\|f(t)\| = 1$ and $f(t) \perp g(t)$ for all $t \in [a,b]$. Then*

(i) *there is a $t \in [a,b]$ for which $g(t)$ and $f(b) - f(a)$ have the same direction (i.e. lie in the same one-dimensional subspace),*

(ii) $\int_a^b g \wedge df = 0$.

Proof. (i) This is essentially a non-smooth version of Rolle's Theorem. Assume $f(a) \neq f(b)$, as the conclusion is obvious otherwise. It is clear that $I = \{t \in [a,b] : f(t) \perp f(b) - f(a)\}$ is non-empty. If the norm of X is smooth at $f(t)$ for some $t \in I$, then clearly $g(t)$ is a scalar multiple of $f(b) - f(a)$. In the non-smooth case, continuity of g and the Intermediate Value Theorem ensure the existence of some t for which $g(t)$ is a scalar multiple of $f(b) - f(a)$.

(ii) Part (i) implies that for any partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a,b]$, we can find points $s_i \in [t_{i-1}, t_i]$ with $g(s_i) \wedge (f(t_{i-1}) - f(t_i)) = 0$. Taking the limit over finer partitions, we see that $\int_a^b g(t) \wedge df(t) = 0$. ■

LEMMA 3.3. *Both $\Gamma_\beta = \{y - \beta^2 z : (y,z)$ is a β-pair$\}$ and $\Delta_\beta = \{z - y : (y,z)$ is a β-pair$\}$ are simple closed rectifiable curves, and the areas they enclose are $A(\Gamma_\beta) = \beta^2(1-\beta^2)A(S)$ and $A(\Delta_\beta) = (1-\beta^2)A(S)$.*

Proof. The map $T \to X : (\lambda, \mu) \mapsto y_\lambda - \beta^2 z_\mu$ is injective and continuous by the argument above. Since its components (with respect to any basis of X) are piecewise monotonic, it is also of bounded variation. This establishes that Γ_β is rectifiable; a similar argument takes care of Δ_β. Obviously both curves are closed.

It is fairly clear from the definition of β-pair that Γ_β is a simple curve. Establishing simplicity of Δ_β means proving that if (y_1, z_1) and (y_2, z_2) are β-pairs with $y_1 - z_1 = y_2 - z_2$, then $y_1 = y_2$ and $z_1 = z_2$.

First consider the strictly convex case. We will show that if $\|y_1\| = \|y_2\| = \beta$, $\|z_1\| = \|z_2\| = 1$ and $y_1 - z_1 = y_2 - z_2$, then either $y_1 \prec z_1$ or $y_2 \prec z_2$. Assume without loss of generality that $y_1 \prec y_2$. Strict convexity implies that, given y_1 and y_2, there are only two possibilities for z_1 and z_2: both are interior points either of the (minor) arc of S joining $\beta^{-1}y_1$ to $\beta^{-1}y_2$ or of the arc joining $-\beta^{-1}y_2$ to $-\beta^{-1}y_1$. In the first case $y_1 \prec z_1$ and in the second case $y_2 \prec z_2$. In neither case is (y_i, z_i) a β-pair.

In the non-strictly convex case, we must also consider the possibility that the intervals $[y_1, y_2]$ and $[z_1, z_2]$ are contained in parallel segments of their respective spheres. If $y_1 - z_1 = y_2 - z_2$, then $\|z_1 - z_2\| = \|y_1 - y_2\|$ and so the length of $[z_1, z_2]$ is strictly less than the length of the segment of S containing it. Without loss of generality we may assume that z_1 is an interior point of this segment, and so there is a unique $f \in D(z_1)$. Then $N(z_1, y_1) = f(y_1) = \pm\beta$ and Lemma 1.1 tells us that (y_1, z_1) is not a β-pair.

To calculate the areas, note first that the orthogonality requirement implies that for any β-pair (y,z), every point in the interval $[\beta^2 z, y]$ has norm at least β^2. Thus as (λ, μ) moves around T, the segment $[\beta^2 z_\mu, y_\lambda]$ sweeps out the annulus $B(0,\beta) \setminus B(0,\beta^2)$, touching almost every point only once. Hence $A(\Gamma_\beta) = (\beta^2 - \beta^4)A(S)$.

Unfortunately, no such argument is valid for Δ_β. Taking $y = (\beta^2, \beta)$ and $z = (1, -1)$ in $\ell_\infty(2)$ shows that the segment $[y,z]$ may well contain points with norm strictly less than β. We therefore need more explicit calculations to determine $A(\Delta_\beta)$. Except possibly for a sign change due to the orientation, the area enclosed by Δ_β is $\frac{1}{2}\oint (z_\mu - y_\lambda) \wedge d(z_\mu - y_\lambda)$.

Out of sympathy for the printer, we omit the subscripts $_\lambda$ and $_\mu$ in the following calculation, which includes an integration by parts and an application of Lemma 3.2. Since $z - y = (1-\beta^2)z - (y - \beta^2 z)$, we have

$$\begin{aligned}
2A(\Delta_\beta) &= \oint (z-y) \wedge d(z-y) \\
&= \oint ((1-\beta^2)z - (y-\beta^2 z)) \wedge d((1-\beta^2)z - (y - \beta^2 z)) \\
&= (1-\beta^2)^2 \oint z \wedge dz + \oint (y - \beta^2 z) \wedge d(y - \beta^2 z) \\
&\quad - (1-\beta^2)\oint z \wedge d(y-\beta^2 z) - (1-\beta^2)\oint (y - \beta^2 z) \wedge dz \\
&= 2(1-\beta^2)^2 A(S) + 2A(\Gamma_\beta) - 2(1-\beta^2)\oint (y-\beta^2 z) \wedge dz \\
&= 2(1-\beta^2)^2 A(S) + 2\beta^2(1-\beta^2)A(S) - 0 \\
&= 2(1-\beta^2)A(S),
\end{aligned}$$

as claimed. ■

A similar calculation also shows that $A(\Gamma_\beta) = (\beta^2 - \beta^4)A(S)$, but the argument given was considerably easier.

PROPOSITION 3.4. *Suppose that $\xi(\beta) \le \xi_2(\beta)$. Then*

(i) *for any β-pair (y,z), $\|y - z\| = \sqrt{1-\beta^2}$ and $N_+(z,y) = \beta^2$,*
(ii) *the norm on X is smooth and strictly convex,*
(iii) *for fixed $z \in S$ and y with $\|y\| = \beta$, the following are equivalent:*

(a) $z \perp y - \beta^2 z$,
(b) $\|y - z\| = \sqrt{1-\beta^2}$,
(c) $N(z,y) = \beta^2$,
(d) $N(y,z) = \beta$,
(e) $y \perp z - y$.

Proof. (i) For any β-pair (y,z), Lemma 1.2 together with our hypothesis tells us that $\|z - y\|/(1 - N_+(z,y)) \le 1/\sqrt{1-\beta^2}$, and Lemma 1.1 that $N_-(z,y) \le \beta^2 \le N_+(z,y)$. Thus, $\|z - y\| \le \sqrt{1-\beta^2}$. But we have just seen that $A(\Delta_\beta) = (1-\beta^2)A(S)$. It follows that for every β-pair (y,z), $\|y - z\| = \sqrt{1-\beta^2}$. This implies that $1 - N_+(z,y) \ge 1 - \beta^2$, whence $N_+(z,y) = \beta^2$.

(ii) Suppose that the norm is not smooth at some $z \in S$, and choose y so that (y,z) is a β-pair. Then $N_-(z,y) < N_+(z,y) = \beta^2$. Let v be the unique point of norm β on the ray $\{\beta^2 z + \mu(y - N_- z) : \mu > 0\}$. Then (v,z) is a β-pair. In fact, $N_-(z,v) = \beta^2 + N_-(z, \mu(y - N_- z)) = \beta^2$ and $N_+(z,v) = \beta^2 + N_+(z, \mu(y - N_- z)) > \beta^2$. The last inequality clearly contradicts (i). So X is smooth.

The inequality $\xi_{X^*}(\sqrt{1-\beta^2}) \le \xi_2(\sqrt{1-\beta^2})$ now follows from Theorem 2.7 and our hypothesis. The preceding argument implies that X^* is smooth, hence that X is strictly convex.

(iii) Since the norm is smooth, we may write N instead of N_+ and N_-. Since reversing the orientation of the plane does not affect our hypotheses, Lemma 1.1 tells us that (a) is equivalent to (c), and that (d) is equivalent to (e). Now fix $z \in S$, and consider the function $N(z, z-y)/\|z-y\|$ as y varies in βS.

Lemma 1.2 and our hypothesis tell us that $N(z, z-y)/\|z-y\| \geq \sqrt{1-\beta^2}$ for all such y. The latter value is the minimum value of the function, being attained whenever (y, z) is a β-pair. Moreover, it is geometrically obvious that, for any y which attains the minimum, the ray from the origin passing through $z - y$ must be a tangent to $B(z, \beta)$. This means of course that $y \perp z - y$. Thus (a) implies (e) (equivalently, (c) implies (d)).

Part (i) tells us that (c) (equivalently (a)) implies (b). Note that, since the norm is regular, there are precisely two points y satisfying (b), two points satisfying (c) and two points y satisfying (d). Thus (b), (c) and (d) are all equivalent. ∎

The following result is obviously weaker than Theorem 3.6; we mention it separately because its proof is so much simpler. We emphasize again that our proof does not use Dvoretzky's Theorem.

COROLLARY 3.5. *Let X be any normed space. Then*

(i) $\xi_X(\beta) \geq \xi_2(\beta)$ *for all* $\beta \in [0, 1)$,
(ii) *if* $\xi_X(\beta) = \xi_2(\beta)$ *for all* $\beta \in [0, 1)$, *X must be an inner product space.*

Proof. (i) This is immediate from Proposition 3.4(i) and Lemma 1.2.

(ii) It follows from Proposition 3.4(iii) that, for any unit vectors x and w, we have $N(x, \beta w) = \beta^2$ if and only if $N(\beta w, x) = \beta$. This being true for all β, it follows that $N(x, w) = N(w, x)$ for all unit vectors. Thus $\langle a, b\rangle = \|a\| N(a, b)$ defines an inner product compatible with the norm. ∎

We intend to show that if X is a normed space whose modulus of squareness satisfies $\xi(\beta) \leq 1/\sqrt{1-\beta^2}$ for some β, then X is an inner product space. The proof of this breaks naturally into two cases, depending on whether or not β is the sine of a rational multiple of π. To understand why this is necessary, consider a 2-dimensional space endowed with a norm whose unit ball is a regular $4n$-gon. Straightforward calculations show that the values of the modulus of smoothness, the modulus of convexity and several other known moduli [2] coincide at certain points with the corresponding moduli for an inner product space. For the modulus of squareness, there is no such problem, but the proof of this case requires special treatment.

If $\|\lambda x + y\|^2 = \lambda^2\|x\|^2 + \|y\|^2$, we will say that x is λP-*orthogonal* to y (cf. [1]). In an inner product space, this is obviously equivalent to B-orthogonality. In general, like B-orthogonality, it need not be a symmetric relation. (We remark that the proof of the main part of Theorem 3.6 is similar to that used in [1] to show, when λ is not the tangent of a rational multiple of π, that if λP-orthogonality implies λI-orthogonality (i.e. $\lambda x + y$ and $\lambda x - y$ have the same norm), or conversely, then X is an inner product space. This is false when λ is the tangent of a rational multiple of π, a counterexample being again any 2-dimensional space whose unit ball is a regular $4n$-gon.)

THEOREM 3.6. *For any normed space X, the following are equivalent:*

(i) *X is an inner product space,*
(ii) $\xi_X(\beta) \leq \xi_2(\beta)$ *for some* $\beta \in (0, 1)$,
(iii) *for some λ, B-orthogonality implies λP-orthogonality.*

Proof. (i)⇒(ii) is straightforward; see [24, Proposition 2] or do a simple calculation using Lemma 1.2.

(ii)⇒(iii). Put $\lambda = \beta/\sqrt{1-\beta^2}$. Proposition 3.4(iii) tells us that the set $\{\beta u + \sqrt{1-\beta^2} v : u, v \in S,\ u \prec v \text{ and } u \perp v\}$ contains S. Arguments which are by now standard show that this set is a closed rectifiable curve. Hence it coincides with S. It follows easily from this that, for any pair of vectors in S, B-orthogonality implies λP-orthogonality.

(iii)⇒(i). First we prove that X is smooth and strictly convex. (We know already that this follows from (ii), so a proof that (ii)⇒(i) could omit this paragraph.) It is well known that strict convexity of the norm of a normed space is equivalent to left uniqueness of B-orthogonality, i.e. the property that $x_1 \perp y$ and $x_2 \perp y$ for $y \neq 0$ implies that x_1 and x_2 are colinear [3, p. 33]. Failure of this property would imply that $x \perp y$ for all x in some nondegenerate segment in S and for some $y \in S$. But it would clearly be impossible for all such x to satisfy $\|\lambda x + \mu y\|^2 = \lambda^2 + \mu^2$. Thus our normed space is strictly convex. Similar reasoning, using right uniqueness of B-orthogonality, shows that the space is smooth.

When X is strictly convex, it is easy to see that, for any $x \in S$, there are precisely two points $y \in S$ for which x is λP-orthogonal to y. Given this regularity of the unit sphere, we now see that x can be λP-orthogonal to y only when it is B-orthogonal. In particular, this implies that if x is λP-orthogonal to y, then it is also λP-orthogonal to $-y$.

We can always find unit vectors u, v which are B-orthogonal to each other [8]. (This is the same as saying that $\{u, v\}$ is an Auerbach basis for X.) It follows that u and v are λP-orthogonal to each other. Thus $\|\lambda u \pm v\|^2 = \|u \pm \lambda v\|^2 = \lambda^2 + 1$. It follows that

$$\|\lambda(\lambda u \pm v) + u \mp \lambda v\| = (\lambda^2 + 1)\|u\| = \sqrt{\lambda^2\|\lambda u \pm v\|^2 + \|u \mp \lambda v\|^2},$$

i.e. that $\lambda u \pm v$ is λP-orthogonal to $u \mp \lambda v$. Similarly, we can show that $u \pm \lambda v$ is λP-orthogonal to $-\lambda u \pm v$, and hence also to $\lambda u \mp v$. Of course, this implies B-orthogonality.

Let C denote the circumference of a euclidean unit ball for which u and v form an orthogonal basis. Then the twelve points $\pm u$, $\pm v$, $(1+\lambda^2)^{-1/2}(\pm u \pm \lambda v)$ and $(1+\lambda^2)^{-1/2}(\pm\lambda u \pm v)$ all lie in $S \cap C$. The same argument applied to $u_1 = (1+\lambda^2)^{-1/2}(\lambda u - v)$ and $v_1 = (1+\lambda^2)^{-1/2}(u + \lambda v)$ shows that $(1+\lambda^2)^{-1/2}(\pm u_1 \pm \lambda v_1)$ and $(1+\lambda^2)^{-1/2}(\pm\lambda u_1 \pm v_1)$ are also in $S\cap C$. Continuing this process with $u_{n+1} = (1+\lambda^2)^{-1/2}(\lambda u_n - v_n)$ and $v_{n+1} = (1+\lambda^2)^{-1/2}(u_n + \lambda v_n)$, we obtain a set $\{u, v, u_1, v_1, u_2, v_2, \ldots\}$ which is contained in $S \cap C$.

If λ is not the tangent of a rational multiple of π, the above set is dense in C, and it follows that $S = C$.

Now we consider the *rational case*: let $\lambda = \tan(k\pi/2n)$, where k and n are relatively prime and $1 \le k < n$.

It is proved implicitly in [9] and explicitly in [5, Lemma 2.4] that there exist mutually B-orthogonal unit vectors u and v so that $u \wedge v$ has minimal absolute value, amongst all pairs with u B-orthogonal to v.

In this case the set $\{u, v, u_1, v_1, u_2, v_2, \ldots\}$ coincides with a regular $4n$-gon, so more work is needed to conclude that $S = C$. Put $\alpha = \pi/(2n)$ and let A_j be the area of the sector of the unit ball determined by two consecutive radii of this $4n$-gon, $u\cos(j-1)\alpha + v\sin(j-1)\alpha$ and $u\cos j\alpha + v \sin j\alpha$. To avoid confusion with the numbering, note that $A_j = A_{j+2n}$. To calculate these areas, let $x(\theta)$ be the unique point of S at an angle of θ with u, and let $y(\theta)$ be the unique point of S which is B-orthogonal to, and comes after, $x(\theta)$. Since S is smooth and strictly convex, it is easy to see that the mapping $\theta \mapsto \lambda x(\theta) + y(\theta)$ is a Jordan rectifiable curve. (We remark that this is also true even when S is not smooth or strictly convex [18].) Note that as θ moves from $(j-1)\alpha$ to $j\alpha$, $(1+\lambda^2)^{-1/2}(\lambda x(\theta) + y(\theta))$ traverses the boundary of the $(j+k)$th sector. For simplicity, we omit the dependence of all functions on the variable of integration θ. We have

$$\begin{aligned}
2(\lambda^2+1)A_{j+k} &= \int_{(j-1)\alpha}^{j\alpha} (\lambda x + y)\wedge d(\lambda x + y)\\
&= \lambda^2 \int_{(j-1)\alpha}^{j\alpha} x\wedge dx + \int_{(j-1)\alpha}^{j\alpha} y \wedge dy + \lambda \int_{(j-1)\alpha}^{j\alpha} x\wedge dy + \lambda \int_{(j-1)\alpha}^{j\alpha} y \wedge dx\\
&= \lambda^2 \int_{(j-1)\alpha}^{j\alpha} x\wedge dx + \int_{(j-1)\alpha}^{j\alpha} y \wedge dy + \lambda \int_{(j-1)\alpha}^{j\alpha} d(x\wedge y) + 0\\
&= \lambda^2 \int_{(j-1)\alpha}^{j\alpha} x\wedge dx + \int_{(j-1)\alpha}^{j\alpha} y \wedge dy + 0 = \lambda^2 A_j + A_{j+n},
\end{aligned}$$

using integration by parts, and the facts that $x(\theta) \perp y(\theta)$ for all θ and that $x(j\alpha)\wedge y(j\alpha)$ is independent of j. A simple adjustment of the indices shows that we also have

$$(\lambda^2+1)A_{j+k+n} = \lambda^2 A_{j+n} + A_j.$$

Given A_j and A_{j+n} one can calculate immediately A_{j+k} and A_{j+n+k}. If A_j were different from A_{j+n}, then A_{j+k} and A_{j+n+k} would lie strictly between A_j and A_{j+n}. Applying this argument (at most) $2n$ times, we see that if A_0 were different from A_n, then both A_0 and A_n would lie strictly between A_0 and A_n. This absurdity forces $A_0 = A_n$, from which it quickly follows that $A_1 = A_2 = \ldots = A_{4n}$.

It is easily shown [4, Lemma 1 or 2] that B-orthogonality is symmetric if (and only if) $|x\wedge y|$ is constant for all $x, y \in S$ for which $x \perp y$. By a simple compactness argument we may choose $u', v' \in S$ with $u' \prec v'$, and $u' \wedge v'$ maximal with respect to all such pairs. A routine argument using [4, Lemma 1] then shows that v' is B-orthogonal to u'.

This leads us to a new $4n$-gon whose radii divide the unit ball into $4n$ sectors of equal areas, $A'_1, \ldots, A'_{4n}$, each of which (when suitably indexed) divides the jth sector of the previous $4n$-gon into two sectors with areas B_j and C_j. Then $B_j + C_j = A_j = A'_j = C_j + B_{j+1}$, and so $B_1 = \ldots = B_{4n}$ and $C_1 = \ldots = C_{4n}$. If $x(j\alpha - \delta)$ is a vertex of the new $4n$-gon, for some $\delta \in (0, \alpha)$, then

$$\begin{aligned}
\lambda^2 B_j + B_{j+n} = (\lambda^2+1)B_{j+k} &= \frac{1}{2}\int_{(j-1)\alpha}^{j\alpha-\delta} (\lambda x + y)\wedge d(\lambda x + y)\\
&= \frac{1}{2}\lambda^2 \int_{(j-1)\alpha}^{j\alpha-\delta} x\wedge dx + \frac{1}{2}\int_{(j-1)\alpha}^{j\alpha-\delta} y\wedge dy + \frac{1}{2}\lambda \int_{(j-1)\alpha}^{j\alpha-\delta} d(x\wedge y)\\
&= \lambda^2 B_j + B_{j+n}\\
&\quad + \frac{1}{2}\lambda(x(j\alpha-\delta)\wedge y(j\alpha-\delta) - x((j-1)\alpha)\wedge y((j-1)\alpha))
\end{aligned}$$

and so $x(j\alpha-\delta)\wedge y(j\alpha-\delta) = x((j-1)\alpha)\wedge y((j-1)\alpha)$. From our choice of u' and v' it follows that $\{x\wedge y : x, y \in S,\ x \prec y,\ x \perp y\}$ is a singleton.

As noted before, this means that B-orthogonality is symmetric. In dimension three or more, this would imply that our space is an inner product space [6]. (For another proof of this, not assuming smoothness, we could refer to [9] or [16].) This is not true in two dimensions ([3, p. 77], or [9]), so we slog on with the proof.

We have now shown that for any $u, v \in S$ with $u \prec v$ and $u \perp v$, we can construct a regular $4n$-gon as above. This, together with the preceding arguments, implies that if u', v' is another such pair, then the area of the segment of the unit ball determined by u and u' is the same as the area of the segment of the unit ball determined by v and v'. The arguments of [10, §3] show, without any modification whatsoever, that this property, together with the symmetry of B-orthogonality, implies that S is an ellipse. ■

Finally, we emphasize that (i)⇔(iii) gives a new characterization of inner product spaces, which makes no mention of the modulus of squareness. We prefer to restate this in the following symmetric form: if there exist non-zero λ and μ for which

$$x, y \in S,\ x \perp y \ \Rightarrow\ \|\lambda x + \mu y\|^2 = \lambda^2 + \mu^2,$$

then X is an inner product space.

The special case $\lambda = \mu$ was first proved in [4, Proposition 2]. Of course, the case when λ/μ is not the tangent of a rational multiple of π is somewhat easier.

4. Open problems. We collect here a number of open problems concerning the content of this paper, some of which have been mentioned already.

PROBLEM 4.1. Characterize those functions which are the modulus of squareness of some normed space.

PROBLEM 4.2. If a function is the modulus of squareness of some normed space, need it be the modulus of squareness of a finite-dimensional (in particular, two-dimensional) normed space?

From now on, ξ is the modulus of squareness of a normed space X.

PROBLEM 4.3. Is ξ always an analytic function?

PROBLEM 4.4. Is $\log \xi$ always a convex function?

PROBLEM 4.5. If X is uniformly convex, must ξ be integrable?

PROBLEM 4.6. Do the techniques of §3 lead to other characterizations of inner product spaces?

PROBLEM 4.7. Calculate exactly the modulus of squareness of $\ell_p(2)$.

PROBLEM 4.8. Is there a sensible *localization* of the modulus of squareness? We mean: in a sense similar to that in which Fréchet smoothness localizes uniform smoothness and local uniform convexity localizes uniform convexity.

PROBLEM 4.9. Is the modulus of squareness related to the notions of girth, inner perimeter and flatness, as defined in [14] and [26]?

PROBLEM 4.10. Find a lower bound for ξ'. In particular, is $\xi' \geq \xi_2'$ always?

Acknowledgements. The first and third authors are grateful to the organizers of the conferences Convexity (Cortona, June 1992) and Functional Analysis and Applications (Gargnano, May 1993), where a number of useful discussions took place. The contribution of the third author was supported at different times by the Free University of Berlin and the Italian National Research Council.

References

[1] J. Alonso and C. Benítez, *Some characteristic and non-characteristic properties of inner product spaces*, J. Approx. Theory 55 (1988), 318–323.

[2] J. Alonso and A. Ullán, *Moduli in normed linear spaces and characterization of inner product spaces*, Arch. Math. (Basel) 59 (1992), 487–495.

[3] D. Amir, *Characterizations of Inner Product Spaces*, Birkhäuser, Basel, 1986.

[4] C. Benítez and M. del Río, *Characterization of inner product spaces through rectangle and square inequalities*, Rev. Roumaine Math. Pures Appl. 29 (1984), 543–546.

[5] C. Benítez and Y. Sarantopoulos, *Characterization of real inner product spaces by means of symmetric bilinear forms*, J. Math. Anal. Appl. 180 (1993), 207–220.

[6] G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. 1 (1935), 169–172.

[7] F. Cabello Sánchez and J. M. F. Castillo, *Isometries of finite dimensional normed spaces*, Extracta Math. 10 (1995), 146–151.

[8] M. M. Day, *Polygons circumscribed about closed convex curves*, Trans. Amer. Math. Soc. 62 (1947), 315–319.

[9] —, *Some characterizations of inner-product spaces*, ibid., 320–337.

[10] M. del Río and C. Benítez, *The rectangular constant for two-dimensional normed spaces*, J. Approx. Theory 19 (1977), 15–21.

[11] J. Gao and K. S. Lau, *On two classes of Banach spaces with uniform normal structure*, Studia Math. 99 (1991), 41–56.

[12] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.

[13] V. I. Gurariĭ, *On differential properties of the modulus of convexity of Banach spaces*, Mat. Issled. 2 (1967), 141–148 (in Russian).

[14] R. E. Harrell and L. A. Karlovitz, *Girths and flat Banach spaces*, Bull. Amer. Math. Soc. 76 (1970) 1288–1291.

[15] J.-B. Hiriart-Urruty, *A short proof of the variational principle for approximate solutions of a minimization problem*, Amer. Math. Monthly 90 (1983), 206–207.

[16] R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. 61 (1947), 265–292.

[17] —, *Uniformly non-square Banach spaces*, Ann. of Math. 80 (1964), 540–550.

[18] J. Joly, *Caractérisations d'espaces hilbertiens au moyen de la constante rectangle*, J. Approx. Theory 2 (1969), 301–311.

[19] M. I. Kadets, *Proof of the topological equivalence of all separable infinite-dimensional Banach spaces*, Functional Anal. Appl. 1 (1967), 53–62 (= Funktsional. Anal. i Prilozhen. 1 (1967), 61–70).

[20] M. A. Khamsi, *Étude de la propriété du point fixe dans les espaces de Banach et les espaces métriques*, thesis, Univ. Paris VI, 1987.
[21] V. I. Liokumovich, *Existence of B-spaces with a non-convex modulus of convexity*, Izv. Vyssh. Uchebn. Zaved. Mat. 12 (1973), 43–49 (in Russian).
[22] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Math. 1200, Springer, Berlin, 1986.
[23] S. Prus, *A remark on a theorem of Turett*, Bull. Polish Acad. Sci. Math. 36 (1988), 225–227.
[24] K. Przesławski and D. Yost, *Lipschitz retracts, selectors and extensions*, Michigan Math. J. 42 (1995), 555–571.
[25] S. Rolewicz, *On drop property*, Studia Math. 85 (1986), 27–35.
[26] J. J. Schäffer, *Inner diameter, perimeter, and girth of spheres*, Math. Ann. 173 (1967), 59–79 and 79–82.
[27] B. Sims, unpublished seminar notes, Kent State Univ., Kent, Ohio, 1986.
[28] C. Zanco and A. Zucchi, *Moduli of rotundity and smoothness for convex bodies*, Boll. Un. Mat. Ital. B (7) 7 (1993), 833–855.

Departamento de Matemáticas
Universidad de Extremadura
06071 Badajoz, Spain
E-mail: cabero@ba.unex.es

Institute of Mathematics
Technical University of Zielona Góra
Podgórna 50
65-246 Zielona Góra, Poland
E-mail: K.Przeslawski@im.pz.zgora.pl

Bâtiment 101—Mathématiques
Université de Lyon 1
Boulevard du 11 Novembre 1918
69622 Villeurbanne Cedex, France
E-mail: yost@jonas.univ-lyon1.fr

Received July 1, 1996
Revised version June 10, 1997
(3702)

Added in proof (October 1997). We learnt in July 1997 that there is some overlap between this article and the work of Ioan Şerb. In particular, the main result of §3, the equivalence of (i) and (ii) in Theorem 3.6, was proved independently by him in *A Day-Nordlander theorem for the tangential modulus of a normed space*, J. Math. Anal. Appl. 209 (1997), 381–391. In *On the behaviour of the tangential modulus of a Banach space II*, Mathematica (Cluj) 38 (61) (1996), 199–207, he proved, earlier than we did, Theorem 2.4(ii).

STUDIA MATHEMATICA 127 (1) (1998)

On the Djrbashian kernel of a Siegel domain

by

ELISABETTA BARLETTA and SORIN DRAGOMIR (Potenza)

Abstract. We establish an inversion formula for the M. M. Djrbashian & A. H. Karapetyan integral transform (cf. [6]) on the Siegel domain $\Omega_n = \{\zeta \in \mathbb{C}^n : \varrho(\zeta) > 0\}$, $\varrho(\zeta) = \operatorname{Im}(\zeta_1) - |\zeta'|^2$. We build a family of Kähler metrics of constant holomorphic curvature whose potentials are the ϱ^α-Bergman kernels, $\alpha > -1$, (in the sense of Z. Pasternak-Winiarski [20]) of Ω_n. We build an anti-holomorphic embedding of Ω_n in the complex projective Hilbert space $\mathbb{CP}(H^2_\alpha(\Omega_n))$ and study (in connection with work by A. Odzijewicz [18]) the corresponding transition probability amplitudes. The Genchev transform (cf. [9]) is shown to be well defined on $L^2(\Omega, \varrho^\alpha)$, for any strip $\Omega \subset \mathbb{C}$, and applied in a problem of approximation by holomorphic functions. Building on work by T. Mazur (cf. [15]) we prove the existence of a complete orthonormal system in $H^2_\alpha(\Omega_n)$ consisting of eigenfunctions of a certain explicitly defined operator V_a, $a \in B_n$.

1. Introduction. Let $\Omega \subset \mathbb{C}^n$ be an open set, $\Omega \neq \emptyset$. Let $W(\Omega)$ be the set of all *weights* on Ω (i.e. $\gamma \in W(\Omega)$ is a Lebesgue measurable function $\gamma : \Omega \to (0, \infty)$). For each $\gamma \in W(\Omega)$ let $L^2H(\Omega, \gamma)$ be the Hilbert space of all functions $f : \Omega \to \mathbb{C}$ for which $\|f\|_\gamma = (\int_\Omega |f|^2 \gamma\, dm)^{1/2} < \infty$, where dm is the Lebesgue measure in $\mathbb{R}^{2n}$. Let $L^2H(\Omega, \gamma)$ be the set of all functions in $L^2(\Omega, \gamma)$ which are holomorphic in Ω. A weight $\gamma \in W(\Omega)$ is *admissible* if for any $z \in \Omega$ there is a neighbourhood V_z of z in Ω and a constant $C_z > 0$ so that $\|\delta_w\|_\gamma \le C_z$ for any $w \in V_z$ (cf. [19], p. 112). Here $\delta_z(f) = f(z)$, $f \in L^2H(\Omega, \gamma)$. The set of all admissible weights on Ω is denoted by $AW(\Omega)$. If $\gamma \in AW(\Omega)$ then (cf. Proposition 2.1 of [19], p. 113) $L^2H(\Omega, \gamma)$ is a closed subspace of $L^2(\Omega, \gamma)$ and the evaluation functional δ_z is continuous on $L^2H(\Omega, \gamma)$ for any $z \in \Omega$. Hence, by the Riesz representation theorem, there is a unique function $K_\gamma(\cdot, z) \in L^2H(\Omega, \gamma)$ (called the γ-*Bergman kernel* of Ω) so that

$$f(z) = \int_\Omega f(\zeta)\overline{K_\gamma(\zeta, z)}\gamma(\zeta)\, dm(\zeta)$$

1991 *Mathematics Subject Classification*: Primary 32H10, 46E22; Secondary 53C55.
Key words and phrases: γ-Bergman kernel, reproducing kernel Hilbert space, Djrbashian kernel, transition probability amplitude, Genchev transform.