References

[1] H. Aimar, *Singular integrals and approximate identities on spaces of homogeneous type*, Trans. Amer. Math. Soc. 292 (1985), 135–153.

[2] S. Bloom, *Pointwise multipliers of weighted BMO spaces*, Proc. Amer. Math. Soc. 105 (1989), 950–960.

[3] R. R. Coifman et G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math. 242, Springer, Berlin, 1971.

[4] —, —, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[5] Y. Gotoh, *On multipliers for* BMO_ϕ *on general domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. 19 (1994), 339–354.

[6] S. Janson, *On functions with conditions on the mean oscillation*, Ark. Mat. 14 (1976), 189–196.

[7] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 14 (1961), 415–426.

[8] P. G. Lemarié, *Algèbres d'opérateurs et semi-groupes de Poisson sur un espace de nature homogène*, Publ. Math. Orsay 84-3 (1984).

[9] R. A. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. Math. 33 (1979), 257–270.

[10] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[11] —, *The equivalence of two conditions for weight functions*, Studia Math. 49 (1974), 101–106.

[12] E. Nakai, *On the restriction of functions of bounded mean oscillation to the lower dimensional space*, Arch. Math. (Basel) 43 (1984), 519–529.

[13] —, *Pointwise multipliers for functions of weighted bounded mean oscillation*, Studia Math. 105 (1993), 105–119.

[14] E. Nakai and K. Yabuta, *Pointwise multipliers for functions of bounded mean oscillation*, J. Math. Soc. Japan 37 (1985), 207–218.

[15] —, —, *Pointwise multipliers for functions of weighted bounded mean oscillation on spaces of homogeneous type*, Math. Japon. 46 (1997), to appear.

[16] S. Spanne, *Some function spaces defined using the mean oscillation over cubes*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 19 (1965), 593–608.

[17] D. A. Stegenga, *Bounded Toeplitz operators on* H^1 *and applications of the duality between* H^1 *and the functions of bounded mean oscillation*, Amer. J. Math. 98 (1976), 573–589.

[18] K. Yabuta, *Pointwise multipliers of weighted BMO spaces*, Proc. Amer. Math. Soc. 117 (1993), 737–744.

Department of Mathematics
Osaka Kyoiku University
Kashiwara, Osaka 582, Japan
E-mail: enakai@cc.osaka-kyoiku.ac.jp

Received April 5, 1996
Revised version February 10, 1997
(3653)

STUDIA MATHEMATICA 125 (1) (1997)

Spreading sequences in JT

by

HELGA FETTER and BERTA GAMBOA de BUEN (Guanajuato)

Abstract. We prove that a normalized non-weakly null basic sequence in the James tree space JT admits a subsequence which is equivalent to the summing basis for the James space J. Consequently, every normalized basic sequence admits a spreading subsequence which is either equivalent to the unit vector basis of l_2 or to the summing basis for J.

1. Introduction. We study subsequences of normalized basic sequences $\{x_i\}_{i=1}^{\infty}$ in the James tree space JT. Amemiya and Ito [1] proved that if $\{x_i\}_{i=1}^{\infty} \subset JT$ is weakly null then it has a subsequence which is equivalent to the unit vector basis of l_2.

We prove, following an idea of Hagler [7], that if $\{x_i\}_{i=1}^{\infty}$ is not weakly null then there is a subsequence equivalent to the summing basis for the James space J. In particular, this yields a classification of all the spreading models of JT, extending the work of Andrew [2] for the space J.

We thank the referee for his detailed revision and his valuable suggestions.

We first introduce some necessary notation and recall the definitions of J and JT constructed by James in [8] and [9] respectively. Most of the material referring to these spaces used here can be found in [5].

DEFINITION 1. The *James space* J is the Banach space of real sequences $b = (b_l)_{l=1}^{\infty}$ with the norm

$$\|b\| = \sup\Big(\sum_{\nu=1}^{M}\Big(\sum_{l=n(\nu)}^{\kappa(\nu)} b_l\Big)^2\Big)^{1/2},$$

where the sup is taken over all finite collections $S_1, \ldots, S_M$ of disjoint intervals of natural numbers with $S_\nu = \{n(\nu), n(\nu)+1, \ldots, \kappa(\nu)\}$.

1991 *Mathematics Subject Classification*: Primary 46B20.
Both authors supported partially by Conacyt Grant 0326P-E.

It follows that the sequence $\xi_n = (\underbrace{0,\ldots,0,1}_{n},0,\ldots)$ is a boundedly complete monotone basis for J which is called the *summing basis*.

DEFINITION 2. Let $\Upsilon = \{(n,i) : n, i \in \mathbb{N},\ 0 \le n < \infty,\ 0 \le i < 2^n\}$ be the standard binary tree. The elements of Υ are called *nodes*, n is the *level* of the node (n,i) and the nodes $(n+1,2i)$ and $(n+1,2i+1)$ are the *offspring* of (n,i).

A set $S = \{t_1,\ldots,t_n\} \subset \Upsilon$ is a *segment* if t_{j+1} is an offspring of t_j for $1 \le j < n$. Any $s \in S$ is called a *descendant* of t_1. If $s, t \in \Upsilon$ and there is no segment containing both of these nodes we say that they are *incomparable*. A *branch* is a set $B = \{t_1, t_2, \ldots\} \subset \Upsilon$ where $t_1 = (0,0)$ and t_{j+1} is an offspring of t_j for all j.

DEFINITION 3. The *James tree space* JT is the Banach space of real sequences $a = (a(n,i))_{(n,i)\in\Upsilon}$ with the norm

$$\|a\| = \sup\Big(\sum_{\nu=1}^{M}\Big(\sum_{(n,i)\in S_\nu} a(n,i)\Big)^2\Big)^{1/2},$$

where the sup is taken over all finite collections $S_1,\ldots,S_M$ of disjoint segments in Υ.

The sequence $\{\eta(n,i)\}_{(n,i)\in\Upsilon}$ with $\eta(n,i) = (a(m,j))_{(m,j)\in\Upsilon}$ where

$$a(m,j) = \begin{cases} 1 & \text{if } (m,j) = (n,i),\\ 0 & \text{otherwise,}\end{cases}$$

is a boundedly complete monotone basis for JT.

We will use the following notation: Γ will stand for the set of branches; for every node $t \in \Upsilon$, $P_t : JT \to JT$ is the projection defined by

$$P_t z = \sum_{\{s\in\Upsilon : s\ge t\}} \langle \eta_s^*, z\rangle \eta_s,$$

where $\{\eta_s^*\}_{s\in\Upsilon}$ is the biorthogonal sequence to $\{\eta_s\}_{s\in\Upsilon}$ and $s \ge t$ means that s is a descendant of t. For every branch $B \subset \Upsilon$, $f_B : JT \to \mathbb{R}$ and $P_B : JT \to JT$ are the functional and projection, respectively, given by

$$f_B(z) = \sum_{t\in B}\langle \eta_t^*, z\rangle \quad\text{and}\quad P_B(z) = \sum_{t\in B}\langle \eta_t^*, z\rangle \eta_t$$

and for a segment S, f_S is defined similarly. It is not difficult to show that for every branch B, $\{P_B(\eta_t)\}_{t\in B}$ is isometrically equivalent to $\{\xi_n\}_{n=1}^\infty$.

It is clear that $\|f_B\| = \|f_S\| = \|P_B\| = \|P_t\| = 1$.

Observe that if $a = (a(n,i))_{(n,i)\in\Upsilon}$ one can write

$$\|a\| = \sup\Big(\sum_{\nu=1}^{M}(f_{S_\nu}(a))^2\Big)^{1/2},$$

where the sup is taken over all finite collections $S_1,\ldots,S_M$ of disjoint segments in Υ.

As we pointed out, we will need the following result due to Amemiya and Ito:

THEOREM 4. *Suppose $\{z_l\}_{l=1}^\infty$ is a normalized weakly null sequence in JT. Then for any $\varepsilon > 0$ there is a subsequence $\{z_{m_l}\}_{l=1}^\infty$ so that if $\sum_{l=1}^\infty b_l^2 < \infty$, then*

$$(1-\varepsilon)\sum_{l=1}^\infty b_l^2 \le \Big\|\sum_{l=1}^\infty b_l z_{m_l}\Big\|^2 \le (2+\varepsilon)\sum_{l=1}^\infty b_l^2.$$

More recently, G. Berg [4] proved that $2+\varepsilon$ can be replaced by $\sqrt{2}+\varepsilon$ and that this result is the best possible.

2. Non-weakly null sequences in JT.

In this section we prove our main result:

THEOREM 5. *Let $\{\zeta_l\}_{l=1}^\infty \subset JT$ be a normalized sequence with no weakly convergent subsequences. Then $\{\zeta_l\}_{l=1}^\infty$ has a subsequence equivalent to the summing basis $\{\xi_l\}_{l=1}^\infty$ in J.*

We start with proving a particular case from which the general case will follow.

PROPOSITION 6. *Let $\{z_l\}_{l=1}^\infty$ be a block basic sequence so that for every $l \in \mathbb{N}$,*

$$(1)\qquad \|z_l\| = 1,\qquad z_l = \sum_{n=p_l}^{q_l}\sum_{i=0}^{2^n-1} a(n,i)\eta(n,i) \quad \text{where } p_l \le q_l < p_{l+1}.$$

Suppose that there exist a sequence $\{r_j\}_{j=1}^\infty$ in $\mathbb{N}$ and for every l a finite sequence of different nodes of level p_l:

$$t_{11}^l,\ldots,t_{1r_1}^l, t_{21}^l,\ldots,t_{2r_2}^l,\ldots,t_{l1}^l,\ldots,t_{lr_l}^l,$$

so that for a fixed pair (j,i), the set $\{t_{ji}^l\}_{l=1}^\infty$ is contained in a branch B_{ji}. Suppose further that:

(i) *$z_l = z(l,1)+\ldots+z(l,l)$, $l = 1,2,\ldots,$ where for $j = 1,\ldots,l$,*

$$z(l,j) = \sum_{i=1}^{r_j} P_{t_{ji}^l}(z_l).$$

(ii) $|f_B(z(l,j))| \le 1/2^{j-1}$ *for every* $B \in \Gamma$, $l = 1, 2, \ldots, j = 1, \ldots, l$.
(iii) $f_{B_{ji}}(z_l) = c_{ji}$ *for* $j = 1, \ldots, l$, $i = 1, \ldots, r_j$, $l = 1, 2, \ldots$
(iv) *There exists a pair* (j_0, i_0) *such that* $c_{j_0 i_0} = c \ne 0$.
(v) $\sum_{j=1}^{\infty} r_j/2^{2j} \le 1$.

Then $\{z_l\}_{l=1}^{\infty}$ *has a subsequence equivalent to the summing basis* $\{\xi_n\}_{n=1}^{\infty}$ *of* J.

Proof. For $\nu = 1, \ldots, M$ let $\lambda(\nu) \le \mu(\nu) < \lambda(\nu+1)$. Define
$$S_\nu = \{t \in B_{j_0 i_0} : p_{\lambda(\nu)} \le \text{level}(t) \le q_{\mu(\nu)}\}.$$
Then $\{S_\nu\}_{\nu=1}^{M}$ is a collection of disjoint segments in JT and we have, for $\sum_{l=1}^{\infty} b_l z_l \in JT$,
$$c^2 \sum_{\nu=1}^{M} \Big(\sum_{l=\lambda(\nu)}^{\mu(\nu)} b_l \Big)^2 = \sum_{\nu=1}^{M} \Big(f_{B_{j_0 i_0}} \Big(\sum_{l=\lambda(\nu)}^{\mu(\nu)} b_l z_l \Big) \Big)^2$$
$$= \sum_{\nu=1}^{M} \Big(f_{S_\nu} \Big(\sum_{l=1}^{\infty} b_l z_l \Big) \Big)^2 \le \Big\| \sum_{l=1}^{\infty} b_l z_l \Big\|^2.$$
Thus by the definition of the norms in J and JT,
$$|c| \Big\| \sum_{l=1}^{\infty} b_l \xi_l \Big\| \le \Big\| \sum_{l=1}^{\infty} b_l z_l \Big\|.$$
Furthermore, the same estimate holds for every subsequence of $\{z_l\}_{l=1}^{\infty}$.

To get the other inequality, we choose a sequence $1 = m_0 < m_1 < \ldots$ so that for $l \ge 1$,
$$\sum_{j=m_l}^{\infty} \frac{1}{2^{2j}} \le \frac{1}{2^{q_{m_{l-1}}}} \cdot \frac{1}{2^l} \tag{2}$$
and
$$\sum_{j=m_l}^{\infty} \sum_{i=1}^{r_j} c_{ij}^2 \le \frac{1}{2^{q_{m_{l-1}}}} \cdot \frac{1}{2^l}. \tag{3}$$
The latter is possible, since for every $l \in \mathbb{N}$,
$$\sum_{j=1}^{l} \sum_{i=1}^{r_j} c_{ij}^2 = \sum_{j=1}^{l} \sum_{i=1}^{r_j} |f_{B_{ji}}(z_l)|^2 \le \|z_l\|^2 = 1. \tag{4}$$

We will show that $\| \sum_{l=1}^{\infty} b_l z_{m_l} \|^2 \le 90 \| \sum_{l=1}^{\infty} b_l \xi_l \|^2$, which will complete the proof. (Clearly, this estimate is not the best possible).

To this end let $S_1, \ldots, S_M$ be disjoint segments in Υ and for $\nu = 1, \ldots, M$ let $n(\nu)$ and $\kappa(\nu)$ be such that $n(\nu) \le \kappa(\nu)$ and S_ν intersects the supports of $z_{m_{n(\nu)}}$ and of $z_{m_{\kappa(\nu)}}$, but S_ν does not intersect the support of z_{m_l} if $l < n(\nu)$ or $l > \kappa(\nu)$. Here by the *support* of z_l we mean the set of nodes t such that $\langle \eta_t^*, z_l \rangle \ne 0$, and we will denote it by $\operatorname{supp} z_l$.

We will use the following decomposition:
$$\Big(f_{S_\nu} \Big(\sum_{l=1}^{\infty} b_l z_{m_l} \Big) \Big)^2 \le 3 b_{m_{n(\nu)}}^2 (f_{S_\nu}(z_{m_{n(\nu)}}))^2$$
$$+ 3 \Big(\sum_{l=n(\nu)+1}^{\kappa(\nu)-1} b_l f_{S_\nu}(z_{m_l}) \Big)^2 + 3 b_{m_{\kappa(\nu)}}^2 (f_{S_\nu}(z_{m_{\kappa(\nu)}}))^2.$$

Since the S_ν's are all disjoint, by the definition of the norm in JT,
$$\sum_{\nu=1}^{M} b_{m_{n(\nu)}}^2 (f_{S_\nu}(z_{m_{n(\nu)}}))^2 \le \sum_{n=1}^{\infty} b_n^2 \sum_{\{\nu : n(\nu) = n\}} (f_{S_\nu}(z_{m_n}))^2$$
$$\le \sum_{n=1}^{\infty} b_n^2 \|z_{m_n}\|^2 = \sum_{n=1}^{\infty} b_n^2.$$
One obtains a similar inequality for $\sum_{\nu=1}^{M} b_{m_{\kappa(\nu)}}^2 (f_{S_\nu}(z_{m_{\kappa(\nu)}}))^2$. Thus
$$3 \sum_{\nu=1}^{M} (b_{m_{n(\nu)}}^2 (f_{S_\nu}(z_{m_{n(\nu)}}))^2 + b_{m_{\kappa(\nu)}}^2 (f_{S_\nu}(z_{m_{\kappa(\nu)}}))^2) \le 6 \sum_{n=1}^{\infty} b_n^2. \tag{5}$$
It remains to estimate the term $(\sum_{l=n(\nu)+1}^{\kappa(\nu)-1} b_l f_{S_\nu}(z_{m_l}))^2$.

Let ν be fixed and $n(\nu) + 1 \le l(1) < \ldots < l(k)$ be such that for $r = 1, \ldots, k$, there exists $t_{j(r)i(r)}^{m_{l(r)}} \in S_\nu$ but $S_\nu \cap \operatorname{supp} z_{m_{l(r)}}$ is not contained in $B_{j(r)i(r)}$, and so that for $l(r) < l < l(r+1)$, $r = 1, \ldots, k-1$, or $l > l(k)$, $S_\nu \cap \operatorname{supp} z_{m_l}$ is either empty or contained in B_{ji} for some (j,i).

Let $n(\nu) + 1 \le \varrho(1) \le \mu(1) < \varrho(2) \le \mu(2) \ldots < \varrho(u) \le \mu(u)$ be so that for $\sigma = 1, \ldots, u$ there is $(j(\sigma), i(\sigma))$ with $S_\nu \cap \operatorname{supp} z_{m_l} \subset B_{j(\sigma)i(\sigma)}$ if $\varrho(\sigma) \le l \le \mu(\sigma)$, $B_{j(\sigma)i(\sigma)} \ne B_{j(\sigma')i(\sigma')}$ if $\sigma \ne \sigma'$ and for $\mu(\sigma) < l < \varrho(\sigma+1)$ or $\mu(u) < l$, $S_\nu \cap \operatorname{supp} z_{m_l}$ is not contained in B_{ji} for any j.

Thus for $n(\nu)+1 \le l \le \kappa(\nu)-1$ either there exists σ with $\varrho(\sigma) \le l \le \mu(\sigma)$ or there is $1 \le r \le k$ with $l = l(r)$.

For $\varrho(\sigma) \le l \le \mu(\sigma)$ it follows that $|f_{S_\nu}(z_{m_l})| = |c_{j(\sigma)i(\sigma)}|$ and since $t_{11}^{m_{\mu(\sigma)}}, \ldots, t_{m_{\mu(\sigma)} r_{\mu(\sigma)}}^{m_{\mu(\sigma)}}$ are all of level $p_{m_{\mu(\sigma)}}$, we see that for $\sigma < u$,
$$j(\sigma+1) > m_{\mu(\sigma)}. \tag{6}$$

Also observe that if $t_{j_1 i_1}^{l_1}$ is a descendant of $t_{j_0 i_0}^{l_0}$ and $(j_1, i_1) \ne (j_0, i_0)$, then $t_{j_1 i_1}^{l_1} \notin B_{ji}$ for $i = 1, \ldots, r_j$, $j = 1, \ldots, l_0$ and thus $j_1 \ge l_0 + 1$, and so by (ii),
$$|f_B(P_{t_{j_1 i_1}^{l_1}}(z_{l_1}))| \le 1/2^{l_0} \quad \text{for every } B \in \Gamma. \tag{7}$$

Thus, since for $r=2,\ldots,k$, $t^{m_{l(r)}}_{j(r)i(r)}$ is a descendant of $t^{m_{l(r-1)}}_{j(r-1)i(r-1)}$, we have

$$|f_{S_\nu}(z_{m_{l(r)}})|\le 1/2^{m_{l(r-1)}}. \tag{8}$$

Now

$$\Big(\sum_{l=n(\nu)+1}^{\kappa(\nu)-1} b_l f_{S_\nu}(z_{m_l})\Big)^2 \le 4b^2_{l(1)}(f_{S_\nu}(z_{m_{l(1)}}))^2 + 4\Big(\sum_{r=2}^{k} b_{l(r)} f_{S_\nu}(z_{m_{l(r)}})\Big)^2 + 4\Big(\sum_{l=\varrho(1)}^{\mu(1)} b_l f_{S_\nu}(z_{m_l})\Big)^2 + 4\Big(\sum_{\sigma=2}^{u}\sum_{l=\varrho(\sigma)}^{\mu(\sigma)} b_l f_{S_\nu}(z_{m_l})\Big)^2.$$

Using (2) and (8) since $l(1)>n(\nu)$, we get

$$\Big(\sum_{r=2}^{k} b_{l(r)} f_{S_\nu}(z_{m_{l(r)}})\Big)^2 \le \sum_{l=1}^{\infty} b_l^2 \sum_{r=2}^{k}\frac{1}{2^{2m_{l(r-1)}}} \le \sum_{l=1}^{\infty} b_l^2 \sum_{j=m_{n(\nu)+1}}^{\infty}\frac{1}{2^{2j}} \le \sum_{l=1}^{\infty} b_l^2 \frac{1}{2^{q_{m_{n(\nu)}}}}\cdot\frac{1}{2^{n(\nu)+1}}. \tag{9}$$

By (3), (6) and the definition of the norm in J,

$$\begin{aligned}\Big(\sum_{l=\varrho(1)}^{\mu(1)} b_l f_{S_\nu}(z_{m_l})\Big)^2 &+ \Big(\sum_{\sigma=2}^{u}\sum_{l=\varrho(\sigma)}^{\mu(\sigma)} b_l f_{S_\nu}(z_{m_l})\Big)^2\\
&\le c^2_{j(1)i(1)}\Big(\sum_{l=\varrho(1)}^{\mu(1)} b_l\Big)^2 + \sum_{\sigma=2}^{u} c^2_{j(\sigma)i(\sigma)}\sum_{\sigma=2}^{u}\Big(\sum_{l=\varrho(\sigma)}^{\mu(\sigma)} b_l\Big)^2\\
&\le c^2_{j(1)i(1)}\Big(\sum_{l=\varrho(1)}^{\mu(1)} b_l\Big)^2 + \sum_{j=m_{n(\nu)+1}}^{\infty}\sum_{i=1}^{r_j} c^2_{ji}\Big\|\sum_{l=1}^{\infty} b_l\xi_l\Big\|^2\\
&\le c^2_{j(1)i(1)}\Big(\sum_{l=\varrho(1)}^{\mu(1)} b_l\Big)^2 + \frac{1}{2^{q_{m_{n(\nu)}}}}\cdot\frac{1}{2^{n(\nu)+1}}\Big\|\sum_{l=1}^{\infty} b_l\xi_l\Big\|^2.\end{aligned}$$

Thus by (ii), (2), (9) and the above, if we write $l(\nu)$, $\varrho(\nu)$, $\mu(\nu)$, $j(\nu)$ and

$i(\nu)$ instead of $l(1)$, $\varrho(1)$, $\mu(1)$, $j(1)$ and $i(1)$ respectively, we get

$$\begin{aligned}\Big(\sum_{l=n(\nu)+1}^{\kappa(\nu)-1} b_l f_{S_\nu}(z_{m_l})\Big)^2 &\le 4b^2_{l(\nu)}\frac{1}{2^{2(j(\nu)-1)}} + 4\sum_{l=1}^{\infty} b_l^2\frac{1}{2^{q_{m_{n(\nu)}}}}\cdot\frac{1}{2^{n(\nu)+1}}\\
&\quad + 4c^2_{j(\nu)i(\nu)}\Big(\sum_{l=\varrho(\nu)}^{\mu(\nu)} b_l\Big)^2\\
&\quad + 4\frac{1}{2^{q_{m_{n(\nu)}}}}\cdot\frac{1}{2^{n(\nu)+1}}\Big\|\sum_{l=1}^{\infty} b_l\xi_l\Big\|^2 \le 4b^2_{l(\nu)}\frac{1}{2^{2(j(\nu)-1)}}\\
&\quad + 8\frac{1}{2^{q_{m_{n(\nu)}}}}\cdot\frac{1}{2^{n(\nu)+1}}\Big\|\sum_{l=1}^{\infty} b_l\xi_l\Big\|^2 + 4c^2_{j(\nu)i(\nu)}\Big(\sum_{l=\varrho(\nu)}^{\mu(\nu)} b_l\Big)^2.\end{aligned}$$

Since for every n the only possible nodes such that $t^{m_{l(\nu)}}_{j(\nu)i(\nu)}\in S_\nu$ with $l(\nu)=n$ are

$$t^{m_n}_{11},\ldots,t^{m_n}_{1r_1},t^{m_n}_{21},\ldots,t^{m_n}_{2r_2},\ldots,t^{m_n}_{m_n1},\ldots,t^{m_n}_{m_nr_{m_n}},$$

summing these estimates over the segments $S_1,\ldots,S_M$ using (v) and (4) we get

$$\begin{aligned}\sum_{\nu=1}^{M}\Big(\sum_{l=n(\nu)+1}^{\kappa(\nu)-1} b_l f_{S_\nu}(z_{m_l})\Big)^2 &\le 4\sum_{n=1}^{\infty}\sum_{\{\nu:l(\nu)=n\}} b^2_{l(\nu)}\frac{1}{2^{2(j(\nu)-1)}}\\
&\quad + 8\sum_{\nu=1}^{M}\frac{1}{2^{q_{m_{n(\nu)}}}}\cdot\frac{1}{2^{n(\nu)+1}}\Big\|\sum_{l=1}^{\infty} b_l\xi_l\Big\|^2\\
&\quad + 4\sum_{j=1}^{\infty}\sum_{i=1}^{r_j}\sum_{\{\nu:(j(\nu),i(\nu))=(j,i)\}} c^2_{j(\nu)i(\nu)}\Big(\sum_{l=\varrho(\nu)}^{\mu(\nu)} b_l\Big)^2\\
&\le 4\sum_{n=1}^{\infty} b_n^2\Big(\sum_{j=1}^{\infty}\frac{r_j}{2^{2(j-1)}}\Big)\\
&\quad + 8\Big\|\sum_{l=1}^{\infty} b_l\xi_l\Big\|^2 + 4\sum_{j=1}^{\infty}\sum_{i=1}^{r_j} c^2_{ji}\Big\|\sum_{l=1}^{\infty} b_l\xi_l\Big\|^2\\
&\le 28\Big\|\sum_{l=1}^{\infty} b_l\xi_l\Big\|^2.\end{aligned}$$

Finally, using (5) we have

$$\Big\|\sum_{l=1}^{\infty} b_l z_{m_l}\Big\|^2 \le 90\Big\|\sum_{l=1}^{\infty} b_l \xi_l\Big\|^2,$$

as desired.

Proof of Theorem 5. Since the basis $\{\eta_t\}_{t\in\Upsilon}$ is boundedly complete, using a standard perturbation argument we may assume that $\{\zeta_l\}_{l=1}^{\infty}$ has a subsequence equivalent to $\{z+z_l\}_{l=1}^{\infty}$ for some $z\in JT$ where $\{z_l\}_{l=1}^{\infty}$ is a block basic sequence which is not weakly null. It is readily seen that we need only prove the following: if $\{z_l\}_{l=1}^{\infty}$ is a normalized block basis of $\{\eta_t\}_{t\in\Upsilon}$ with $\lim_l f_B(z_l)=c\neq 0$ for some $B\in\Gamma$ then a subsequence of $\{z_l\}_{l=1}^{\infty}$ is equivalent to the summing basis of J and thus the corresponding subsequence of $\{z+z_l\}_{l=1}^{\infty}$ is equivalent to the summing basis of J as well. We may also assume that $\{z_l\}_{l=1}^{\infty}$ is of the form (1) as in Proposition 6.

Let B_t denote any $B\in\Gamma$ with $t\in B_t$ and let

$$A(l,0)=\{t\in\Upsilon:\operatorname{level}(t)=p_l \text{ and } f_{B_t}(z_l)=0 \text{ for all } B_t\in\Gamma\},$$

$$A(l,j)=\{t\in\Upsilon:\operatorname{level}(t)=p_l \text{ and } |f_{B_t}(z_l)|\le 1/2^{j-1} \text{ for all } B_t\in\Gamma$$
$$\text{and } \exists B_t\in\Gamma \text{ so that } |f_{B_t}(z_l)|>1/2^j\} \quad \text{for } j\ge 1.$$

Then $\bigcup_{j=0}^{\infty}A(l,j)=\{t\in\Upsilon:\operatorname{level}(t)=p_l\}$ and for $j\ge1$ the cardinality of $A(l,j)$, denoted by $\#A(l,j)$, is less than or equal to 2^{2j}. In fact, if $j\ge1$, $t_1,\dots,t_r\in A(l,j)$ are different nodes and $B_{t_1},\dots,B_{t_r}\in\Gamma$ are such that $|f_{B_{t_i}}(z_l)|>1/2^j$, then

$$\frac{r}{2^{2j}}<\sum_{i=1}^{r}|f_{B_{t_i}}(z_l)|^2\le\|z_l\|^2\le 1.$$

Now we will apply repeatedly the fact that if $\{t_n\}_{n=1}^{\infty}$ is a sequence of different nodes in Υ, then either there exists a subsequence contained in a branch or there is a subsequence $\{t_{m_n}\}_{n=1}^{\infty}$ so that t_{m_n} is incomparable with $t_{m_{n'}}$ if $n\neq n'$.

Consider the sequence $\{A(l,1)\}_{l=1}^{\infty}$. Since $\#A(l,1)\le 4$ for $l=1,2,\dots$, there exist $r_1\le 4$ and an infinite set N_1 such that $N_1\subset\mathbb{N}$ and $\#A(l,1)=r_1$ for every $l\in N_1$.

If $r_1=0$, let $N_1''=N_1'=N_1$.

If $r_1>0$, then let $\{t_{11}^l,\dots,t_{1r_1}^l\}=A(l,1)$ for every $l\in N_1$. Now let $N_1'\subset N_1$ be an infinite set such that for every fixed i, $1\le i\le r_1$, either the sequence $\{t_{1i}^l\}_{l\in N_1'}$ is contained in a branch called B_{1i} or t_{1i}^l is incomparable with $t_{1i}^{l'}$ if $l,l'\in N_1'$ and $l\neq l'$. Let further

$$I_1=\{1\le i\le r_1:\exists B_{1i}\in\Gamma,\ \{t_{1i}^l\}_{l\in N_1'}\subset B_{1i}\}.$$

If $I_1=\emptyset$, let $N_1''=N_1'$.

If $I_1\neq\emptyset$, let N_1'' be an infinite subset of N_1' such that for $i\in I_1$, $\lim_{l\in N_1''}f_{B_1}(z_l)=c_{1i}$ exists. This is possible because $\{f_{B_{1i}}(z_l)\}_{l=1}^{\infty}$ is a bounded sequence.

Proceeding inductively we construct a sequence $\{r_k\}_{k=1}^{\infty}\subset\mathbb{N}$ and sets N_k,N_k',N_k'', $k=1,2,\dots$, $I_1,I_2,\dots\subset\mathbb{N}$, so that

(a) $\mathbb{N}\supset N_1\supset N_1'\supset N_1''\supset N_2\supset\dots$, N_k is infinite for $k=1,2,\dots$

(b) For $l\in N_k$, $k=1,2,\dots$, either $A(l,k)=\emptyset$ or we may write $A(l,k)=\{t_{k1}^l,\dots,t_{kr_k}^l\}$.

(c) If $A(l,k)\neq\emptyset$ and $1\le i\le r_k$, then either $\{t_{ki}^l\}_{l\in N_k'}$ is contained in a branch B_{ki} or t_{ki}^l is incomparable with $t_{ki}^{l'}$ for $l,l'\in N_k'$ and $l\neq l'$.

(d) $I_k=\{1\le i\le r_k:\{t_{ki}^l\}_{l\in N_k'}\subset B_{ki}\}$.

(e) If $i\in I_k$, then

$$\lim_{\substack{l\to\infty\\ l\in N_k''}} f_{B_{ji}}(z_l)=c_{ji} \quad \text{exists}$$

for $j=1,\dots,k$, $i\in I_j$.

Let $m_1<m_2<\dots$ where $m_l\in N_l''$ for all i. By passing to a subsequence of $\{z_{m_l}\}_{l=1}^{\infty}$ and perturbing we may assume that

$$z_{m_l}=w_l+u_l$$

where $w_l=w(l,1)+\dots+w(l,l)$ and for $j=1,\dots,l$,

(i) $w(l,j)=\displaystyle\sum_{\{i\le r_j:i\in I_j\}}P_{t_{ji}^{m_l}}(z_{m_l})=\sum_{\{t=t_{ji}^{m_l}\in A(m_l,j):i\in I_j\}}P_t(z_{m_l})$.

(ii) $|f_B(w(l,j))|\le 1/2^{j-1}$ for every $B\in\Gamma$.

(iii) $f_{B_{ji}}(w_l)=c_{ji}$ for $j=1,2,\dots,l$, $i\in I_j$, $l=1,2,\dots$

Also, since $\|w_l\|\le1$, if $s_j=\#\{i:i\in I_j\}=\#\{t_{ji}^{m_l}\in A(m_l,j):i\in I_j\}$ then we have

(iv) $\displaystyle\sum_{j=1}^{l}\frac{s_j}{2^{2j}}\le\|w_l\|^2\le 1.$

From our construction we infer that $\{u_l\}_{l=1}^{\infty}$ is a bounded block basis of $\{\eta_t\}_{t\in\Upsilon}$ which satisfies, for all $B\in\Gamma$,

$$\lim_l f_B(u_l)=0.$$

Since it is known that a bounded block basis $\{x_n\}_{n=1}^{\infty}$ of $\{\eta_t\}_{t\in\Upsilon}$ is weakly null if and only if $\lim_i f_B(x_i)=0$, it follows that $\{u_l\}_{l=1}^{\infty}$ is weakly null.

By Proposition 6 and Theorem 4 there exists a further subsequence $\{z_{m_l}'\}_{l=1}^{\infty}$ with $z_{m_l}'=w_l'+u_l'$ so that $\{w_l'\}_{l=1}^{\infty}$ is equivalent to the summing basis for J and $\{u_l'\}_{l=1}^{\infty}$ is either norm null or equivalent to the unit vector basis of l_2. The theorem now follows easily.

As an application we are able to describe the spreading basic sequences in JT and its spreading models.

Recall that a basic sequence is *spreading* if it is equivalent to all of its subsequences. The theory of spreading models can be found in [3].

COROLLARY 7. *Let $\{x_l\}_{l=1}^{\infty}$ be a normalized basic sequence in JT. Then $\{x_l\}_{l=1}^{\infty}$ has a subsequence which is equivalent to either the summing basis for J or to the unit vector basis of l_2. In particular, these two spaces are the only spreading models of JT and every normalized basic sequence in JT admits a spreading subsequence.*

Theorem 5 and the corollary apply to the space $(J \oplus J \oplus \ldots)_{l_2}$ since the latter is a subspace of JT. Thus the above results improve those previously proved by the authors in [6].

References

[1] I. Amemiya and T. Ito, *Weakly null sequences in James spaces on trees*, Kodai Math. J. 4 (1981), 418–425.

[2] A. Andrew, *Spreading basic sequences and subspaces of James' quasireflexive space*, Math. Scand. 48 (1981), 109–118.

[3] B. Beauzamy et J.-T. Lapresté, *Modèles étalés des espaces de Banach*, Travaux en Cours, Hermann, Paris 1984.

[4] G. Berg, *On James spaces*, Ph.D. thesis, The University of Texas, Austin, Texas, 1996.

[5] H. Fetter and B. Gamboa de Buen, *The James Forest*, London Math. Soc. Lecture Note Ser. 236, Cambridge Univ. Press, 1997.

[6] —, —, *The spreading models of the space $\mathcal{J} = (J \oplus J \oplus \ldots)_{l_2}$*, Bol. Soc. Mat. Mexicana 2 (3) (1996), 139–146.

[7] J. Hagler, *A counterexample to several questions about Banach spaces*, Studia Math. 60 (1977), 289–308.

[8] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. 52 (1950), 518–527.

[9] —, *A separable somewhat reflexive Banach space with nonseparable dual*, Bull. Amer. Math. Soc. 80 (1974), 738–743.

Centro de Investigación en Matemáticas
Apartado Postal 402
36000 Guanajuato, Guanajuato
México
E-mail: fetter@fractal.cimat.mx
gamboa@fractal.cimat.mx

Received July 4, 1996
Revised version February 19, 1997
(3705)

STUDIA MATHEMATICA 125 (1) (1997)

Estimates of Fourier transforms in Sobolev spaces

by

V. I. KOLYADA (Odessa)

Abstract. We investigate the Fourier transforms of functions in the Sobolev spaces $W_1^{r_1,\ldots,r_n}$. It is proved that for any function $f \in W_1^{r_1,\ldots,r_n}$ the Fourier transform $\widehat{f}$ belongs to the Lorentz space $L^{n/r,1}$, where $r = n(\sum_{j=1}^n 1/r_j)^{-1} \le n$. Furthermore, we derive from this result that for any mixed derivative $D^s f$ $(f \in C_0^\infty,\ s = (s_1, \ldots, s_n))$ the weighted norm $\|(D^s f)^\wedge\|_{L^1(w)}$ $(w(\xi) = |\xi|^{-n})$ can be estimated by the sum of L^1-norms of all pure derivatives of the same order. This gives an answer to a question posed by A. Pełczyński and M. Wojciechowski.

1. Introduction. For any function $f \in L^1(\mathbb{R}^n)$ its Fourier transform is the function $\widehat{f}$ defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n.$$

For the Fourier transform of a function $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$ (see [13], Ch. 1), we have the following classical inequalities ([2], Ch. 1):

• *the Hausdorff–Young inequality*

$$\|\widehat{f}\|_{p'} \le \|f\|_p, \quad 1 \le p \le 2,\ \frac{1}{p} + \frac{1}{p'} = 1; \tag{1}$$

• *the Hardy–Littlewood–Paley inequality*

$$\Big(\int_{\mathbb{R}^n} |\xi|^{n(p-2)} |\widehat{f}(\xi)|^p\, d\xi\Big)^{1/p} \le c\|f\|_p, \quad 1 < p \le 2. \tag{2}$$

It is well known that (2) is not true for $p = 1$, $n \ge 1$. On the other hand, by Hardy's inequality we know that for any $f \in H^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)|}{|\xi|^n}\, d\xi \le c\|f\|_{H^1}. \tag{3}$$

Furthermore, the inequality (2) can be strengthened in terms of rearrangements.

1991 *Mathematics Subject Classification*: 42B15, 46E35, 46E40.