STUDIA MATHEMATICA 125 (1) (1997)

Bernstein and van der Corput–Schaake type inequalities on semialgebraic curves

by

M. BARAN and W. PLEŚNIAK (Kraków)

Abstract. We show that in the class of compact, piecewise C^1 curves K in $\mathbb{R}^n$, the semialgebraic curves are exactly those which admit a Bernstein (or a van der Corput–Schaake) type inequality for the derivatives of (the traces of) polynomials on K.

0. Introduction. The Markov and Bernstein inequalities estimating the growth of the derivative of a polynomial in terms of its values on an interval or a circle are still a fascinating object of investigations. For an interesting account of these classical results and their refinements in the one-dimensional case, we refer the reader to [MiRa]. A corresponding theory in the several variables case is relatively new and was essentially developed in the last decade by W. Pawłucki and W. Pleśniak ([PP1], [PP2], [Pl3]), P. Goetgheluck ([Goe1], [Goe2]), A. Jonsson ([Jon1], [Jon2]), J. Siciak ([Si3]), M. Baran ([Ba1]–[Ba4], [BaPl), L. Bos, N. Levenberg, P. Milman, B. A. Taylor ([BM1], [BM2], [BLT], [BLMT]), A. Zeriahi ([Z]), A. Goncharov ([Gon]) and others. In particular, it was found that Markov's inequality is closely connected with the classical problem of the existence of a continuous linear operator extending C^∞ functions from a (sufficiently regular) compact set in $\mathbb{R}^n$ to the whole $\mathbb{R}^n$ (see [PP2], [Pl3], [Z], [BM2], [Gon]). As concerns Bernstein's inequality, following Baran's papers [Ba2], [Ba3], it is closely related to the equilibrium measure of a compact set in $\mathbb{R}^n$. These two discoveries emphasize once more the important role of both the Markov and Bernstein inequalities in real and complex analysis.

1991 *Mathematics Subject Classification*: 41A17, 41A25, 14P10, 14P05, 32C10, 32F05, 32F07.

Key words and phrases: Bernstein and van der Corput–Schaake type inequalities, semialgebraic curves, algebraic manifolds, pluricomplex Green function, Lipschitz functions.

Research partially supported by grant No 2 P03A 057 08 from KBN (Committee for Scientific Research). The first author has also been supported by the European Programme PECO of the French Government.

Markov or Bernstein type inequalities on curves in $\mathbb{R}^n$ do not directly follow from their versions for "big" subsets of $\mathbb{R}^n$, and the paper [FeNa] seems to be a first attempt at such investigations. This paper is motivated by the papers [BLT] and [BLMT] which are important from the Constructive Theory of Functions point of view, since they establish a characterization of an algebraic submanifold K of $\mathbb{R}^n$ with the aid of a tangential Markov inequality as well as a Bernstein theorem for Lipschitz classes of functions. Following [BLMT], a compact m-dimensional submanifold K of $\mathbb{R}^n$ with boundary $(1 \le m \le n-1)$ is said to admit a *tangential Markov inequality* with exponent one if there exists a positive constant M depending only on K such that for all polynomials p,

$$\|D_T p\|_K \le M(\deg p)\|p\|_K.$$

Here $D_T p$ denotes any tangential derivative of p along K and $\|p\|_K := \sup|p|(K)$. According to [BLT] and [BLMT], a $\mathcal{C}^1$ submanifold K of $\mathbb{R}^n$ admits such an inequality if and only if K is algebraic. However, the reader of those two papers has to be careful, since, in both, the $\mathcal{C}^1$ regularity of a given submanifold should probably be replaced by $\mathcal{C}^\infty$ regularity. Indeed, the following simple example (due to the first author) shows that $\mathcal{C}^1$ regularity does not suffice for an algebraic curve K to admit a tangential Markov inequality.

Example 0.1. Consider the algebraic curve

$$K_l = \{(x,y) \in \mathbb{R}^2 : y^{2l} = (1-x^2)^{2l-1}\}, \qquad l \in \mathbb{N}.$$

Then, for the polynomials $P_k(x,y) = \left[\frac{1}{k}T_k'(x)\right]^2 y$, where T_k is the Chebyshev polynomial $\cos(k \arccos x)$ for $x \in [-1,1]$, one has

$$|D_T P_k(1,0)| = |D_2 P_k(1,0)| = k^2 \ge k^{2-1/l}\|P_k\|_{K_l}.$$

In connection with the above example, the question arises about the size of the exponent ℓ with which a curve K satisfies the inequality

$$(M_T) \qquad |D_T p(x)| \le M(\deg p)^\ell \|p\|_K$$

for all polynomials p. If K is not algebraic, then, in general, K need not satisfy such an inequality with any finite ℓ. For an example, see [BLMT, Section 6]. On the other hand, it was shown in [BLMT, Proposition 6.1] that if K is an arc of a smooth algebraic curve $\widetilde{K}$ in $\mathbb{R}^2$, then K satisfies (M_T) with exponent $\ell = 2$. In the proof of that proposition the authors claim that $\widetilde{K} \subset \{(x,y) \in \mathbb{R}^2 : p(x,y) = 0\}$ for some irreducible polynomial p with $\nabla p = (p_x, p_y) \ne (0,0)$ on $\widetilde{K}$. This, however, need not be true, which is seen by the following example (also due to M. Baran).

Example 0.2. Take the (irreducible) polynomial

$$p(x,y) = [2y^2 - 3x(1-2x)]^2 - x^2[8(1-2x)^2 + 1].$$

Then $\nabla p(0,0) = (0,0)$. At the same time, the curve $\{(x,y) \in \mathbb{R}^2 : p(x,y) = 0\}$ has the explicit representation

$$2y^2 = x[3(1-2x) + [8(1-2x)^2+1]^{1/2}],$$

whence it is $\mathbb{R}$-analytic.

It seems that the same gap occurs in the proof of the main theorem of [BLT]. In order to avoid the above mentioned difficulties, in this paper we propose an essentially different (and technically simpler than that of [BLT] and [BLMT]) approach to Bernstein and Markov type inequalities in $\mathbb{R}^n$ that applies to curves with an analytic parametrization.

Definition 0.3. Let K be a compact curve in $\mathbb{R}^n$ and let $I = [-1,1]$. Then K is said to admit an *analytic parametrization* if there exist $r \in \mathbb{N}$, $\alpha > 1$ and $\mathbb{R}$-analytic maps $\phi_j = (\phi_{j1}, \ldots, \phi_{jn}) : \alpha I \to K$, $j = 1, \ldots, r$, such that each $\phi_j|_I$ is a bijection onto $\phi_j(I)$ and

$$\bigcup_{j=1}^{r} \phi_j(I) = K.$$

Let us give some examples of curves admitting an analytic parametrization.

Example 0.4. The natural parametrization $h(t) = t$ of the line segment $I = [-1,1]$ does not fit the requirements of Definition 0.3. The point is that although h is an analytic bijection of I onto I, there is no $\alpha > 1$ such that $h((-\alpha,\alpha)) \subset I$. If we replace h with the parametrization $\phi : \mathbb{R} \ni t \to \phi(t) = \sin\frac{\pi}{2}t \in I$ then I becomes a curve with an analytic parametrization.

Example 0.5. The curve

$$K = \{(x,y) \in \mathbb{R}^2 : y^2 = (1-x^2)^3\}$$

is not of class $\mathcal{C}^1$. Nevertheless, it admits the analytic parametrization

$$\phi(t) = (\cos \pi t, \sin^3 \pi t).$$

Example 0.6. The curve

$$K = \{(x,y) \in \mathbb{R}^2 : y^2 = x^2(1-x^2)\}$$

has a double point at $(0,0)$, whence it cannot be a topological manifold. At the same time, it admits the global analytic parametrization given by

$$\phi(t) = (\cos \pi t, \cos \pi t \sin \pi t).$$

Notice that any curve K in $\mathbb{R}^n$ such that $K = h(I)$, where h is an analytic function in an open neighbourhood of I, admits an analytic parametrization in the sense of Definition 0.3. (It suffices to take $\phi(t) = \sin\frac{\pi}{2}t$.) In the class of curves admitting an analytic parametrization, we are able to characterize semialgebraicity of K in terms of Bernstein or van der Corput–Schaake type inequalities as well as Lipschitz type conditions for the pluricomplex Green function associated with K (see Theorem 2.1). We recall that a subset K of $\mathbb{R}^n$ is said to be *algebraic* if $K = \{x \in \mathbb{R}^n : f(x) = 0 \text{ for any } f \in \mathcal{I}(K)\}$, where

$$\mathcal{I}(K) = \{f \in \mathbb{R}[x_1, \ldots, x_n] : \forall x \in K,\ f(x) = 0\}.$$

K is said to be *semialgebraic* if it is a finite union of subsets of $\mathbb{R}^n$ of the form

$$\{x \in \mathbb{R}^n : f_i(x) = 0 \text{ and } g_j(x) > 0 \text{ for } i = 1, \ldots, l \text{ and } j = 1, \ldots, m\},$$

where f_i and g_j are in $\mathbb{R}[x_1, \ldots, x_n]$. The semialgebraic sets in $\mathbb{R}^n$ form the smallest family of subsets of $\mathbb{R}^n$ which contains all algebraic sets and is closed under projections. For more information on properties of semialgebraic sets we refer the reader to [BeRi].

By Puiseux's theorem (see e.g. [Ł]), any semialgebraic curve in $\mathbb{R}^n$ is piecewise $\mathcal{C}^1$ and, moreover, it admits an analytic parametrization. For this reason, the class of curves we consider here is well adapted to our problem.

The proof of our main result (Theorem 2.1) is given in Section 2. It is preceded by some technical preliminaries, collected in Section 1.

1. Preliminaries. In our study the crucial role is played by the so called *Joukowski function*

$$g(w) = \frac{1}{2}\left(w + \frac{1}{w}\right), \quad w \in \mathbb{C} \setminus \{0\},$$

which establishes a biholomorphism between $\{w \in \mathbb{C} : |w| > 1\}$ and $\mathbb{C} \setminus [-1, 1]$. The inverse function $h = g^{-1} : \mathbb{C} \setminus [-1, 1] \to \mathbb{C} \setminus \{|w| \le 1\}$ has the form

$$h(z) = z + (z^2 - 1)^{1/2}$$

if we choose an appropriate branch of the square root. We shall need the following formulae (see [Ba2, Proposition 1.13]):

(1.1.1) If $\alpha \in (-1, 1)$, $\varepsilon > 0$ and $\beta \in \mathbb{R}$, then

$$\frac{1}{\varepsilon}\log|h(\alpha + i\varepsilon\beta)| \le |\beta|(1 - \alpha^2)^{-1/2}.$$

(1.1.2) If $\alpha \in (-1, 1)$, $0 < \varepsilon \le 1/2$, $\beta \in \mathbb{R}$, and $|\beta| \le 1 - |\alpha|$, then

$$(1 - \varepsilon)|\beta|(1 - \alpha^2)^{-1/2} \le \frac{1}{\varepsilon}\log|h(\alpha + i\varepsilon\beta)|.$$

Let now E be a compact subset of $\mathbb{C}^n$. We set

$$V_E(z) = \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^n),\ u|_E \le 0\}, \quad z \in \mathbb{C}^n,$$

where

$$\mathcal{L}(\mathbb{C}^n) = \{u \in \mathrm{PSH}(\mathbb{C}^n) : \sup_{z \in \mathbb{C}^n}[u(z) - \log(1 + |z|)] < \infty\}$$

is the *Lelong class* of plurisubharmonic functions with minimal growth. The function V_E is called the (*plurisubharmonic*) *extremal function* associated with E (see [Si2]). It is a multidimensional counterpart of the classical *Green function* for $\mathbb{C} \setminus \widehat{E}$, where $\widehat{E}$ is the polynomial hull of E, since by the pluripotential theory due to E. Bedford and B. A. Taylor, its upper semicontinuous regularization V_E^* is a solution of the homogeneous complex *Monge–Ampère equation*, which reduces in the one-dimensional case to the *Laplace equation* (for references, see [K]). By [Si2],

$$V_E(z) = \sup\left\{\frac{1}{\deg p}\log|p(z)| : p \text{ is a polynomial with } \deg p \ge 1 \text{ and } \|p\|_E \le 1\right\}. \tag{1.2}$$

In other words, $V_E = \log \Phi_E$, where Φ_E is the (*polynomial*) *extremal function* introduced by Siciak [Si1]. If E is a compact subset of $\mathbb{R}^n$ then by [Ba1, Th. 1.3],

$$\exp V_E(z) = \sup\{|h(p(z))|^{1/\deg p} : p \in \mathbb{R}[z],\ \deg p \ge 1,\ \|p\|_E \le 1\}. \tag{1.3}$$

A subset E of $\mathbb{C}^n$ is said to be *pluripolar* if one can find a plurisubharmonic function u on $\mathbb{C}^n$ such that $E \subset \{u = -\infty\}$. By Josefson [Jos], E is pluripolar if and only if it is *locally pluripolar*, i.e. for each point $a \in E$ there exist an open neighbourhood U of a and a function u plurisubharmonic on U such that $E \cap U \subset \{u = -\infty\}$.

Let now A be an analytic subset of $\mathbb{C}^n$ such that the set A_{reg} of regular points of A is a complex submanifold of $\mathbb{C}^n$ of pure dimension m. Let K be a compact subset of A. Then, since A is pluripolar in $\mathbb{C}^n$, so is the set K and hence $V_K^* \equiv \infty$ (see [Si2]). However, by definition, $V_K = 0$ on K and V_K may be finite at some points of $\mathbb{C}^n$ as well. Now, K is said to be (*pluri*)*polar in* A if there is a plurisubharmonic function u on A (i.e. u is plurisubharmonic on A_{reg} and locally bounded above on A) such that $K \cap A_{\mathrm{reg}} \subset \{u = -\infty\}$. We shall need the "only if" part of the following important criterion of Sadullaev [Sa]:

(1.4) A is algebraic if and only if V_K is locally bounded in A for some (and hence for each) non-pluripolar compact subset K of A.

Given a subset E of $\mathbb{C}^n$, let $\mathcal{P}_k(E)$ denote the vector space of the restrictions to E of all polynomials in $\mathcal{P}_k(\mathbb{C}^n)$, the space of polynomials of degree at most k in $(z_1, \ldots, z_n) \in \mathbb{C}^n$. Let $\delta_k = \delta_k(E)$ be the dimension of the space $\mathcal{P}_k(E)$ and let $\{\widehat{e}_1, \ldots, \widehat{e}_{\delta_k}\}$ be a basis of $\mathcal{P}_k(E)$. If $1 \le l \le \delta_k$, let $\{\zeta_1^{(l)}, \ldots, \zeta_l^{(l)}\} \subset K$ be a system of *extremal points* of E of order l, i.e.

$$V_l(E) := |V(\zeta_1^{(l)}, \ldots, \zeta_l^{(l)})| = \sup\{|V(x_1, \ldots, x_l)| : \{x_1, \ldots, x_l\} \subset K\},$$

where $V(x_1,\ldots,x_l) := \det[\widehat{e}_i(x_j)]$ is the *generalized Vandermonde determinant*. We claim that

$$V_l(E) > 0 \quad \text{for } 1 \le l \le \delta_k \tag{1.5}$$

(cf. [Si1, Proposition 4.3]). For, if $l = 1$, we have $V_1(E) > 0$. Suppose that $V_l(E) = |V(\zeta_1^{(l)},\ldots,\zeta_l^{(l)})| > 0$ for $1 \le l < \delta_k$. Then, for $x \in E$,

$$W(x) := V(\zeta_1^{(l)},\ldots,\zeta_l^{(l)},x) = V(\zeta_1^{(l)},\ldots,\zeta_l^{(l)})\widehat{e}_{l+1} + \sum_{j=1}^{l} c_j\widehat{e}_j(x).$$

Since $V_l(E) \ne 0$, $W(x) \not\equiv 0$ on E, whence $V_{l+1}(E) \ge \sup_{x\in E}|W(x)| > 0$, as claimed.

Let $X = X(E)$ denote the Zariski closure of E, i.e. X is the smallest algebraic subset of $\mathbb{C}^n$ that contains E. We claim that

$$\delta_k(E) = O(k^{\dim X(E)}), \tag{1.6.1}$$

where the exponent $\dim X(E)$ is best possible. For, since $\mathcal{P}_k(E) = \mathcal{P}_k(X)$, it is enough to consider the case of $E = X$ being an algebraic set in $\mathbb{C}^n$. Let $\imath : \mathbb{C}^n \ni (x_1,\ldots,x_n) \to [1,x_1,\ldots,x_n] \in \mathbb{P}^n$ be the canonical inclusion of $\mathbb{C}^n$ into the projective space $\mathbb{P}^n$ and let X^* denote the closure of $\imath(X)$ in $\mathbb{P}^n$. The ring $\mathbb{C}[x_0,x_1,\ldots,x_n]$ of all polynomials in $\mathbb{C}^{n+1}$ has a natural gradation $\mathbb{C}[x] = \bigotimes_{k=0}^{\infty}\mathbb{C}[x]_k$, where $\mathbb{C}[x]_k$ is the space of homogeneous polynomials of degree k. Correspondingly, the ideal $\mathcal{I} := \mathcal{I}(X^*)$ of homogeneous polynomials in $\mathbb{C}[x_0,x_1,\ldots,x_n]$ that vanish on X^* has the natural gradation $\mathcal{I} = \bigotimes_{k=0}^{\infty}\mathcal{I}_k$, where $\mathcal{I}_k = \mathcal{I}\cap\mathbb{C}[x]_k$. Consequently, the homogeneous coordinate ring $S(X^*) = \mathbb{C}[x_0,x_1,\ldots,x_n]/\mathcal{I}$ has the natural gradation $S(X^*) = \bigotimes_{k=0}^{\infty}\mathbb{C}[x]_k/\mathcal{I}_k$. Put $\mathcal{P}_k^*(X^*) := \mathbb{C}[x]_k/\mathcal{I}_k$ and consider the map

$$\psi : \mathcal{P}_k^*(X^*) \ni f(x_0,x_1,\ldots,x_n) \mapsto f(1,x_1,\ldots,x_n) \in \mathcal{P}_k(X).$$

Then ψ is $\mathbb{C}$-linear. Moreover, if $f(1,x_1,\ldots,x_n)$ vanishes on X then the (homogeneous) polynomial f is equal to 0 on $\imath(X)$, and hence on X^*, since $\imath(X)$ is dense in X^*. Thus the map ψ is injective. It is also surjective. For, let $g \in \mathcal{P}_k(X)$ and take $f(x_0,x_1,\ldots,x_n) := x_0^k g(x_1/x_0,\ldots,x_n/x_0)$. Then $\psi(f) = g$. Hence ψ is a $\mathbb{C}$-linear isomorphism of $\mathcal{P}_k^*(X^*)$ onto $\mathcal{P}_k(X)$. Consequently, $\dim\mathcal{P}_k(X) = \dim\mathcal{P}_k^*(X^*) =: h_{X^*}(k)$, where h_{X^*} is called the *Hilbert function* of the projective variety X^*. Now, by the Hilbert–Serre theorem (see e.g. [Hart, pp. 51–52] or [Har, Proposition 13.2]), there is a polynomial p_{X^*} of degree $\dim X^*$, called the *Hilbert polynomial* of X^*, such that for k large enough, $h_{X^*}(k) = p_{X^*}(k)$. Thus $\dim\mathcal{P}_k(X) = O(k^{\dim X^*})$, where the exponent $\dim X^* = \dim X$ is best possible.

The above result also admits an inverse. Namely,

(1.6.2) if E is an irreducible closed analytic subset of $\mathbb{C}^n$ of pure dimension m such that $\delta_k(E) = \dim\mathcal{P}_k(E) = O(k^m)$, for k large enough, then E is a pure m-dimensional algebraic subset of $\mathbb{C}^n$.

For, let X be the Zariski closure of E. Since $\dim\mathcal{P}_k(E) = O(k^m)$ for $k \gg 0$, it follows from (1.6.1) that $\dim X \le m$. Since $E \subset X$ and $\dim E = m$, we have $\dim X = m$. Hence it easily follows (see e.g. [Chir, Proposition 3, p. 61]) that E must be an algebraic subset of $\mathbb{C}^n$, and $E = X$.

Thus we have proved that

(1.7) an irreducible closed analytic subset E of $\mathbb{C}^n$ of pure dimension m is algebraic if and only if $\delta_k(E) = O(k^m)$.

R e m a r k. The authors owe the proof of (1.6.1) and (1.6.2) to Dr. Z. Jelonek. They also thank him for valuable discussions concerning algebraic geometry.

We find it interesting that (1.6.1) can also be proved by purely "analytic" methods. To show this, assume that X is an algebraic subset of $\mathbb{C}^n$ of dimension m. Let x_0 be a regular point of X and let U_0 be a coordinate neighbourhood of x_0 which is mapped by a biholomorphic map h onto some open ball B_0 in $\mathbb{C}^m$ of finite radius, centred at $0 \in \mathbb{C}^m$. We set

$$Q_k := \{f\circ h^{-1} : f \in \mathcal{P}_k(X)\}.$$

Let B be a closed ball centred at $0 \in \mathbb{C}^m$ such that $B \subset B_0$. By a uniform version of the Bernstein–Walsh–Siciak theorem (see [Pl1, Lemma 1]), there exist constants $M > 0$ and $a \in (0,1)$ such that

$$\operatorname{dist}_B(\widetilde{f},\mathcal{P}_l(\mathbb{C}^m)) := \inf\{\sup|\widetilde{f}-p|(B) : p \in \mathcal{P}_l(\mathbb{C}^m)\} \le M\|\widetilde{f}\|_{B_0}a^l$$

for each $\widetilde{f} \in Q_k$ and $l = 1,2,\ldots$ Now, if $\widetilde{f} \in Q_k$, there exists $F \in \mathcal{P}_k(\mathbb{C}^n)$ such that $F_{|X} = f$. Hence, since $\frac{1}{\deg F}\log|F| \in \mathcal{L}(\mathbb{C}^n)$, by the definition of V_B, and since $h^{-1}(B)$ is non-(pluri)polar in X, we get

$$\begin{aligned}\|\widetilde{f}\|_{B_0} &= \|f\circ h^{-1}\|_{B_0} = \|f\|_{h^{-1}(B_0)} \le \|f\|_{h^{-1}(B)}\exp(k\sup_{h^{-1}(B_0)} V_{h^{-1}(B)})\\ &= \|f\|_{h^{-1}(B)}A^k = \|\widetilde{f}\|_B A^k,\end{aligned}$$

where by Sadullaev's criterion (1.4), $A := \exp(\sup_{h^{-1}(B_0)} V_{h^{-1}(B)}) < \infty$. Hence we obtain

$$\operatorname{dist}_B(\widetilde{f},\mathcal{P}_l(\mathbb{C}^m)) \le M\|\widetilde{f}\|_B A^k a^l.$$

Set $l = rk$, where r is so chosen that $Aa^r < 1/M$. Then, if $\widetilde{f} \ne 0$,

$$\operatorname{dist}_B(\widetilde{f},\mathcal{P}_{rk}(\mathbb{C}^m)) < \|\widetilde{f}\|_B \quad \text{for } k = 1,2,\ldots$$

We claim that

$$\dim\mathcal{P}_k(X) = \dim Q_k \le \dim\mathcal{P}_{rk}(\mathbb{C}^m).$$

For, if $\dim Q_k > \dim \mathcal{P}_{rk}(\mathbb{C}^n)$, then by the Kreĭn–Krasnosel'skiĭ–Milman theorem (see e.g. [Sin, Chap. II, Lemma 6.1]), one can find $\widetilde{f}_0 \in Q_k \setminus \{0\}$ such that $\widetilde{f}_0 \perp \mathcal{P}_{rk}(\mathbb{C}^m)$. This means that

$$\|\widetilde{f}_0\|_B \le \|\widetilde{f}_0 + p\|_B, \quad \forall p \in \mathcal{P}_{rk}(\mathbb{C}^m).$$

Hence we get $\|\widetilde{f}_0\|_B \le \operatorname{dist}_B(\widetilde{f}_0, \mathcal{P}_{rk}(\mathbb{C}^n)) < \|\widetilde{f}_0\|_B$, contradiction. Consequently,

$$\dim \mathcal{P}_k(X) \le \dim \mathcal{P}_{rk}(\mathbb{C}^m) = \binom{rk+m}{m} \le \frac{(r+m)^m}{m!} k^m = O(k^m).$$

Remark. The idea of making use of the Kreĭn–Krasnosel'skiĭ–Milman theorem for estimating the dimension of Q_k goes back to [Pl2].

2. Bernstein and van der Corput–Schaake type inequalities on semialgebraic curves. The main result of our paper is the following

THEOREM 2.1. *Let K be a compact curve in $\mathbb{R}^n$ with an analytic parametrization $\{\phi_j\}$ (with parameters r and α). Then the following conditions are equivalent*:

(i) *K is semialgebraic*;

(ii) *There exist positive constants M_1 and δ_0 such that $V_K(\phi_j(\zeta)) \le M_1\delta$ if $\operatorname{dist}(\zeta, I) \le \delta \le \delta_0$, $j = 1, \ldots, r$*;

(iii) *K has property P* (cf. [Pl3]): *there exist positive constants M_2 and C such that for each $j = 1, \ldots, r$ and $p \in \mathbb{C}[z_1, \ldots, z_n]$,*

$$|p(\phi_j(\zeta))| \le M_2 \|p\|_K \quad \text{if } \operatorname{dist}(\zeta, I) \le C/\deg p;$$

(iv) *K admits a Bernstein type inequality* (cf. [Bern]): *There exists a constant $M_3 > 0$ such that for each $j = 1, \ldots, r$ and $p \in \mathbb{C}[x_1, \ldots, x_n]$,*

$$|(p \circ \phi_j)'(t)| \le M_3(\deg\, p)\|p\|_K, \quad t \in I;$$

(iv′) *K admits a van der Corput–Schaake type inequality* (cf. [CS1], [CS2]): *There exists a constant $M_4 > 0$ such that for each $j = 1, \ldots, r$ and $p \in \mathbb{R}[x_1, \ldots, x_n]$,*

$$|(p \circ \phi_j)'(t)| \le M_4(\deg p)[\|p\|_K^2 - p^2(\phi_j(t))]^{1/2}, \quad t \in I.$$

Proof. (i)⇒(ii). Due to the definition of a semialgebraic set, since $V_E \le V_F$ if $F \subset E$, we may assume that $K = \phi(I)$, where $\phi = (\phi_1, \ldots, \phi_n) : \alpha I \to K$ is $\mathbb{R}$-analytic and $\phi(I)$ is contained in an algebraic set $A \subset \mathbb{R}^n$ of dimension 1. We put

$$R_l = [\limsup_{k\to\infty} \sqrt[k]{\operatorname{dist}_I(\phi_l, \mathcal{P}_k(\mathbb{C}))}]^{-1}, \quad l = 1, \ldots, n,$$

and

$$R = \min\left\{\frac{1+R_l}{2}, h\left(\frac{1+\alpha}{2}\right)\right\},$$

where h is the inverse of the Joukowski function (see Section 1). Since the map ϕ is analytic, by the classical Bernstein theorem, $R > 1$ and ϕ extends holomorphically to an open neighbourhood U of the ellipse

$$\mathcal{E}_R = \{\zeta : |h(\zeta)| \le R\} = \{\zeta : |\zeta - 1| + |\zeta + 1| \le 2g(R)\}.$$

Moreover, in view of Definition 0.3, U can be chosen such that $\phi(U \cap \mathbb{R}) = K$. Set

(2.1) $$R^* = \sqrt{2g(R) - 1}.$$

Then $R^*\mathcal{E}_{R^*} \subset \mathcal{E}_R$ and the function $\phi(R^* g(\zeta))$ is holomorphic in an open neighbourhood of the annulus $\{1 \le |\zeta| \le R^*\}$. Moreover, by the identity principle, $\phi(R^* g(\{1 \le |\zeta| \le R^*\}))$ is a (compact) subset of the algebraic set $\widetilde{A} = \operatorname{loc}\{f \in \mathbb{C}[z_1, \ldots, z_n] : f = 0 \text{ on } A\} \subset \mathbb{C}^n$. Now, if we set

$$M = \frac{1}{\log R^*} \sup_{|\zeta| = R^*} V_K(\phi(R^* g(\zeta))),$$

then, since K is a non-polar subset of $\widetilde{A}$, by Sadullaev's theorem (1.4) we have $M < \infty$. Let now $u \in \mathcal{L}(\mathbb{C}^n)$, $u|_K \le 0$. Then the function

$$v(\zeta) = u(\phi(R^* g(\zeta))) - M \log|\zeta|$$

is subharmonic in an open neighbourhood of the annulus $\{1 \le |\zeta| \le R^*\}$. Since $v(\zeta) \le 0$ for $|\zeta| = 1$, and $u(\phi(R^* g(\zeta))) \le V_K(\phi(R^* g(\zeta)))$ if $|\zeta| = R^*$, by the maximum principle we get $v(\zeta) \le 0$ for $1 \le |\zeta| \le R^*$, whence we derive the inequality

(2.2) $$V_K(\phi(z)) \le M \log|h(z/R^*)|, \quad z \in R^*\mathcal{E}_{R^*}.$$

Choose now $\delta_0 = (R^* - 1)/2$, $0 \le \varrho \le \delta \le \delta_0$ and $\zeta = t + \varrho e^{i\theta}$, where $t \in I$ and $\theta \in \mathbb{R}$. Then

$$\frac{|t| + \varrho}{R^*} \le \frac{1 + R^*}{2R^*} < 1$$

and hence, by putting $\varepsilon = 1$ in (1.1.1) we get

$$\begin{aligned} V_K(\phi(t + \varrho e^{i\theta})) &\le M \log\left|h\left(\frac{t + \varrho e^{i\theta}}{R^*}\right)\right| \\ &\le M \frac{\varrho}{R^*}|\sin\theta|\left[1 - \left(\frac{|t| + \varrho|\cos\theta|}{R^*}\right)^2\right]^{-1/2} \\ &\le M \frac{\delta}{R^*}\left[1 - \left(\frac{1 + R^*}{2R^*}\right)^2\right]^{-1/2} \le \frac{M}{(R^* - 1)^{1/2}}\delta, \end{aligned}$$

which yields (ii).

(i)⇒(iv′). Choose $\theta \in \arccos([-1/R^*, 1/R^*])$ where R^* is defined by (2.1). Put $t = R^* \cos\theta$ and $\widetilde{t} = R^* \sin\theta$. For $B \to 1_+$, set $\varepsilon = \frac{1}{2}(B - B^{-1})$ and $t_\varepsilon = (1+\varepsilon^2)^{1/2} t$. Then $R^* g(Be^{i\theta}) = t_\varepsilon + i\varepsilon\widetilde{t}$, and we can write

$$\phi(R^* g(Be^{i\theta})) = \phi(t_\varepsilon + i\varepsilon\widetilde{t}) = \phi(t_\varepsilon) + i\varepsilon\widetilde{t}\phi'(t_\varepsilon) + O(\varepsilon^2).$$

Hence, if p is a polynomial with real coefficients, we get

$$\begin{aligned} p(\phi(R^* g(Be^{i\theta}))) &= p(\phi(t_\varepsilon) + i\varepsilon\widetilde{t}\phi'(t_\varepsilon) + O(\varepsilon^2)) \\ &= p(\phi(t_\varepsilon)) + i\varepsilon\widetilde{t}\operatorname{grad} p(\phi(t_\varepsilon)) \cdot \phi'(t_\varepsilon) + O(\varepsilon^2). \end{aligned} \tag{2.3}$$

Assume now that $\|p\|_K = 1$ and $0 < b < 1$. Then by (1.3) and (2.2) we have

$$\frac{1}{\deg p} \log |h(bp(\phi(R^* g(\zeta))))| \le M \log|\zeta|,$$

where $\zeta = Be^{i\theta}$. Dividing the above inequality by $\varepsilon\widetilde{t}$ and letting $B \to 1_+$, by (1.1.1), (1.1.2) and (2.3) we get

$$b|(p\circ\phi)'(t)| \le \frac{M \deg p}{[(R^*)^2 - 1]^{1/2}} [1 - b^2 p^2(\phi(t))]^{1/2}.$$

(Here we have used the fact that the function $\Gamma(\beta) = \lim_{\varepsilon\to 0} \varepsilon^{-1} \log|h(\alpha + i\varepsilon\beta)|$ defined by (1.1.1) and (1.1.2) is homogeneous, which permits us to replace ε by $C\varepsilon$ with $C > 0$ so chosen that the assumption $|\beta| \le 1 - |\alpha|$ be satisfied.) Now, letting b tend to 1 gives

$$|(p\circ\phi)'(t)| \le \frac{M \deg p}{[(R^*)^2 - 1]^{1/2}} [1 - p^2(\phi(t))]^{1/2}.$$

The implication (ii)⇒(iii) easily follows from (1.2). To prove that (iii) implies (iv) it suffices to apply Cauchy's Integral Formula: if $p \in \mathbb{C}[z_1, \ldots, z_n]$ and $t \in I$, then

$$(p\circ\phi)'(t) = \frac{1}{2\pi i} \int\limits_{|\zeta - t| = \delta} \frac{p(\phi(\zeta))}{(\zeta - t)^2}\, dt,$$

whence by (iii) we easily obtain (iv) with $M_3 = M_2/C$.

Since (iv) also easily follows from (iv′), in order to end the proof of our theorem it suffices to show that (iv) implies (i). To see this, assume (iv) and suppose that K is not semialgebraic. Then, for at least one $j \in \{1, \ldots, r\}$, say $j = 1$, $K_1 := \phi_1(I) \subset K$ is not semialgebraic. Let $\mathcal{P}_k(K_1)$ be the vector space of the restrictions to K_1 of all polynomials $p \in \mathcal{P}_k(\mathbb{C}^n)$ and let $\delta_k = \dim \mathcal{P}_k(K_1)$. Let $\widehat{e}_1, \ldots, \widehat{e}_{\delta_k}$ be a basis of the space $\mathcal{P}_k(K_1)$. For $\xi = (\xi_1, \ldots, \xi_{\delta_k}) \in K_1^{\delta_k}$, let

$$V(\xi) := \det[\widehat{e}_i(\xi_j)]$$

and let $\xi^{(\delta_k)} \in K_1^{\delta_k}$ be such that

$$|V(\xi^{(\delta_k)})| = \sup\{|V(\xi)| : \xi \in K_1^{\delta_k}\} > 0,$$

where the last inequality follows from (1.5). Let now $t_j = \phi_1^{-1}(\xi_j^{(\delta_k)}) \cap I$. Without loss of generality we may assume that $|t_1 - t_2| = \min\{|t_i - t_j| : i \ne j\}$. Then $|t_1 - t_2| \le 2/(\delta_k - 1)$. Consider the polynomials

$$Q_k(x) = V(x, \xi_2^{(\delta_k)}, \ldots, \xi_{\delta_k}^{(\delta_k)})/V(\xi^{(\delta_k)}).$$

We have $\deg Q_k \le k$, $Q_k(\xi_j^{(\delta_k)}) = \delta_{1j}$ (Kronecker's symbol) and $\|Q_k\|_{K_1} = 1 = Q_k(\xi_1^{(\delta_k)})$. Hence by (iv) we get

$$1 = |Q_k(\phi(t_1)) - Q_k(\phi(t_2))| \le M_3 k |t_1 - t_2| \le 2M_3 k/(\delta_k - 1).$$

Since K_1 cannot be contained in an algebraic variety of dimension 1, by (1.7) for each j we can find $k_j \in \mathbb{N}$ such that $\delta_{k_j} > jk_j$ and, consequently, we get

$$1 \le 2M_3 k_j/(jk_j - 1),$$

which is impossible for j large enough. This completes the proof of the theorem.

In the proof of "(iv) or (iv′) implies (i)" we only need to know that K is of class $\mathcal{C}^1$. Hence, since by Puiseux's theorem any semialgebraic curve is piecewise $\mathcal{C}^1$ and admits an analytic parametrization, we get the following characterization of semialgebraic curves.

COROLLARY 2.2. *A compact, piecewise $\mathcal{C}^1$ curve K in $\mathbb{R}^n$ is semialgebraic if and only if it admits the Bernstein type inequality* (iv) *of Theorem* 2.1 *or, equivalently, the van der Corput–Schaake type inequality* (iv′).

Our version of Bernstein's inequality on the derivatives of polynomials also permits us to give a generalization of a classical theorem of Bernstein.

COROLLARY 2.3. *If K is a compact semialgebraic curve in $\mathbb{R}^n$ then it admits the Bernstein theorem: for all f in $\mathcal{C}(K)$, if for some $0 < \alpha < 1$, $\operatorname{dist}_K(f, \mathcal{P}_k(\mathbb{C}^n)) = O(k^{-\alpha})$ as $k \to \infty$, then $f \in \operatorname{Lip}_\alpha^\phi(K)$. Here*

$$\operatorname{Lip}_\alpha^\phi(K) = \{f \in \mathcal{C}(K) : |f(\phi(t)) - f(\phi(s))| \le M|t - s|^\alpha \text{ for all } t, s \in I\},$$

where M may depend on f.

The proof of the above corollary goes essentially along the same lines as in the classical case (see e.g. [L, Chap. 4, Theorem 4]). For the convenience of the reader, we reproduce it here.

Proof of Corollary 2.3. For $k = 0, 1, \ldots$, let $p_k \in \mathcal{P}_k(\mathbb{C}^n)$ be a polynomial such that

$$E_k(f) := \operatorname{dist}_K(f, \mathcal{P}_k(\mathbb{C}^n)) = \|f - p_k\|_K$$

and let $t, s \in I$, $t \neq s$. Then we get

$$|f \circ \phi(t) - f \circ \phi(s)| \leq |f \circ \phi(t) - p_k \circ \phi(t)| + |f \circ \phi(s) - p_k \circ \phi(s)| + |p_k \circ \phi(t) - p_k \circ \phi(s)| \leq 2E_k(f) + |t - s| \cdot |(p_k \circ \phi)'(x)|,$$

for some $x \in (0,1)$. Setting $k = 2^{l+1}$, by the inequality (iv) of Theorem 2.1 we can write

$$\begin{aligned}\|(p_{2^{l+1}} \circ \phi)'\|_I &= \Big\|(p_1 \circ \phi)' - (p_0 \circ \phi)' + \sum_{i=0}^{l} (p_{2^{i+1}} \circ \phi - p_{2^i} \circ \phi)'\Big\|_I \\ &\leq 2ME_0(f) + 2M \sum_{i=0}^{l} 2^{i+1} E_{2^i}(f) \\ &\leq 2M\Big[E_0(f) + 2E_1(f) + 4 \sum_{i=1}^{l} \sum_{j=2^{i-1}+1}^{2^i} E_j(f)\Big] \\ &\leq 8M \sum_{i=0}^{2^l} E_i(f).\end{aligned}$$

Now, if we select l in such a way that $2^l \leq h^{-1} < 2^{l+1}$ where $h = |t - s|$, we shall have

$$\begin{aligned}|f \circ \phi(t) - f \circ \phi(s)| &\leq O(2^{-\alpha(l+1)}) + O(1)h\Big(E_0(f) + \sum_{1 \leq i \leq 1/h} i^{-\alpha}\Big) \\ &\leq O(h^\alpha) + O(1)h + O(1)h \int_0^{1/h} x^{-\alpha}\, dx = O(h^\alpha).\end{aligned}$$

References

[Ba1] M. Baran, *Plurisubharmonic extremal function and complex foliations for the complement of convex sets in* $\mathbb{R}^n$, Michigan Math. J. 39 (1992), 395–404.

[Ba2] —, *Complex equilibrium measure and Bernstein type theorems for compact sets in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. 123 (1995), 485–494.

[Ba3] —, *Bernstein type theorems for compact sets in* $\mathbb{R}^n$ *revisited*, J. Approx. Theory 79 (1994), 190–198.

[Ba4] —, *Markov inequality on sets with polynomial parametrization*, Ann. Polon. Math. 60 (1994), 69–79.

[BaPl] M. Baran and W. Pleśniak, *Markov's exponent of compact sets in* $\mathbb{C}^n$, Proc. Amer. Math. Soc. 123 (9) (1995), 2785–2791.

[BeRi] R. Benedetti and J.-J. Risler, *Real Algebraic and Semi-Algebraic Sets*, Hermann, Paris, 1990.

[Bern] S. N. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, Mém. Acad. Roy. Belg. 4 (2) (1912), 1–103.

[BLMT] L. Bos, N. Levenberg, P. Milman and B. A. Taylor, *Tangential Markov inequalities characterize algebraic submanifolds of* $\mathbb{R}^N$, Indiana Univ. Math. J. 44 (1995), 115–138.

[BLT] L. Bos, N. Levenberg and B. A. Taylor, *Characterization of smooth, compact algebraic curves in* $\mathbb{R}^2$, in: Topics in Complex Analysis, P. Jakóbczak and W. Pleśniak (eds.), Banach Center Publ. 31, Inst. Math., Polish Acad. Sci., Warszawa, 1995, 125–134.

[BM1] L. Bos and P. Milman, *On Markov and Sobolev type inequalities on compact subsets in* $\mathbb{R}^n$, in: Topics in Polynomials in One and Several Variables and Their Applications, T. Rassias *et al.* (eds.), World Scientific, Singapore, 1992, 81–100.

[BM2] —, —, *Sobolev–Gagliardo–Nirenberg and Markov type inequalities on subanalytic domains*, Geom. Funct. Anal. 5 (1995), 853–923.

[Chir] E. M. Chirka, *Complex Analytic Sets*, Kluwer Acad. Publ., Dordrecht, 1989.

[CS1] J. G. van der Corput und G. Schaake, *Ungleichungen für Polynome und trigonometrische Polynome*, Compositio Math. 2 (1935), 321–361.

[CS2] —, —, *Berichtigung zu: Ungleichungen für Polynome und trigonometrische Polynome*, ibid. 3 (1936), 128.

[FeNa] C. Fefferman and R. Narasimhan, *Bernstein's inequality on algebraic curves*, Ann. Inst. Fourier (Grenoble) 43 (1993), 1319–1348.

[Goe1] P. Goetgheluck, *Inégalité de Markov dans les ensembles effilés*, J. Approx. Theory 30 (1980), 149–154.

[Goe2] —, *Polynomial inequalities on general subsets of* $\mathbb{R}^N$, Colloq. Math. 57 (1989), 127–136.

[Gon] A. Goncharov, *A compact set without Markov's property but with an extension operator for* C^∞*-functions*, Studia Math. 119 (1996), 27–35.

[Har] J. Harris, *Algebraic Geometry*, Springer, New York, 1992.

[Hart] R. Hartshorne, *Algebraic Geometry*, Springer, New York, 1987.

[Jon1] A. Jonsson, *Markov's inequality on compact sets*, in: Orthogonal Polynomials and Their Applications, C. Brezinski, L. Gori and A. Ronveaux (eds.), J. G. Bultzer AG, 1991, 309–313.

[Jon2] —, *Markov's inequality and zeros of orthogonal polynomials on fractal sets*, J. Approx. Theory 78 (1994), 87–97.

[Jos] B. Josefson, *On the equivalence between locally and globally polar sets for plurisubharmonic functions in* $\mathbb{C}^n$, Ark. Mat. 16 (1978), 109–115.

[K] M. Klimek, *Pluripotential Theory*, Oxford Univ. Press, London, 1991.

[L] G. G. Lorentz, *Approximation of Functions*, Holt, Rinehart and Winston, New York, 1966.

[Ł] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, Birkhäuser, Basel, 1991.

[MiRa] G. V. Milanović and T. M. Rassias, *On Markov–Duffin–Schaeffer inequalities*, J. Nat. Geometry 5 (1994), 29–41.

[PP1] W. Pawłucki and W. Pleśniak, *Markov's inequality and* C^∞ *functions on sets with polynomial cusps*, Math. Ann. 275 (1986), 467–480.

[PP2] —, —, *Extension of* C^∞ *functions from sets with polynomial cusps*, Studia Math. 88 (1988), 279–287.

[Pl1] W. Pleśniak, *On superposition of quasianalytic functions*, Ann. Polon. Math. 26 (1972), 73–84.
[Pl2] —, *Remarques sur une généralisation de l'inégalité de S. Bernstein*, C. R. Acad. Sci. Paris Sér. A 284 (1977), 1211–1213.
[Pl3] —, *Markov's inequality and the existence of an extension operator for C^∞ functions*, J. Approx. Theory 61 (1990), 106–117.
[Sa] A. Sadullaev, *An estimate for polynomials on analytic sets*, Math. USSR-Izv. 20 (1983), 493–502.
[Si1] J. Siciak, *On some extremal functions and their applications in the theory of analytic functions of several complex variables*, Trans. Amer. Math. Soc. 105 (1962), 322–357.
[Si2] —, *Extremal plurisubharmonic functions in $\mathbb{C}^n$*, Ann. Polon. Math. 39 (1981), 175–211.
[Si3] —, *Rapid polynomial approximation on compact sets in $\mathbb{C}^n$*, Univ. Iagell. Acta Math. 30 (1993), 145–154.
[Sin] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer, Berlin, 1970.
[Z] A. Zeriahi, *Inégalités de Markov et développement en série de polynômes orthogonaux des fonctions C^∞ et $\mathcal{A}^\infty$*, in: Proc. Special Year of Complex Analysis of the Mittag-Leffler Institute 1987–88, J. F. Fornaess (ed.), Princeton Univ. Press, Princeton, N.J., 1993, 693–701.

Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: baran@im.uj.edu.pl
plesniak@im.uj.edu.pl

Received December 2, 1996
Revised version February 24, 1997 (3789)

STUDIA MATHEMATICA 125 (1) (1997)

An irreducible semigroup of idempotents

by

ROMAN DRNOVŠEK (Ljubljana)

Abstract. We construct a semigroup of bounded idempotents with no nontrivial invariant closed subspace. This answers a question which was open for some time.

Let $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on a (real or complex) Banach space X. A (multiplicative) semigroup $\mathcal{S}$ in $\mathcal{B}(X)$ is said to be *irreducible* if the only closed subspaces of X invariant under all members of $\mathcal{S}$ are $\{0\}$ and X. Otherwise, $\mathcal{S}$ is called *reducible*. An operator $P \in \mathcal{B}(X)$ is called *idempotent* if $P^2 = P$. In this note we answer the following question negatively:

Is every semigroup of idempotents reducible?

It seems that this problem was first considered by Heydar Radjavi [3]. He proved that a semigroup $\mathcal{S}$ of idempotents on a Hilbert space is reducible provided $\mathcal{S}$ contains a nonzero finite-rank operator. The above question was explicitly mentioned in the paper [1] by P. Fillmore, G. MacDonald, M. Radjabalipour and H. Radjavi, where it was shown that a finitely generated semigroup of idempotents on a Banach space is reducible. Recently, the problem has also been mentioned in the survey article [4] by H. Radjavi.

Our construction of an irreducible semigroup of idempotents was inspired by the construction of a weakly dense semigroup of nilpotent operators (see [2]).

THEOREM. *There exists a semigroup $\mathcal{S}$ of idempotents on the Hilbert space l^2, which is weakly dense in $\mathcal{B}(l^2)$. In particular, the semigroup $\mathcal{S}$ is irreducible.*

Proof. If A and B are $k \times k$ matrices, then let $P_{A,B}$ be the $3k \times 3k$ matrix

---

1991 *Mathematics Subject Classification*: Primary 47A15, 47D03.
Key words and phrases: invariant subspaces, idempotents, operator semigroups.
This work was supported by the Research Ministry of Slovenia.