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Two-sided estimates of the approximation numbers of certain Volterra integral operators

by

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Abstract. We consider the Volterra integral operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ defined by

$$(Tf)(x) = v(x)\int_0^x u(t)f(t)\,dt.$$

Under suitable conditions on u and v, upper and lower estimates for the approximation numbers $a_n(T)$ of T are established when $1 < p < \infty$. When $p = 2$, these yield

$$\lim_{n\to\infty} n a_n(T) = \pi^{-1}\int_0^\infty |u(t)v(t)|\,dt.$$

We also provide upper and lower estimates for the ℓ^α and weak ℓ^α norms of $(a_n(T))$ when $1 < \alpha < \infty$.

1. Introduction. In this paper we study the approximation numbers of the Volterra integral operator T given by

$$(Tf)(x) = v(x)\int_0^x u(t)f(t)\,dt \tag{1.1}$$

for $x \in \mathbb{R}^+ := [0,\infty)$ and $f \in L^p(\mathbb{R}^+)$. Here $1 < p < \infty$, and u, v are real-valued functions, with $u \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^+)$ and $v \in L^p(\mathbb{R}^+)$; as usual, $p' = p/(p-1)$. The paper is a continuation of our earlier work [4], in which we gave a necessary and sufficient condition for $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ to be compact and also provided a scheme for obtaining upper and lower estimates for the approximation numbers of T. As an illustrative example we showed that when $u(x) = e^{Ax}$ and $v(x) = e^{-Bx}$, where $0 < A < B$, then the nth approximation number $a_n(T)$ of T is bounded above and below by positive multiples of n^{-1}. However, the general scheme mentioned above

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was somewhat cumbersome to use in all but the simplest cases, and it was clearly desirable to have easily checkable sufficient conditions on the weights u and v which would enable the behaviour of the $a_n(T)$ to be determined without great labour. This is just what we do here.

We give conditions under which

$$\lim_{n\to\infty} na_n(T) = \frac{1}{\pi}\int_0^\infty |u(t)v(t)|\,dt \tag{1.2}$$

when $p=2$; and when $p\neq 2$,

$$\frac{1}{4}\alpha_p \int_0^\infty |u(t)v(t)|\,dt \le \liminf_{n\to\infty} na_n(T) \le \limsup_{n\to\infty} na_n(T) \le \alpha_p \int_0^\infty |u(t)v(t)|\,dt \tag{1.3}$$

for some constant α_p, depending only on p.

Both (1.2) and (1.3) hold for the compact map T if, for example, $|u|/|v|^{p-1}$ is non-decreasing and bounded away from zero on $\mathbb{R}^+$. Thus for the case of exponential weights mentioned above, our analysis shows that when $p=2$ we have the asymptotic formula

$$\lim_{n\to\infty} na_n(T) = \pi^{-1}(B-A)^{-1},$$

which sharpens the inequalities obtained in [4]. The general results (1.2) and (1.3) also hold if $\sum_{k\in\mathbb{Z}}\sigma_k<\infty$, where

$$\sigma_k = \Big(\int_{\xi_k}^{\xi_{k+1}} U^{p/p'}(t)|v(t)|^p\,dt\Big)^{1/p}, \quad U(x)=\int_0^x |u(t)|^{p'}\,dt$$

and $\xi_k\in\mathbb{R}^+$ is defined by $U(\xi_k)=2^{kp'/p}$. The particular case $p=2$, $u=1$ of this recovers the special instance $\nu=1$ of the asymptotic formula given by Newman and Solomyak [8] for the singular values of the operator $T_{\nu,v}$ given by

$$(T_{\nu,v}f)(x) = \frac{v(x)}{x^\nu\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)\,dt \quad (\nu>1/2).$$

While this operator is more general than our map T in the sense that it contains the term $(x-t)^{\nu-1}$, by way of compensation we have the function u and can also deal with $p\neq 2$ so far as two-sided estimates are concerned. Moreover, the Hilbert-space methods of [8] are very different from our own.

We also show that the sequence (σ_k) plays a key role in the behaviour of T: its ℓ^∞ norm is equivalent to $\|T\|$; T is compact if, and only if, $\sigma_k\to 0$ as $k\to\infty$; the ℓ^α and the weak ℓ^α norm of (σ_k) give upper and lower estimates

of the ℓ^α and weak ℓ^α norm respectively of $(a_k(T))$; and if $(\sigma_k)\in\ell^1$ then (1.2) and (1.3) hold. These results were established in [8] for the operator $T_{\nu,v}$ acting in L^2; in that case, the ℓ^α norm of $(a_k(T_{\nu,v}))$ is simply the Schatten α-norm of $T_{\nu,v}$.

For background information on singular values and approximation numbers of integral operators we refer to [2] and [7]; more developments will be found in [5], [9] and the references listed in [8].

It is a pleasure to thank Michael Solomyak for his encouragement: his insistence that our techniques in [4] could be pushed further was vital. The ICMS Edinburgh Workshop on Harmonic Analysis and Partial Differential Equations (July, 1994) and the Paseky Conference on Function Spaces, Differential Operators and Nonlinear Analysis (September, 1995) provided further stimulation.

2. Preliminaries. Throughout the paper we shall assume that $p\in(1,\infty)$, that p' is defined by $1/p'+1/p=1$, and that u and v are given real-valued functions which satisfy

$$u\in L^{p'}_{\mathrm{loc}}(\mathbb{R}^+) \tag{2.1}$$

and

$$v\in L^p(\mathbb{R}^+). \tag{2.2}$$

At the cost of some technical complications it would be possible to establish the main results of the paper under the assumption that $v\in L^p(x,\infty)$ for all $x>0$, rather than the condition (2.2) which is used in [4]; certain changes in the arguments of [4] would then be required. As we are mainly concerned to present the central ideas in as simple a form as possible, rather than to aim for maximum generality, we shall suppose that (2.2) holds.

Given any interval I in $\mathbb{R}^+$ and any $f\in L^p(I)$, we write

$$\|f\|_{p,I} = \Big(\int_I |f(t)|^p\,dt\Big)^{1/p}.$$

For any $a\in\mathbb{R}^+$ we set

$$J_a = \sup_{x\ge a}\Big\{\Big(\int_a^x |u(y)|^{p'}\,dy\Big)^{1/p'}\Big(\int_x^\infty |v(z)|^p\,dz\Big)^{1/p}\Big\}. \tag{2.3}$$

The integral operator T that we shall study is, as explained in the Introduction, given by

$$(Tf)(x) = v(x)\int_0^x u(t)f(t)\,dt \tag{2.4}$$

for $f \in L^p(\mathbb{R}^+)$ and $x \in \mathbb{R}^+$. It is shown in Theorems 1 and 2 of [4] that T is a bounded linear map from $L^p(\mathbb{R}^+)$ to itself if, and only if, $J_0 < \infty$; and that if $J_0 < \infty$, then T is compact if, and only if, $\lim_{a\to\infty} J_a = 0$. Our interest in this paper lies in the approximation numbers of T, viewed as a map from $L^p(\mathbb{R}^+)$ to itself. We recall that given any $m \in \mathbb{N}$, the mth *approximation number* of T, $a_m(T)$, is defined to be

$$a_m(T) = \inf \|T - F\| \tag{2.5}$$

where the infimum is taken over all bounded linear maps $F : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ with rank less than m ($m \in \mathbb{N}$). When $p = 2$ these numbers coincide with the *singular values* of T. General information on approximation numbers may be found in [3]; here we merely note that our map T is compact if, and only if, $a_m(T) \to 0$ as $m \to \infty$.

Next, we introduce some quantities used in [4] which will play an important part here also. Given any interval $I \subset \mathbb{R}^+$ and any $f \in L^p(\mathbb{R}^+)$, put

$$l(I, f; u, v) = \int_I\int_I \Big|v(x)v(y)\int_x^y f(t)u(t)\,dt\Big|^p dx\,dy. \tag{2.6}$$

If no ambiguity is possible, we shall write $l(I, f)$ for $l(I, f; u, v)$. Note that

$$l(I, f) = \int_I\int_I \Big|\int_x^y f(t)u(t)\,dt\Big|^p d\mu(x)\,d\mu(y), \tag{2.7}$$

where μ is the finite measure defined by

$$d\mu(x) = |v(x)|^p dx, \quad \text{so that} \quad \mu(I) = \int_I |v(x)|^p\,dx. \tag{2.8}$$

With

$$F(x) = \int_0^x f(t)u(t)\,dt \quad (x \in \mathbb{R}^+), \tag{2.9}$$

it follows that

$$l(I, f) = \int_I\int_I |F(y) - F(x)|^p\,d\mu(x)\,d\mu(y). \tag{2.10}$$

Moreover, application of Hölder's inequality to (2.7) shows that

$$l(I, f) \le \|f\|_{p,I}^p \|u\|_{p',I}^p \mu^2(I). \tag{2.11}$$

Define

$$L(I) = L(I; u, v) = (\sup\{l(I, f)/\mu(I) : \|f\|_{p,I} \le 1\})^{1/p}. \tag{2.12}$$

By (2.11),

$$L(I) \le \|u\|_{p',I}\mu^{1/p}(I) = \|u\|_{p',I}\|v\|_{p,I}. \tag{2.13}$$

LEMMA 1. *Define*

$$\mu(x) = \int_0^x |v(t)|^p\,dt \quad (x \ge 0), \tag{2.14}$$

let $I = [a, b) \subset \mathbb{R}^+$, *put* $A = \mu(a)$, $B = \mu(b)$, $I^* = [A, B)$ *and set*

$$w = \frac{|u \circ \mu^{-1}|}{|v \circ \mu^{-1}|^{p-1}}. \tag{2.15}$$

Then $w \in L^{p'}([A, X])$ *for all* $X \in [A, B)$, *and*

$$L(I; u, v) = L(I^*; w, 1). \tag{2.16}$$

Proof. Let $X \in [A, B)$. Then

$$\int_A^X w^{p'}(\tau)\,d\tau = \int_a^{\mu^{-1}(X)} |u(t)|^{p'}\,dt. \tag{2.17}$$

If $f \in L^p(\mathbb{R}^+)$, then from (2.7), after natural changes of variable, we see that

$$l(I, f; u, v) = \int_A^B\int_A^B \Big|\int_X^Y (f \circ \mu^{-1})(\tau)\Big\{\frac{|(u \circ \mu^{-1})(\tau)|}{|(v \circ \mu^{-1})(\tau)|^p}\Big\}\,d\tau\Big|^p dX\,dY. \tag{2.18}$$

Moreover,

$$\|f\|_{p,I}^p = \int_A^B \Big|\frac{(f \circ \mu^{-1})(X)}{(v \circ \mu^{-1})(X)}\Big|^p dX = \|\widetilde{f}\|_{p,I^*}^p, \tag{2.19}$$

where $\widetilde{f}(X) = (f \circ \mu^{-1})(X)/(v \circ \mu^{-1})(X)$. Thus (2.12), (2.18) and (2.19) give

$$\begin{aligned} L(I; u, v) &= \Big[\sup\Big\{(B - A)^{-1}\int_A^B\int_A^B\Big|\int_X^Y \widetilde{f}(\tau)w(\tau)\,d\tau\Big|^p dX\,dY : \|\widetilde{f}\|_{p,I^*}^p \le 1\Big\}\Big]^{1/p} \\ &= L(I^*; w, 1). \end{aligned} \tag{2.20}$$

When u and v are constant over an interval I, $L(I; u, v)$ has a simple form, as the following lemma shows.

LEMMA 2. *Let* u, v *be positive and constant over an interval* $I \subset \mathbb{R}^+$ *with end-points* a *and* b, $a < b$. *Then*

$$L(I; u, v) = \alpha_p u_0 v_0 (b - a), \tag{2.21}$$

where $u_0 = u(a)$, $v_0 = v(a)$ *and*

$$\alpha_p = L([0, 1]; 1, 1). \tag{2.22}$$

Proof. The proof follows easily after making the change of variables

$$x = a + (b-a)X, \quad y = a + (b-a)Y, \quad t = a + (b-a)z. \tag{2.23}$$

It is easy to determine α_2. From [4], Lemma 6, we see that

$$\alpha_2^2 = \sup\Big\{2\|F\|^2_{2,[0,1]} : \int_0^1 F(t)\,dt = 0,\ \|F'\|_{2,[0,1]} = 1\Big\} = \frac{2}{\pi^2},$$

the final step following from the variational characterisation of the least positive eigenvalue of the Neumann Laplacian. Hence

$$\alpha_2 = \sqrt{2}/\pi. \tag{2.24}$$

To conclude this preliminary section we introduce a quantity which is often easier to handle than $L(I)$ but is equivalent to it. This is

$$J(I) = J(I;u,v) := \inf\{\max(A_c, B_c) : c \in (a,b)\}, \tag{2.25}$$

where I has end-points a and b,

$$\begin{aligned} A_c &= \sup_{a<s<c}\Big\{\Big(\int_s^c |u(t)|^{p'}\,dt\Big)^{1/p'}\Big(\int_a^s |v(t)|^p\,dt\Big)^{1/p}\Big\} \\ &= \sup_{a<s<c}\Big\{\Big(\int_{\mu(s)}^{\mu(c)} \frac{|(u\circ\mu^{-1})(t)|^{p'}}{|(v\circ\mu^{-1}(t))|^p}\,dt\Big)^{1/p'}\Big(\int_{\mu(a)}^{\mu(s)} dt\Big)^{1/p}\Big\} \\ &= \sup_{A<S<\mu(c)}\Big\{\Big(\int_S^{\mu(c)} w^{p'}(t)\,dt\Big)^{1/p'}(S-A)^{1/p}\Big\}, \end{aligned} \tag{2.26}$$

$$\begin{aligned} B_c &= \sup_{c<s<b}\Big\{\Big(\int_c^s |u(t)|^{p'}\,dt\Big)^{1/p'}\Big(\int_s^b |v(t)|^p\,dt\Big)^{1/p}\Big\} \\ &= \sup_{\mu(c)<S<B}\Big\{\Big(\int_{\mu(c)}^S w^{p'}(t)\,dt\Big)^{1/p'}(B-S)^{1/p}\Big\}, \end{aligned} \tag{2.27}$$

and $A = \mu(a)$, $B = \mu(b)$. In [4], Lemma 6, and the remark following that lemma, it is shown that there are positive constants K_1 and K_2, depending only on p, such that for all intervals $I \subseteq \mathbb{R}^+$,

$$K_1 L(I;u,v) \le J(I;u,v) \le K_2 L(I;u,v). \tag{2.28}$$

Since A_c increases from 0 as c increases from a, and B_c decreases to 0 as c increases to b, it follows that

(2.29) $\max(A_c, B_c)$ attains its minimum on (a,b) when $A_c = B_c$.

Two final pieces of notation will be of use to us. For non-negative expressions (functions or functionals) F_1, F_2 the symbol $F_1 \lesssim F_2$ means that $F_1 \le CF_2$ for some constant $C \in (0,\infty)$ independent of any variables in F_1 and F_2. If $F_1 \lesssim F_2$ and $F_2 \lesssim F_1$ we shall write $F_1 \asymp F_2$.

3. Estimates for the approximation numbers of T. To obtain these we need additional notation and results from [4]. The quantity $L(I)$ is defined by (2.12) for any interval $I \subset \mathbb{R}^+$; if I has end-points a and b ($a<b$), we shall also denote $L(I)$ by $L(a,b)$, when convenient to do so. Given any $\varepsilon > 0$, we define numbers c_k by the rule that

$$c_0 = 0, \quad c_{k+1} = \inf\{t > c_k : L(c_k,t) > \varepsilon\}, \tag{3.1}$$

with the convention that $\inf\emptyset = \infty$. The numbers c_k are said to form an (ε,L)-*sequence*. For a given ε the (ε,L)-sequence may be finite or infinite; in [4], pp. 482–483, it is shown that if it is infinite, then the map T (given by (2.4)) is not compact. For the compact maps T with which we shall be dealing, it follows that there exists $N \in \mathbb{N}$, called the *length* of the (ε,L)-sequence, such that

$$c_0 < c_1 < \ldots < c_N < c_{N+1} = \infty;$$

and in [4], p. 482, it is shown that

$$\begin{aligned} &L(c_k, c_{k+1}) = \varepsilon \quad \text{for } k = 0,1,\ldots,N-1,\\ &L(c_N, c_{N+1}) \le \varepsilon. \end{aligned} \tag{3.2}$$

If the (ε,L)-sequence is infinite, then $\lim_{k\to\infty} c_k = \infty$ (see [4], p. 483). Given any $\varepsilon > 0$ and any $x \in [0,\infty]$, we shall write

$$N(x,\varepsilon) = \max\{k \in \mathbb{N}_0 : c_k \le x\}. \tag{3.3}$$

Two preparatory lemmas are needed before the first key result.

Lemma 3. *Let I be any interval in $\mathbb{R}^+$ and let $v_1, v_2 \in L^p(\mathbb{R}^+)$. Then*

$$|L(I;u,v_1) - L(I;u,v_2)| \le 3\|v_1 - v_2\|_{p,I}\|u\|_{p',I} \tag{3.4}$$

where $L(I;u,v_j)$ is defined by (2.12) *and, as always, u satisfies* (2.1).

Proof. In what follows the suprema are taken over all f with $\|f\|_{p,I} \le 1$. We have

$$\begin{aligned} &|L(I;u,v_1) - L(I;u,v_2)| \\ &\quad \le \sup\Big\{\Big|\Big(\int_I\int_I \Big|v_1(x)v_1(y)\int_x^y f(t)u(t)\,dt\Big|^p dx\,dy\Big)^{1/p}\|v_1\|_{p,I}^{-1} \\ &\qquad - \Big(\int_I\int_I \Big|v_2(x)v_2(y)\int_x^y f(t)u(t)\,dt\Big|^p dx\,dy\Big)^{1/p}\|v_2\|_{p,I}^{-1}\Big|\Big\} \end{aligned}$$

$$= \|v_1\|_{p,I}^{-1}\|v_2\|_{p,I}^{-1} \sup \Big\{ \Big| \Big(\iiint_{I\,I\,I} \Big| v_1(x)v_1(y)v_2(z) \int_x^y f(t)u(t)\,dt \Big|^p dx\,dy\,dz \Big)^{1/p}$$

$$- \Big(\iiint_{I\,I\,I} \Big| v_1(z)v_2(x)v_2(y) \int_x^y f(t)u(t)\,dt \Big|^p dx\,dy\,dz \Big)^{1/p} \Big| \Big\}$$

$$\le \|v_1\|_{p,I}^{-1}\|v_2\|_{p,I}^{-1} \sup \Big(\iiint_{I\,I\,I} |v_2(z)v_1(x)v_1(y)$$

$$- v_1(z)v_2(x)v_2(y)|^p \Big| \int_x^y f(t)u(t)\,dt \Big|^p dx\,dy\,dz \Big)^{1/p}.$$

Now write

$$v_2(z)v_1(x)v_1(y) - v_1(z)v_2(x)v_2(y)$$
$$= \{v_1(x) - v_2(x)\}v_1(y)v_2(z) + v_2(x)v_2(z)\{v_1(y) - v_2(y)\}$$
$$+ v_2(x)v_2(y)\{v_2(z) - v_1(z)\}$$

and estimate $\int_x^y f(t)u(t)\,dt$ by means of Hölder's inequality. The result follows immediately.

We now investigate the effect on L of changing u as well as v.

LEMMA 4. *Let I be any interval in $\mathbb{R}^+$ and let $v_1, v_2 \in L^p(\mathbb{R}^+)$, $u_1, u_2 \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^+)$. Then*

$$(3.5)\quad |L(I;u_1,v_1) - L(I;u_2,v_2)| \le \|u_1-u_2\|_{p',I}\|v_1\|_{p,I} + 3\|u_2\|_{p',I}\|v_1-v_2\|_{p,I}.$$

Proof. Since $L(I;\cdot,v)$ is a norm on $L^{p'}(I)$ and depends on $|u|$, $|v|$ rather than u, v, with the help of Lemma 3 we deduce that

$$|L(I;u_1,v_1) - L(I;u_2,v_2)|$$
$$\le |L(I;u_1,v_1) - L(I;u_2,v_1)| + |L(I;u_2,v_1) - L(I;u_2,v_2)|$$
$$\le L(I;|u_1-u_2|,v_1) + 3\|u_2\|_{p',I}\|v_1-v_2\|_{p,I}$$
$$\le \|u_1-u_2\|_{p',I}\|v_1\|_{p,I} + 3\|u_2\|_{p',I}\|v_1-v_2\|_{p,I},$$

the final step following from (2.13).

THEOREM 5. *For all $x \in \mathbb{R}^+$,*

$$(3.6)\quad \lim_{\varepsilon\to 0+} \varepsilon N(x,\varepsilon) = \alpha_p \int_0^x |u(t)v(t)|\,dt,$$

where $N(x,\varepsilon)$ is defined by (3.3).

Proof. Without loss of generality we may suppose that $u, v \ge 0$ a.e. Fix $x \in \mathbb{R}^+$. For each $\eta > 0$, there are step-functions u_η, v_η on $[0,x]$ such that

$$\|u-u_\eta\|_{p',[0,x]}, \quad \|v-v_\eta\|_{p,[0,x]} < \eta.$$

We may assume that

$$(3.7)\quad u_\eta = \sum_{j=1}^m \xi_j \chi_{W(j)}, \quad v_\eta = \sum_{j=1}^m \eta_j \chi_{W(j)},$$

where the $W(j)$ are disjoint closed subintervals of $[0,x]$ with characteristic functions $\chi_{W(j)}$.

Now let $\varepsilon > 0$ and consider the (ε, L)-sequence $c_0, c_1, \ldots, c_{N+1}$, with $0 = c_1 < c_1 < \ldots < c_N < c_{N+1} = \infty$ and with the properties given in (3.2). Let $I_k(\varepsilon) = [c_k, c_{k+1}]$ if $k < N(x,\varepsilon)$, and $I_k(\varepsilon) = [c_k, x]$ if $k = N(x,\varepsilon)$. If $W(j)$ is contained in some $I_k(\varepsilon)$ for arbitrarily small values of ε, then $L(W(j);u,v) = 0$. It follows that $uv = 0$ a.e. on $W(j)$. To see this, we simply observe that by (2.24)–(2.27), either $v = 0$ a.e. on $W(j)$ or $\|u\|_{p',(y,c)}\|v\|_{p,(c_j,y)} = 0$ for all $y \in [c_j, c]$, where

$$\mu([c_k,c]) = \frac{1}{2}\mu(W(j)),$$

and similarly on $[c, c_{k+1}]$. Note that, as observed in [4], pp. 477 and 480, $J(I) \asymp \max(A_c, B_c)$ for this choice of c.

Let $\varepsilon_k = \inf\{\varepsilon > 0$: there exists j such that $I_j(\varepsilon) \supset W_k\}$ and put $\delta = \min\{\varepsilon_k : \varepsilon_k > 0\}$. Then if $0 < \varepsilon < \delta$, it follows that for all j and k, $W(k) \not\subseteq I_j(\varepsilon)$. Also, with $\sum_k'$ denoting summation over those $k \in \{1,\ldots,m\}$ for which $\int_{W(k)} u(t)v(t)\,dt \ne 0$, we have

$$(3.8)\quad \Big| \int_0^x u(t)v(t)\,dt - \int_0^x \sum_k{}' \xi_k\eta_k\chi_{W(k)}(t)\,dt \Big|$$
$$= \Big| \int_0^x \{u(t)v(t) - u_\eta(t)v_\eta(t)\}\,dt \Big|$$
$$\le \|u\|_{p',(0,x)}\|v-v_\eta\|_{p,(0,x)} + \|u-u_\eta\|_{p',(0,x)}\|v_\eta\|_{p,(0,x)}$$
$$\le \{\|u\|_{p',(0,x)} + \|v\|_{p,(0,x)} + \eta\}\eta.$$

However, $\int_{W(k)} u(t)v(t)\,dt \ne 0$ implies that $\varepsilon_k > 0$. Thus if $0 < \varepsilon < \delta$, we see with the help of Lemma 2 that

$$(3.9)\quad \alpha_p\xi_k\eta_k|W(k)| \le \sum_{I_j(\varepsilon)\subset W_k} L(I_j(\varepsilon);\xi_k,\eta_k) + L(I_{j_1(k)}(\varepsilon)\cap W(k);\xi_k,\eta_k)$$
$$+ L(I_{j_2(k)}(\varepsilon)\cap W(k);\xi_k,\eta_k),$$

where $j_1(k)$, $j_2(k)$ are such that the left-hand end-point of $W(k)$ is interior to $I_{j_1(k)}(\varepsilon)$, and the right-hand end-point of $W(k)$ is interior to $I_{j_2(k)}(\varepsilon)$. Of course, one of these terms involving $j_1(k)$ and $j_2(k)$ may be zero as the corresponding interval might be void.

Hence, with the aid of Lemma 4, we see that

$$\begin{aligned}
(3.10)\quad & \alpha_p \sum_k{}' \xi_k\eta_k|W(k)| \\
&\le \sum_{j\le N(\varepsilon,x)} L(I_j(\varepsilon);u_\eta,v_\eta) \\
&\quad + \sum_k\{L(I_{j_1(k)}(\varepsilon);u_\eta,v_\eta)+L(I_{j_2(k)}(\varepsilon);u_\eta,v_\eta)\} \\
&\le \sum_{j=0}^{N(x,\varepsilon)} L(I_j(\varepsilon);u,v) + \sum_k\{L(I_{j_1(k)}(\varepsilon);u,v)+L(I_{j_2(k)}(\varepsilon);u,v)\} \\
&\quad + \sum_{j=0}^{N(x,\varepsilon)} \{\|u-u_\eta\|_{p',I_j(\varepsilon)}\|v\|_{p,I_j(\varepsilon)} + 3\|u\|_{p',I_j(\varepsilon)}\|v-v_\eta\|_{p,I_j(\varepsilon)}\} \\
&\quad + \sum_k\sum_{\ell=1}^{2}\{\|u-u_\eta\|_{p',I_{j_\ell}(\varepsilon)}\|v\|_{p,I_{j_l}(\varepsilon)} + 3\|u\|_{p',I_{j_l}(\varepsilon)}\|v-v_\eta\|_{p,I_{j_l}(\varepsilon)}\} \\
&\le \{N(x,\varepsilon)+1+2m\}\varepsilon \\
&\quad + 3\{\|u-u_\eta\|_{p',(0,x)}\|v\|_{p,(0,x)} + 3\|u\|_{p',(0,x)}\|v-v_\eta\|_{p,(0,x)}\} \\
&= \{N(x,\varepsilon)+1+2m\}\varepsilon + O(\eta).
\end{aligned}$$

Combination of (3.8)–(3.10) now shows that

$$\alpha_p\int_0^x u(t)v(t)\,dt \le \{N(x,\varepsilon)+1+2m\}\varepsilon + O(\eta).$$

Thus

$$\alpha_p\int_0^x u(t)v(t)\,dt \le \liminf_{\varepsilon\to0+}\varepsilon N(x,\varepsilon) + O(\eta),$$

and so

$$(3.11)\qquad \alpha_p\int_0^x u(t)v(t)\,dt \le \liminf_{\varepsilon\to0+}\varepsilon N(x,\varepsilon).$$

Next, let $0<\varepsilon<\delta$ and put $\mathcal{K}=\{j:$ there exists k with $I_j(\varepsilon)\subset W(k)\}$. Then $\#\mathcal{K}\ge N(x,\varepsilon)-2m$, so that, using Lemma 4 again,

$$\begin{aligned}
(3.12)\quad & \{N(x,\varepsilon)-2m\}\varepsilon \\
&\le \sum_{j\in\mathcal{K}} L(I_j(\varepsilon);u,v)) \le \sum_{j\in\mathcal{K}} L(I_j(\varepsilon);u_\eta,v_\eta)+O(\eta) \\
&\le \alpha_p\sum_j \xi_j\eta_j|I_j(\varepsilon)| + O(\eta) = \alpha_p\int_0^x u_\eta(t)v_\eta(t)\,dt + O(\eta) \\
&= \alpha_p\int_0^x u(t)v(t)\,dt + O(\eta).
\end{aligned}$$

This shows that

$$(3.13)\qquad \limsup_{\varepsilon\to0+}\varepsilon N(x,\varepsilon) \le \alpha_p\int_0^x u(t)v(t)\,dt$$

and completes the proof of the theorem.

COROLLARY 6. *Suppose that*

$$(3.14)\qquad \liminf_{n\to\infty} n a_n(T) < \infty.$$

Then $uv\in L^1(\mathbb{R}^+)$.

Proof. In view of Theorem 5 we see that for all $x>0$,

$$\begin{aligned}
\alpha_p\int_0^x |u(t)v(t)|dt &= \lim_{\varepsilon\to0+}\varepsilon N(x,\varepsilon) \le \liminf_{\varepsilon\to0+}\varepsilon N(\infty,\varepsilon) \\
&\le \mathrm{const}\cdot\liminf_{n\to\infty} n a_n(T),
\end{aligned}$$

the final step following from Lemma 8 of [4]. Hence $uv\in L^1(\mathbb{R}^+)$.

The next group of results show what can be established if it is supposed that the function w is non-decreasing.

LEMMA 7. *Let $I=[a,b)\subset\mathbb{R}^+$, put $A=\mu(a)$, $B=\mu(b)$, $I^*=[A,B]$ and assume that the function w given by (2.15) is non-decreasing on I^*. Then*

$$(3.15)\qquad \sup_{A\le Y\le B}\Big\{\Big(\int_A^Y w^{p'}(t)\,dt\Big)^{1/p'}(B-Y)^{1/p}\Big\} \asymp \sup_{A\le X\le B}\{(B-X)w(X)\},$$

where the constants implicit in the symbol $\asymp$ are independent of A and B. Hence

$$(3.16)\qquad J(I^*;w,1)\lesssim\int_{I^*} w(t)\,dt,\qquad J(I;u,v)\lesssim\int_I |u(t)v(t)|\,dt.$$

Proof. Since w is non-decreasing, if $x, y \in I$ and $x < y$, $X = \mu(x)$, $Y = \mu(y)$, then

$$(Y - X)^{1/p'} w(X) \le \Big(\int_X^Y w^{p'}(t)\,dt\Big)^{1/p'} \le \Big(\int_A^Y w^{p'}(t)\,dt\Big)^{1/p'}.$$

Thus

$$\sup_{A \le Y \le B}\Big\{\Big(\int_A^Y w^{p'}(t)\,dt\Big)^{1/p'}(B-Y)^{1/p}\Big\} \ge w(X)\sup_{X\le Y\le B}\{(Y-X)^{1/p'}(B-Y)^{1/p}\} = (B-X)w(X)p^{-1/p}(p')^{-1/p'}. \tag{3.17}$$

On the other hand,

$$\Big(\int_A^Y w^{p'}(t)\,dt\Big)^{1/p'} \le \{\sup_{A\le X\le B}(B-X)w(X)\}\Big(\int_A^Y (B-X)^{-p'}\,dX\Big)^{1/p'} \le \{\sup_{A\le X\le B}(B-X)w(X)\}(p'-1)^{-1/p'}(B-Y)^{-1/p},$$

which leads to

$$\sup_{A\le Y\le B}\Big\{\Big(\int_A^Y w^{p'}(t)\,dt\Big)^{1/p'}(B-Y)^{1/p}\Big\} \le (p/p')^{1/p'}\sup_{A\le X\le B}\{(B-X)w(X)\}.$$

The lemma follows.

Corollary 8. *Suppose that w is non-decreasing on $[0,\mu(\infty)]$. Then $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ is compact if, and only if,*

$$w(X) = o\Big(\frac{1}{\mu(\infty)-X}\Big) \quad \text{as } X \to \mu(\infty)-;$$

that is, if, and only if,

$$|u(x)|/|v(x)|^{p-1} = o\Big(1/\int_x^\infty |v(t)|^p\,dt\Big) \quad \text{as } x \to \infty.$$

Proof. The result follows immediately from Lemma 7 and the fact that T is compact if, and only if, $J_a \to 0$ as $a \to \infty$ (see (2.3)).

Theorem 9. *Suppose that $|u|/|v|^{p-1}$ is non-decreasing and that $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ is compact. Then*

$$\lim_{\varepsilon\to 0+} \varepsilon N(\infty,\varepsilon) = \alpha_p \int_0^\infty |u(t)v(t)|\,dt. \tag{3.18}$$

When $p = 2$,

$$\lim_{n\to\infty} n a_n(T) = \frac{1}{\pi}\int_0^\infty |u(t)v(t)|\,dt. \tag{3.19}$$

When $p \ne 2$,

$$\frac{1}{4}\alpha_p \int_0^\infty |u(t)v(t)|\,dt \le \liminf_{n\to\infty} n a_n(T) \le \limsup_{n\to\infty} n a_n(T) \le \alpha_p \int_0^\infty |u(t)v(t)|\,dt. \tag{3.20}$$

Proof. Suppose that $\liminf_{\varepsilon\to 0+}\varepsilon N(\infty,\varepsilon) < \infty$. Then from the proof of Corollary 6 we see that $uv \in L^1(\mathbb{R}^+)$. Thus given any $\eta > 0$, there exists $x \in \mathbb{R}^+$ such that $\int_x^\infty |u(t)v(t)|dt < \eta$. On using (2.28) and (3.16), we obtain

$$\{N(\infty,\varepsilon) - N(x,\varepsilon) - 1\}\varepsilon \le \sum_{k=N(x,\varepsilon)+1}^{N(\infty,\varepsilon)} L(I_k;u,v) \asymp \sum_{k=N(x,\varepsilon)+1}^{N(\infty,\varepsilon)} J(I_k;u,v) \lesssim \int_x^\infty |u(t)v(t)|dt < \eta.$$

Moreover, from Theorem 5 we know that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$,

$$\Big|\varepsilon N(x,\varepsilon) - \alpha_p \int_0^x |u(t)v(t)|\,dt\Big| < \eta.$$

From these last two estimates (3.18) follows immediately.

If $\liminf_{\varepsilon\to 0+}\varepsilon N(\infty,\varepsilon) = \infty$, then since $\varepsilon \lesssim \int_{I_k}|u(t)v(t)|\,dt$ by (3.16), it follows that $\varepsilon N(\infty,\varepsilon) \lesssim \int_0^\infty |u(t)v(t)|\,dt$ and we conclude that $uv \notin L^1(\mathbb{R}^+)$. Thus (3.18) again holds.

When $p = 2$, we know from Lemmas 7 and 8 of [4] that

$$a_{N+2}(T) \le \varepsilon/\pi, \quad a_N(T) \ge \varepsilon/\pi$$

where $N = N(\infty,\varepsilon)$; (3.19) follows immediately from this and (3.18). The case $p \ne 2$ is treated in the same way, again using Lemmas 7 and 8 of [4].

The following theorem is a variant of Theorem 9.

Theorem 10. *Suppose that w is of bounded variation on $[0,\mu(\infty))$ and that its variation W ($W(x) = V_0^x w$ in standard notation) is in $L^1([0,\mu(\infty)])$. Then the conclusions of Theorem 9 hold.*

Proof. Given $\eta > 0$, let x be such that $\int_{\mu(x)}^{\mu(\infty)} W(t)\,dt < \eta$. Then as in the proof of Theorem 9,

$$\{N(\infty,\varepsilon) - N(x,\varepsilon) - 1\}\varepsilon \lesssim J([\mu(x),\mu(\infty)); w, 1)$$
$$\lesssim J([\mu(x),\mu(\infty)); W, 1) \lesssim \int_{\mu(x)}^{\mu(\infty)} W(t)\,dt < \eta.$$

Furthermore, (3.6) still holds and

$$\int_x^\infty |u(t)v(t)|\,dt = \int_{\mu(x)}^{\mu(\infty)} w(t)\,dt \le \int_{\mu(x)}^{\mu(\infty)} W(t)\,dt < \eta.$$

The proof may now be completed just as for Theorem 9.

THEOREM 11. *Suppose that $|u|/|v|^{p-1}$ is non-decreasing on $\mathbb{R}^+$. Then there is a positive constant δ such that as $\varepsilon \to 0_+$,*

$$N(\infty,\varepsilon) - 1 = O\Big(\varepsilon^{-1} \int_0^{\phi^{-1}(\varepsilon\delta)} w(t)\,dt\Big),$$

where

$$\phi(X) = \sup_{X<Y<\mu(\infty)} \{(\mu(\infty) - Y)w(Y)\}.$$

Proof. By Lemma 7,

$$J([X,\mu(\infty)); w, 1) \lesssim \phi(X),$$

and with $c_M = \phi^{-1}(\varepsilon)$,

$$L([c_M,\mu(\infty)); w, 1) \le c\phi(c_M) = c\varepsilon$$

for some constant c. Let $\varepsilon' = c\varepsilon$. Then $L([c_M,\mu(\infty)); w, 1) \le \varepsilon'$.

For $k = 1,\ldots,M$ define c_k by $L([c_{k-1},c_k]; w, 1) = \varepsilon'$, $c_0 \ge 0$; we see that $N(\infty,\varepsilon') \le M+1$. Hence, using (3.16) we have

$$\varepsilon'(N(\infty,\varepsilon') - 1) \le \sum_{k=1}^{M} L([c_{k-1},c_k]; w, 1)$$
$$\lesssim \int_0^{\phi^{-1}(\varepsilon)} w(t)\,dt = \int_0^{\phi^{-1}(\varepsilon'/c)} w(t)\,dt,$$

which proves the theorem.

As an illustration of these results, suppose that $u(x) = e^{Ax}$, $v(x) = e^{-Bx}$ for all $x \in \mathbb{R}^+$, where $0 < A < B$. This is the case discussed in detail in [4], where it was shown that for all p, $a_n(T) \asymp n^{-1}$. Here we observe that u/v^{p-1}

is non-decreasing, that $uv \in L^1(\mathbb{R}^+)$ and that T is compact since $J_0 < \infty$ and $\lim_{a\to\infty} J_a = 0$ (see (2.3)). Hence by Theorem 9, if $p = 2$,

$$\lim_{a\to\infty} n a_n(T) = \frac{1}{(B-A)\pi},$$

and if $p \ne 2$,

$$(B-A) n a_n(T)/\alpha_p \in [1/4, 1].$$

We now give some results which avoid the hypothesis that w is non-decreasing and enable us to link up with the work of Newman and Solomyak [8]. To explain these we begin by setting

$$U(x) = \int_0^x |u(t)|^{p'}\,dt \quad (x \in \mathbb{R}^+) \tag{3.21}$$

and define $\xi_k \in \mathbb{R}^+$ by

$$U(\xi_k) = 2^{kp'/p}. \tag{3.22}$$

Here k may be any integer if $u \notin L^{p'}(\mathbb{R}^+)$; if $u \in L^{p'}(\mathbb{R}^+)$ it is supposed that ξ_k is defined for all $k \in \mathbb{Z}$ less than a certain number. Corresponding to each admissible k we set

$$\sigma_k = \Big\{\int_{\xi_k}^{\xi_{k+1}} U^{p/p'}(t)|v(t)|^p\,dt\Big\}^{1/p}, \qquad Z_k = (\xi_k, \xi_{k+1}), \tag{3.23}$$

so that

$$2^k \int_{\xi_k}^{\xi_{k+1}} |v(t)|^p\,dt \le \sigma_k^p \le 2^{k+1} \int_{\xi_k}^{\xi_{k+1}} |v(t)|^p\,dt. \tag{3.24}$$

For non-admissible k we set $\sigma_k = 0$.

LEMMA 12. *Let $k_0, k_1, k_2 \in \mathbb{Z}$ with $k_0 < k_1 < k_2$, and let $I_j = (a_j, b_j)$ $(j = 0,1,\ldots,l)$ be non-overlapping intervals in $\mathbb{R}^+$ with end-points a_j, b_j such that $I_j \subset Z_{k_2}$ $(j = 1,\ldots,l)$, $a_0 \in Z_{k_0}$, $b_0 \in Z_{k_2}$; let $x_j \in I_j$ $(j = 0,1,\ldots,l)$ and $x_0 \in Z_{k_1}$. Then if $\alpha \ge 1$,*

$$S := \sum_{j=0}^{l} \Big(\int_{a_j}^{x_j} |u(t)|^{p'}\,dt\Big)^{\alpha/p'} \Big(\int_{x_j}^{b_j} |v(t)|^p\,dt\Big)^{\alpha/p} \le 2^{\alpha/p}(2^{\alpha/p}+1) \max_{k_1\le n\le k_2} \sigma_n^\alpha. \tag{3.25}$$

Proof. We have, with the aid of Jensen's inequality (see [6], Theorem 19, p. 28) and Hölder's inequality,

$$\begin{aligned} S &\le \Big(\int_{\xi_{k_0}}^{\xi_{k_1+1}} |u(t)|^{p'}\,dt\Big)^{\alpha/p'}\Big(\int_{\xi_{k_1}}^{\xi_{k_2+1}} |v(t)|^{p}\,dt\Big)^{\alpha/p} \\ &\quad + \sum_{j=1}^{l}\Big(\int_{I_j} |u(t)|^{p'}\,dt\Big)^{\alpha/p'}\Big(\int_{I_j} |v(t)|^{p}\,dt\Big)^{\alpha/p} \\ &\le (2^{(k_1+1)p'/p} - 2^{k_0p'/p})^{\alpha/p'}\Big(\sum_{n=k_1}^{k_2}\frac{\sigma_n^p}{2^n}\Big)^{\alpha/p} \\ &\quad + \Big\{\sum_{j=1}^{l}\Big(\int_{I_j} |u(t)|^{p'}\,dt\Big)^{1/p'}\Big(\int_{I_j} |v(t)|^{p}\,dt\Big)^{1/p}\Big\}^{\alpha} \\ &\le 2^{(k_1+1)\alpha/p}\Big(\frac{2}{2^{k_1}}\max_{k_1\le n\le k_2}\sigma_n^p\Big)^{\alpha/p} \\ &\quad + \Big(\int_{Z_{k_2}} |u(t)|^{p'}\,dt\Big)^{\alpha/p'}\Big(\int_{Z_{k_2}} |v(t)|^{p}\,dt\Big)^{1/p} \\ &\le 2^{2\alpha/p}\max_{k_1\le n\le k_2}\sigma_n^{\alpha} + 2^{(k_2+1)\alpha/p}\sigma_{k_2}^{\alpha}/2^{k_2\alpha/p}. \end{aligned}$$

The result follows.

LEMMA 13. *The quantity J_0 defined by (2.3) satisfies*

(3.26) $$J_0 \le 2^{1/p}(2^{1/p}+1)\sup_k \sigma_k \le 2^{2/p}(2^{1/p}+1)J_0.$$

Proof. By Lemma 12,

$$J_0 \le 2^{1/p}(2^{1/p}+1)\sup_k \sigma_k.$$

Also,

$$\sigma_k^p \le 2^{k+1}\int_{\xi_k}^{\xi_{k+1}} |v(t)|^p\,dt \le 2^{k+1}J_0^p/U^{p/p'}(\xi_k) = 2J_0^p.$$

COROLLARY 14. *The norm of $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ satisfies*

$$\|T\| \asymp \|(\sigma_k)\|_\infty,$$

where $\|(\sigma_k)\|_\infty$ means the ℓ^∞ norm of the sequence (σ_k).

Proof. This is immediate from Lemma 13, and Theorem 1 of [4].

COROLLARY 15. *The map $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ is compact if, and only if, $\lim_{n\to\infty}\sigma_n = 0$.*

Proof. As in the proof of Lemma 13, and with the quantities J_a defined by (2.3), we see that

$$J_{\xi_k} \le 2^{1/p}(2^{1/p}+1)\sup_{n\ge k}\sigma_n$$

and

$$\sigma_n^p \le 2J_{\xi_k}^p(1-2^{-p'/p})^{-p/p'}, \quad n \ge k+1.$$

Hence $\lim_{a\to\infty} J_a = 0$ if, and only if, $\lim_{n\to\infty}\sigma_n = 0$. The corollary now follows from Theorem 2 of [4].

THEOREM 16. *Suppose that $\sum_{n\in\mathbb{Z}}\sigma_n$ converges. Then the conclusions of Theorem 9 hold.*

Proof. From Lemma 12 we have

$$\{N(\infty,\varepsilon) - N(\xi_k,\varepsilon) - 1\}\varepsilon \le \sum_{m=N(\xi_k,\varepsilon)-1}^{N(\infty,\varepsilon)} L(I_m; u, v) \lesssim \sum_{n=k}^{\infty}\sigma_n.$$

Together with Theorem 5 this shows that

$$0 \le \lim_{\varepsilon\to 0+}\varepsilon N(\infty,\varepsilon) - \alpha_p\int_0^{\xi_k}|u(t)v(t)|\,dt \lesssim \sum_{n=k}^{\infty}\sigma_n.$$

Hence

$$\lim_{\varepsilon\to 0+}\varepsilon N(\infty,\varepsilon) = \alpha_p\int_0^{\infty}|u(t)v(t)|\,dt.$$

The rest of the theorem, namely (3.19) and (3.20), follows as in the proof of Theorem 9.

With some more effort, various norms of the sequence of approximation numbers of T can be estimated by means of corresponding norms of the sequence (σ_k). Some preparatory lemmas are needed.

LEMMA 17. *Given any interval $I \subset \mathbb{R}^+$ with end-points a and b, let*

$$A(I) = \sup_{x\in I}\Big\{\Big(\int_x^b |u(t)|^{p'}\,dt\Big)^{1/p'}\Big(\int_a^x |v(t)|^p\,dt\Big)^{1/p}\Big\},$$

$$B(I) = \sup_{x\in I}\Big\{\Big(\int_a^x |u(t)|^{p'}\,dt\Big)^{1/p'}\Big(\int_x^b |v(t)|^p\,dt\Big)^{1/p}\Big\}.$$

(In the notation of (2.25), (2.26), $A(I) = A_b$, $B(I) = B_a$.) Then for all k,

$$A(\overline{Z}_k \cup \overline{Z}_{k+1}) \ge 2^{1/p}(1-2^{-p'/p})^{1/p'}\sigma_k,$$

$$B(\overline{Z}_k \cup \overline{Z}_{k+1}) \ge 2^{-1/p}(1-2^{-p'/p})^{1/p'}\sigma_{k+1}.$$

Proof. The estimate for A follows, with the aid of (3.24), from

$$A(\overline{Z}_k \cup \overline{Z}_{k+1}) \ge (U(\xi_{k+2}) - U(\xi_{k+1}))^{1/p'} \Big(\int_{\xi_k}^{\xi_{k+1}} |v(t)|^p \, dt \Big)^{1/p}$$
$$\ge \{2^{(k+2)p'/p} - 2^{(k+1)p'/p}\}^{1/p'} \sigma_k 2^{-(k+1)/p}$$

and that for B, in the same way, from

$$B(\overline{Z}_k \cup \overline{Z}_{k+1}) \ge (2^{(k+1)p'/p} - 2^{kp'/p})^{1/p'} \sigma_{k+1} 2^{-(k+2)/p}.$$

LEMMA 18. *Let I be an interval in $\mathbb{R}^+$ with end-points a and b, define $J(I)$ by* (2.25), *let $\varepsilon > 0$ and suppose that*

$$S(\varepsilon) := \{k \in \mathbb{Z} : Z_k \subset I,\ \sigma_k > 2^{1/p}(1 - 2^{-p'/p})^{-1/p'} \varepsilon\} \tag{3.27}$$

has at least 4 distinct elements. Then $J(I) > \varepsilon$.

Proof. Let $c \in (a, b)$. Since $\# S \ge 4$, at least one of the intervals (a, c), (c, b) contains 2 members of S. If (a, c) has this property and $k_1 = \min\{k : k \in S\}$, then $Z_{k_1} \cup Z_{k_1+1} \subset (a, c)$ and by Lemma 17,

$$A((a,c)) \ge A(\overline{Z}_{k_1} \cup \overline{Z}_{k_1+1}) \ge 2^{1/p}(1 - 2^{-p'/p})^{1/p'} \sigma_k \ge \varepsilon 2^{2/p} > \varepsilon.$$

A similar argument shows that if (c, b) contains 2 members of S, then $B((c,b)) > \varepsilon$. Hence $\max\{A((a,c)), B((c,b))\} > \varepsilon$ and the result follows from (2.25) and (2.29).

COROLLARY 19. *Let $\varepsilon > 0$ and suppose that $\# S(\varepsilon_p \varepsilon) \ge 4$, where S is defined by* (3.27) *and ε_p equals 1 if $p = 2$ and equals 2 otherwise. Then $L(I) > \varepsilon$.*

Proof. This is immediate from Lemma 18 and [4], Lemma 6.

LEMMA 20. *Let $\varepsilon > 0$ and let $N = N(\varepsilon)$ be the length of the (ε, L)-sequence (c_k) (see the discussion just before* (3.2)*). Then*

$$\#\{k \in \mathbb{Z} : \sigma_k > c\varepsilon\} \le 5N(\varepsilon) + 3,$$

where

$$c = \varepsilon_p 2^{1/p}(1 - 2^{-p'/p})^{-1/p'}. \tag{3.28}$$

Proof. Evidently

$$\#\{k \in \mathbb{Z} : c_i \in \overline{Z}_k \text{ for some } i,\ 1 \le i \le N\} \le 2N,$$

and for every $k \in \mathbb{Z}$ not included in the above set, $\overline{Z}_k \subset I_i = (c_i, c_{i+1})$ for some i, $1 \le i \le N$. Then by Corollary 19,

$$\#\{k \in \mathbb{Z} : \sigma_k > c\varepsilon\} \le 2N + 3(N + 1) = 5N + 3.$$

LEMMA 21. *Let ν_p equal $1/\sqrt{2}$ if $p = 2$ and $1/4$ otherwise, and let c be given by* (3.28). *Then for all $t > 0$,*

$$\#\{k \in \mathbb{Z} : \sigma_k > t\} \le 5\#\{k \in \mathbb{N} : a_k(T) \ge \nu_p t/c\} + 3.$$

Proof. By Theorem 9 of [4],

$$\#\{k \in \mathbb{N} : a_k(T) \ge \nu_p \varepsilon\} \ge N(\varepsilon).$$

Hence by Lemma 20,

$$\#\{k \in \mathbb{Z} : \sigma_k > t\} \le 5N(t/c) + 3 \le 5\#\{k \in \mathbb{N} : a_k(T) \ge \nu_p t/c\} + 3.$$

LEMMA 22. *For all $\alpha > 0$,*

$$\|(\sigma_k)\|^\alpha_{\ell^\alpha(\mathbb{Z})} \le 5(c/\nu_p)^\alpha \|(a_k(T))\|^\alpha_{\ell^\alpha(\mathbb{N})} + 3\|(\sigma_k)\|^\alpha_{\ell^\infty(\mathbb{Z})}.$$

Here the (quasi-) norms have their natural meaning, and c is given by (3.28).

Proof. By Proposition II.1.8 of [1],

$$\|(\sigma_k)\|^\alpha_{\ell^\alpha(\mathbb{Z})} = \alpha \int_0^\infty t^{\alpha-1} \#\{k \in \mathbb{Z} : \sigma_k > t\} \, dt.$$

Put $S = \|(\sigma_k)\|_{\ell^\infty(\mathbb{Z})}$. Then from Lemma 21 we have

$$\|(\sigma_k)\|^\alpha_{\ell^\alpha(\mathbb{Z})} = \alpha \int_0^S t^{\alpha-1} \#\{k \in \mathbb{Z} : \sigma_k > t\} \, dt$$
$$\le 5\alpha \int_0^\infty t^{\alpha-1} \#\{k \in \mathbb{N} : a_k(T) \ge \nu_p t/c\} \, dt + 3S^\alpha$$
$$= 5(c/\nu_p)^\alpha \|(a_k(T))\|^\alpha_{\ell^\alpha(\mathbb{N})} + 3S^\alpha.$$

COROLLARY 23. *For any $\alpha > 0$, there exists a constant C such that*

$$\|(\sigma_k)\|_{\ell^\alpha(\mathbb{Z})} \le C\|(a_k(T))\|_{\ell^\alpha(\mathbb{N})}.$$

Proof. By Corollary 14, $\|(\sigma_k)\|_{\ell^\infty(\mathbb{Z})} \asymp \|T\|$. Since

$$\|T\| = a_1(T) \le \|(a_k(T))\|_{\ell^\alpha(\mathbb{N})},$$

the result follows directly from Lemma 22.

To obtain an inequality reverse to that of Corollary 23, let $\varepsilon > 0$, let the intervals $I_k = [c_k, c_{k+1}]$ formed by the (ε, L)-sequence be grouped into families $\mathcal{F}_j$ $(j = 1, 2, \ldots)$ such that each $\mathcal{F}_j$ consists of the maximal number of those intervals satisfying the hypotheses of Lemma 12: they lie within (ξ_{k_0}, ξ_{k_2+1}) for some k_0, k_2 and the next interval I_k intersects Z_{k_2+1}. Lemma 12 and (2.25)–(2.28) tell us that there is a constant $c > 0$ such that

$$\varepsilon \# \mathcal{F}_j \le c \max_{k_0 \le k \le k_2} \sigma_k = c\sigma_{k_j}, \quad \text{say}.$$

Thus

$$\#\mathcal{F}_j \le [c\sigma_{k_j}/\varepsilon] = n_j \text{ (say)} = \sum_{n=1}^{n_j} 1,$$

and

$$N(\varepsilon) = N(\infty,\varepsilon) = \sum_j \#\mathcal{F}_j \le \sum_j \sum_{n=1}^{n_j} 1 \tag{3.29}$$
$$= \sum_{n=1}^{\infty} \sum_{j, n_j \ge n} 1 = \sum_{n=1}^{\infty} \#\{j : c\sigma_{k_j}/\varepsilon \ge n\}$$
$$\le \sum_{n=1}^{\infty} \#\{k : \sigma_k \ge n\varepsilon/c\}.$$

Thus if $(\sigma_k) \in \ell^\alpha$ for some $\alpha \in (1,\infty)$,

$$\alpha \int_0^\infty t^{\alpha-1} N(t)\,dt \tag{3.30}$$
$$\le \alpha \int_0^\infty \sum_{n=1}^{\infty} t^{\alpha-1} \#\{k : \sigma_k > nt/c\}\,dt$$
$$= \alpha c^\alpha \int_0^\infty \sum_{n=1}^{\infty} n^{-\alpha} s^{\alpha-1} \#\{k : \sigma_k > s\}\,ds \lesssim \|(\sigma_k)\|^\alpha_{\ell^\alpha(\mathbb{Z})}.$$

Moreover, by Theorem 9 of [4],

$$\#\{k \in \mathbb{N} : a_k(T) > \varepsilon\eta_p\} \le N(\varepsilon) + 1,$$

where $\eta_p = 1$ $(p \ne 2)$, $\eta_2 = 1/\sqrt{2}$. Hence

$$\|(a_k(T))\|^\alpha_{\ell^\alpha} = \alpha \int_0^\infty t^{\alpha-1} \#\{k \in \mathbb{N}_0 : a_k(T) > t\}\,dt$$
$$\le \alpha \int_0^{\|T\|} t^{\alpha-1}\{N(t/\eta_p) + 1\}\,dt$$
$$\lesssim \|(\sigma_k)\|^\alpha_{\ell^\alpha(\mathbb{Z})} + \|T\|^\alpha \lesssim \|(\sigma_k)\|^\alpha_{\ell^\alpha(\mathbb{Z})},$$

the last two steps following from (3.30) and Corollary 14.

We summarise this conclusion and that of Corollary 23 in the following

THEOREM 24. *Let $\alpha \in (1,\infty)$. Then*

$$\|(\sigma_k)\|_{\ell^\alpha(\mathbb{Z})} \asymp \|(a_k(T))\|_{\ell^\alpha(\mathbb{N})}.$$

A similar result can be obtained for the weak ℓ^α spaces ℓ^α_w ($\ell^{\alpha,\infty}$ in the Lorentz scale). We recall that $\ell^\alpha_w(\mathbb{Z})$ is the space of all sequences $x = (x_k)$ such that

$$\|x\|_{\ell^\alpha_w(\mathbb{Z})} := \sup_{t>0}(t(\#\{k \in \mathbb{Z} : |x_k| > t\})^{1/\alpha}),$$

and that $\|\cdot\|_{\ell^\alpha_w(\mathbb{Z})}$ is a norm on $\ell^\alpha_w(\mathbb{Z})$ when $\alpha > 1$. The space $\ell^\alpha_w(\mathbb{N})$ is defined analogously. Insofar as the σ_k are concerned, see the convention about admissible k's made after (3.22).

THEOREM 25. *Let $\alpha \in (1,\infty)$. Then*

$$\|(\sigma_k)\|_{\ell^\alpha_w(\mathbb{Z})} \asymp \|(a_k(T))\|_{\ell^\alpha_w(\mathbb{N})}.$$

Proof. First suppose that $\sigma = (\sigma_k) \in \ell^\alpha_w$. Then from (3.29),

$$t^\alpha N(t) \le \sum_{n=1}^{\infty} t^\alpha \#\{k : \sigma_k > n\varepsilon/c\} \le \sum_{n=1}^{\infty} \|\sigma\|^\alpha_{\ell^\alpha_w} (c/n)^\alpha \lesssim \|\sigma\|^\alpha_{\ell^\alpha_w}.$$

From Theorem 9 of [4],

$$\|(a_k(T))\|^\alpha_{\ell^\alpha_w} \lesssim \sup_{t>0} t^\alpha N(t) \lesssim \|\sigma\|^\alpha_{\ell^\alpha_w}.$$

Now suppose that $(a_k(T)) \in \ell^\alpha_w$. Lemma 20 and [4], Theorem 9, imply that

$$\#\{k : \sigma_k > t\} \le 5\#\{k : a_k(T) \ge \nu_p t/c\} + 3$$

and so

$$\sup_{t>0} t^\alpha \#\{k : \sigma_k > t\} \le 5 \sup_{t>0} t^\alpha \{\#\{k : a_k(T) \ge \nu_p t/c\} + 3\}.$$

Since $\#\{k : a_k(T) \ge \nu_p t/c\} \ge N(t/c) \ge 1$, it follows that

$$\sup_{t>0} t^\alpha \#\{k : \sigma_k > t\} \lesssim \sup_{t>0} t^\alpha \#\{k : a_k(T) \ge \nu_p t/c\},$$

and thus

$$\|(\sigma_k)\|_{\ell^\alpha_w(\mathbb{Z})} \lesssim \|(a_k(T))\|_{\ell^\alpha_w(\mathbb{N})}.$$

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Some Ramsey type theorems for normed and quasinormed spaces

by

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Abstract. We prove that every bounded, uniformly separated sequence in a normed space contains a "uniformly independent" subsequence (see definition); the constants involved do not depend on the sequence or the space. The finite version of this result is true for all quasinormed spaces. We give a counterexample to the infinite version in $L_p[0,1]$ for each $0 < p < 1$. Some consequences for nonstandard topological vector spaces are derived.

0. Introduction. We are concerned with the following problem: given a bounded sequence in a quasinormed space V whose terms are uniformly far apart, can we pass to a subsequence such that each term is uniformly far from the subspace spanned by the remaining terms?

If V is a normed space, it is well known that the answer is "yes". We strengthen this result by showing that the distance of each term from the subspace spanned by the other terms can be determined rather uniformly; in particular, it need not depend on the geometry of the given sequence. The finite version of this result turns out to be true for all quasinormed spaces, and it is tempting to conjecture that the infinite result is also true for all quasinormed spaces. However, we give a counterexample to this conjecture.

Before continuing the discussion we introduce some definitions and notation:

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