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Standard exact projective resolutions relative to a countable class of Fréchet spaces

by

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Abstract. We will show that for each sequence of quasinormable Fréchet spaces $(E_n)_{n\in\mathbb{N}}$ there is a Köthe space $\lambda(A)$ such that

$$\mathrm{Ext}^1(\lambda(A),\lambda(A))=\mathrm{Ext}^1(\lambda(A),E_n)=0$$

and there are exact sequences of the form

$$\ldots\to\lambda(A)\to\lambda(A)\to\lambda(A)\to\lambda(A)\to E_n\to 0.$$

If, for a fixed $n\in\mathbb{N}$, E_n is nuclear or a Köthe sequence space, the resolution above may be reduced to a short exact sequence of the form

$$0\to\lambda(A)\to\lambda(A)\to E_n\to 0.$$

The result has some applications in the theory of the functor Ext^1 in various categories of Fréchet spaces by providing a substitute for non-existing projective resolutions.

Let us recall that $\mathrm{Ext}^1(E,F)=0$ for Fréchet spaces E, F means that every short exact sequence

$$0\to F\xrightarrow{j} G\xrightarrow{q} E\to 0$$

of Fréchet spaces splits (i.e., q has a continuous linear right inverse). We will prove the following main result:

MAIN THEOREM. *Let (E_n) and (F_n) be two sequences of quasinormable Fréchet spaces. There exists a Köthe space $\lambda(A)$ such that:*

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(0) $\lambda(A) \oplus \lambda(A) \simeq \lambda(A, \lambda(A)) \simeq \lambda(A)$;
(1) $\mathrm{Ext}^1(\lambda(A), \lambda(A)) = 0$;
(2) $\mathrm{Ext}^1(\lambda(A), E_n) = 0$;
(3) *For every n there is an exact sequence*

$$(*) \qquad \ldots \to \lambda(A) \to \lambda(A) \to \lambda(A) \to \lambda(A) \to F_n \to 0.$$

(4) *For every F_n which is a reduced projective limit of Banach spaces l_1 we have a short exact sequence*

$$0 \to \lambda(A) \to \lambda(A) \to F_n \to 0.$$

Moreover, the space $\lambda(A)$ may be chosen Schwartz, whenever all F_n are Schwartz spaces.

The result is a far reaching refinement of an unpublished result of the second named author [Kr, 2.2.4]. In view of Th. 2.1 below the obtained theorem is optimal (recall that a quotient of a quasinormable space is quasinormable). Although we believe that the result is interesting in itself we explain some external motivations for it.

There are two main sources of motivation for the result. First of all, it provides a substitute for non-existing projective resolutions for the class of quasinormable Fréchet spaces. Let us recall that a locally convex space (lcs) X is called *projective* in the class $\mathcal{L}$ of lcs if for any $Y \in \mathcal{L}$ and any topological quotient map $q : Y \to X$, the operator q has a linear continuous right inverse. Moreover, the following topologically exact diagram is called a *projective resolution* of X in $\mathcal{L}$:

$$\ldots \to P_3 \to P_2 \to P_1 \to X \to 0$$

if P_i are projective in $\mathcal{L}$. Gejler proved [G1] (see also [G2]) that, contrary to the Banach case and the LB-space case (see [K1], comp. [D1]), in many classes of Fréchet spaces (like the classes of nuclear, Schwartz or Montel spaces) there are no infinite dimensional projective spaces. In particular, there are no projective resolutions. The latter fact produces an annoying asymmetry in the theory of the functor Ext^1 for Fréchet spaces (as developed in [P1], [V6] or [V5]): we may use injective resolutions but we cannot apply homological constructions based on projective resolutions.

Our result gives a resolution $(*)$ for any countable class $\mathcal{L} := \{F_n : n \in \mathbb{N}\}$ of quasinormable Fréchet spaces, and (if we take $E_n = F_n$) $(*)$ is "relatively projective" for $\mathcal{L}$ in the sense of condition (1) and (2), which suffices for applications.

The authors are mainly interested in applications to the theory of the functor Ext^1 in the category of "graded" Fréchet spaces [DV]. Using our Main Theorem we can obtain an essential ingredient of that theory: the fact that if $\mathrm{Ext}^1(E, F) = 0$, then $\mathrm{Ext}^1(E, G) = 0$ for any "graded" quotient G of F. The theory of Ext^1 for "graded" Fréchet spaces allows to give a proper splitting theory of short exact sequences containing spaces $C^\infty(\Omega)$ and a splitting theory of differential complexes. It is also a base of a structural theory of the space $C^\infty(\Omega)$. For more details see [DV].

The Main Theorem could also be applied to the classical theory of Ext^1 in the category of Fréchet spaces. For example, we can get an analogue of [V5, 1.1] (and the asymmetry mentioned above disappears). The fact that our resolution is "short" in case of locally-l_1-spaces amounts to the known equality $\mathrm{Ext}^2(E, F) = 0$ for E nuclear or locally projective [V5, 1.4] or [V6, 1.2, 1.6], which in turn implies some permanence properties for Ext^1 (see [V5, 1.5 and 1.7] or [V6, 1.6 and 1.7]). The theory of the functor Ext^1 for Fréchet spaces has proved its importance in the structure theory of Fréchet spaces (see [A2], [V2], [V3], [V5], [VW1], [VW2] etc.).

The second source of motivation comes from the problem whether a Fréchet space of a certain class is a quotient of a "nice" Köthe sequence space. This problem serves to reduce some questions on general Fréchet spaces to questions on Köthe spaces, where we have methods of calculations with matrices at our disposal. There are some known positive results ([A1], [VWd], [W]), in particular, it is known that each nuclear, Schwartz or quasinormable Fréchet space is a quotient of a nuclear, Schwartz, or quasinormable Köthe sequence space, respectively, see [W], [VWd] and [MV1].

We make another step in this direction: in our result F is a quotient of $\lambda(A)$, where also the corresponding kernel is "nice", in the locally projective case even isomorphic to $\lambda(A)$ and the spaces involved have nice splitting properties. Moreover, for any countable class of F we find a universal $\lambda(A)$. Our proof in the Schwartz locally projective case is modelled after a proof of a result of that type due to the second named author [Kr, 2.2.4].

Now, we summarize the content of the paper. In Section 0 we introduce some (known) definitions and in Section 1 we describe a construction of Köthe sequence spaces $\lambda_\Phi(\alpha)$. In Section 2 we prove a splitting theorem for $\lambda_\Phi(\alpha)$ and in Section 3 we construct some auxiliary short exact sequences containing $\lambda_\Phi(\alpha)$. Section 4 is devoted to the proof of the main result (based on results of Section 2 and 3) in the case of (F_n) being Schwartz spaces and projective limits of l_1 Banach spaces. Section 5 shows how the general case follows from the special one.

We should point out that our method of proof uses certain elements of the proof of [Kr, 2.2.4] and some other results from that unpublished paper of the second named author. Nevertheless, we give here a selfcontained proof of our main result.

0. Preliminaries. By an operator we always mean a linear continuous map. A Köthe space will be defined as follows:

$$\lambda(A) := \Big\{x = (x_i)_{i\in I} \in \mathbb{K}^I : \|x\|_k := \sum_{i\in I} |x_i| a_{i,k} < \infty\Big\},$$

where I is an arbitrary set and $A = (a_{i,k})$ is a matrix of positive numbers such that $a_{i,k} \le a_{i,k+1}$ for $i \in I$, $k \in \mathbb{N}$. In our definition $\lambda(A)$ always is a Fréchet space with a continuous norm. A Köthe sequence space is separable iff I is countable and then we assume $I = \mathbb{N}$. If $(E, (\|\cdot\|_n^E)_{n\in\mathbb{N}})$ is a Fréchet space, then

$$\lambda(A,E) := \Big\{x = (x_i) \in E^I : \|x\|_k^{\lambda(A,E)} := \sum_{i\in I} \|x_i\|_k^E a_{i,k} < \infty\Big\}.$$

We call $\lambda(A)$ *shift stable*, *stable* or *tensor stable* if

$$\lambda(A) \simeq \lambda(A) \times \mathbb{K}, \quad \lambda(A) \simeq \lambda(A) \times \lambda(A), \quad \lambda(A) \simeq \lambda(A, \lambda(A)),$$

respectively. If $\lambda(A)$ is tensor stable, then the Pełczyński decomposition method implies (see [V4, Lemma 1.1] applied to $A(E) := \lambda(A,E)$):

PROPOSITION 0.1. *Let $\lambda(A)$ be tensor stable. If E is a Fréchet space isomorphic to a complemented subspace of $\lambda(A)$, and if E contains a complemented subspace isomorphic to $\lambda(A)$, then $E \simeq \lambda(A)$.*

We will be interested in Köthe spaces of the form $\lambda(B) = l_1(J) \widetilde{\otimes}_\pi \lambda(A)$, where $\lambda(A)$ is a separable Köthe sequence space and

$$l_1(J) = \Big\{x = (x_i) \in \mathbb{K}^J : \|x\| := \sum_{i\in J} |x_i| < \infty\Big\}.$$

It is easily seen that for $I = J \times \mathbb{N}$ we have

$$B = (b_{j,n,k})_{(j,n)\in I, k\in\mathbb{N}}, \qquad b_{j,n,k} := a_{n,k}.$$

A matrix A is called *regular* if $I = \mathbb{N}$ and $a_{i,k}/a_{i,k+1}$ is decreasing as $i \to \infty$. A projective limit $\operatorname{proj}(E_n, i_n^k)$ is called *reduced* if

$$\forall k\ \exists l\ \forall m > l: \quad \overline{i_k(E)} \supseteq i_k^m(E_m),$$

where $i_k : E \to E_k$ is the canonical map. Finally, we call a sequence of Fréchet spaces and operators

$$\ldots \xrightarrow{T_k} G_k \xrightarrow{T_{k-1}} G_{k-1} \xrightarrow{T_{k-2}} G_{k-2} \to \ldots$$

exact if $\operatorname{im} T_l = \ker T_{l-1}$ for each $l \in \mathbb{N}$. For example,

$$(0.1) \qquad 0 \to F \xrightarrow{j} G \xrightarrow{q} E \to 0$$

is said to be *short* (*topologically*) *exact* if j is a topological embedding, q is a topological quotient map and $\ker q = \operatorname{im} j$.

One can find more about the functors Ext^k for Fréchet spaces in [V6], [V5], [MV] or [P1].

For other unexplained functional analytic notions see [J] and [K].

1. G_∞-spaces with increasing transition functions. From now on we denote by $\alpha = (\alpha_i)$ an increasing unbounded sequence of positive numbers and by $\Phi = (\phi_1, \phi_2, \ldots)$ an increasing sequence of functions $\phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$(1.1) \qquad r^2 \le \phi_1(r) \le \phi_2(r) \le \ldots \qquad \text{for any } r \in \mathbb{R}_+.$$

Let us define $\phi^n := \phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_1$. We call a Köthe sequence space $\lambda(A)$, where

$$a_{i,0} := \alpha_i, \qquad a_{i,k+1} := \phi_{k+1}(a_{i,k}),$$

a *G_∞-space* with increasing transition functions and we denote it by $\lambda_\Phi(\alpha)$.

Remark. We should point out that without changing the space $\lambda_\Phi(\alpha)$ we may assume that: (i) (1.1) holds only for large r (i.e., for $r > R$); (ii) (ϕ_i) are continuous; (iii) (ϕ_i) are strictly increasing; (iv) $\alpha_0 = 1$; (v) $\phi_i(1) = 1$.

There are two typical examples of G_∞-spaces with increasing transition functions:

(a) any power series space of infinite type $\Lambda_\infty(\overline{\alpha}) \simeq \lambda_\Phi(\alpha)$, where $\alpha_i := 2^{\overline{\alpha}_i}$, $\phi_i(r) := r^2$;

(b) any Dragilev space of infinite type $L_f(\overline{\alpha}, \infty) \simeq \lambda_\Phi(\alpha)$, where

$$\alpha_i := \exp(f(\overline{\alpha}_i)), \qquad \phi_i(r) := \exp(f(2f^{-1}(\log r))).$$

The following easy proposition summarizes elementary properties of $\lambda_\Phi(\alpha)$.

PROPOSITION 1.1. *Let α and Φ be as above.*

(1) *$\lambda_\Phi(\alpha)$ is a Schwartz space.*

(2) *$\lambda_\Phi(\alpha)$ is regular whenever $\phi_i(r)/r$ is increasing for each $i \in \mathbb{N}$.*

(3) *If $\sup_{n\in\mathbb{N}} \alpha_{n+1}/\alpha_n < \infty$, then $\lambda_\Phi(\alpha)$ and $l_1(J) \widetilde{\otimes}_\pi \lambda_\Phi(\alpha)$ are shift stable.*

(4) *If $\sup_{n\in\mathbb{N}} \alpha_{2n}/\alpha_n < \infty$, then $\lambda_\Phi(\alpha)$ and $l_1(J) \widetilde{\otimes}_\pi \lambda_\Phi(\alpha)$ are stable.*

(5) *If $\sup_{n\in\mathbb{N}} \alpha_{n^2}/\alpha_n < \infty$, then $\lambda_\Phi(\alpha)$ and $l_1(J) \widetilde{\otimes}_\pi \lambda_\Phi(\alpha)$ are tensor stable.*

Remark. We call sequences α satisfying conditions from (3), (4) and (5) *shift stable*, *stable* and *tensor stable*, respectively.

2. A splitting theory for $\lambda_\Phi(\alpha)$. We characterize those Fréchet spaces E for which $\operatorname{Ext}^1(\lambda_\Phi(\alpha), E) = 0$. The characterization will be given in terms of the so-called Ω-type conditions introduced in [VW1, Def. 3.2] (cf. [MV1]).

From now on $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function and $(U_n)_{n\in\mathbb{N}}$ is a decreasing 0-neighbourhood basis in E.

The space E satisfies the condition (Ω_ψ) if

$$\forall u\ \exists k\ \forall K\ \exists C>0\ \forall r>0: \quad U_k \subseteq C\psi(r)U_K + \frac{1}{r}U_u. \tag{2.1}$$

The space E satisfies the condition (Ω_Φ) if

$$\forall u\ \exists k\ \forall K\ \exists n, C>0\ \forall r>0: \quad U_k \subseteq C\phi^n(r)U_K + \frac{C}{r}U_u. \tag{2.2}$$

It is clear that (Ω_ψ) implies (Ω_Φ) whenever $\phi_1 \geq \psi$ and (Ω_Φ) implies (Ω_ψ) whenever

$$\forall n\ \exists R\ \forall r>R: \quad \psi(r) \geq \phi^n(r). \tag{2.3}$$

Thus we get by [MV1, Th. 7] and [V5, 5.11]:

THEOREM 2.1. *For any Fréchet space E the following conditions are equivalent.*

(1) *E is quasinormable.*
(2) *There is Φ such that E satisfies (Ω_Φ).*
(3) *There is ψ such that E satisfies (Ω_ψ).*
(4) *There is a non-Banach Fréchet space F such that* $\mathrm{Ext}^1(F,E)=0$.

The following theorem is the main result of the present section:

THEOREM 2.2 [Kr, 2.2.2]. *Let E be a Fréchet space and let α be shift stable. The following conditions are equivalent*:

(1) *The space E satisfies (Ω_Φ).*
(2) $\mathrm{Ext}^1(\lambda_\Phi(\alpha),E)=0$.
(3) $\mathrm{Ext}^1(l_1(J)\mathbin{\widetilde{\otimes}_\pi}\lambda_\Phi(\alpha),E)=0$.

For the sake of completeness we give the proof based on the following splitting result of Vogt [V6, 3.1, 3.4, 2.5] (comp. also [KrV]):

THEOREM 2.3. *Let $\lambda(A)$ be a Köthe sequence space and let E be an arbitrary Fréchet space.*

(a) $\mathrm{Ext}^1(\lambda(A),E)=0$, *whenever $(\lambda(A),E)$ satisfies the following condition (S_1)*:

$$\exists p\ \forall u\ \exists k\ \forall m,K,R>0\ \exists n,S\ \forall i\in\mathbb{N}: \quad a_{i,m}U_k \subseteq Sa_{i,n}U_K + \frac{a_{i,p}}{R}U_u. \tag{2.4}$$

(b) *If* $\mathrm{Ext}^1(\lambda(A),E)=0$, *then the pair $(\lambda(A),E)$ satisfies the following condition (S_2)*:

$$\forall u\ \exists k,p\ \forall m,K\ \exists n,S\ \forall i\in\mathbb{N}: \quad a_{i,m}U_k \subseteq Sa_{i,n}U_K + Sa_{i,p}U_u. \tag{2.5}$$

Proof of 2.2. We may assume that $\alpha_i \geq 1$, $\phi_i(1)=1$ and apply 2.3.

(2)⇒(1). We apply (S_2) for $\lambda(A)=\lambda_\Phi(\alpha)$ and $m=p+1$. Dividing (2.5) by $a_{i,m}$ we obtain

$$U_k \subseteq S\frac{a_{i,n}}{a_{i,p+1}}U_K + S\frac{a_{i,p}}{a_{i,p+1}}U_u.$$

Since $\phi_{p+1}(r)\geq r^2$ and $\alpha_i \geq 1$, we get $a_{i,p}/a_{i,p+1} \leq 1/a_{i,p} \leq 1/\alpha_i$ and $a_{i,p+1}\geq 1$. Moreover, shift stability implies that for large i we get

$$a_{i,n}=\phi^n(\alpha_i) \leq \phi^n(\alpha_{i-1}^2) \leq \phi^{n+1}(\alpha_{i-1}) = a_{i-1,n+1}.$$

Hence increasing S, we obtain, for all i,

$$U_k \subseteq Sa_{i-1,n+1}U_K + \frac{S}{\alpha_i}U_u.$$

Let $\alpha_{i-1}\leq r<\alpha_i$. Then $a_{i-1,n+1}\leq \phi^{n+1}(r)$ and we get

$$\forall u\ \exists k\ \forall K\ \exists n,S>0\ \forall r\geq \alpha_0: \quad U_k \subseteq S\phi^{n+1}(r)U_K + \frac{S}{r}U_u.$$

In order to get the same inclusion for all $r>0$, it suffices to increase S appropriately.

(1)⇒(3). We take $p=1$ and we choose k for any u according to (Ω_Φ). Then for K we find n,C again as in (Ω_Φ). We put $r:=CRa_{i,m}$ in (Ω_Φ):

$$U_k \subseteq C\phi^n(CRa_{i,m})U_K + \frac{1}{Ra_{i,m}}U_u.$$

Since $a_{i,m}/a_{i,m+1}\to 0$ as $i\to\infty$ (see Prop. 1.1(1)), for large i we obtain

$$\begin{aligned} a_{i,m}\phi^n(CRa_{i,m}) &\leq a_{i,m}\phi^n(a_{i,m+1}) \leq (\phi^n(a_{i,m+1}))^2 \\ &\leq \phi^{n+1}(a_{i,m+1}) \leq a_{i,m+n+2}. \end{aligned}$$

Hence putting S suitably large we obtain, for all i,

$$a_{i,m}U_k \subseteq Sa_{i,m+n+2}U_K + \frac{a_{i,p}}{R}U_u.$$

Now taking $B=(b_{j,i,k})_{(j,i)\in I,k\in\mathbb{N}}$, $I=J\times\mathbb{N}$, $b_{j,i,k}=a_{i,k}$, we obtain (S_1) for $(\lambda(B),E)$, where $\lambda(B)=l_1(J)\mathbin{\widetilde{\otimes}_\pi}\lambda_\Phi(\alpha)$.

(3)⇒(2). This is obvious, since $\lambda_\Phi(\alpha)$ is a complemented subspace of $l_1(J)\mathbin{\widetilde{\otimes}_\pi}\lambda_\Phi(\alpha)$.

COROLLARY 2.4 [Kr, 1.2.5 and 1.1.2]. *For arbitrary α and Φ as in Section 1 we have*

$$\mathrm{Ext}^1(\lambda_\Phi(\alpha),\lambda_\Phi(\alpha))=0 \quad \textit{and} \quad \mathrm{Ext}^1(l_1(J)\mathbin{\widetilde{\otimes}_\pi}\lambda_\Phi(\alpha), l_1(J)\mathbin{\widetilde{\otimes}_\pi}\lambda_\Phi(\alpha))=0.$$

Proof. We assume that $\alpha_0=1$ and $\phi_i(1)=1$. It suffices to show that both $\lambda_\Phi(\alpha)$ and $l_1(J)\mathbin{\widetilde{\otimes}_\pi}\lambda_\Phi(\alpha)$ satisfy (Ω_Φ). Since the two cases are nearly identical we consider only $\lambda_\Phi(\alpha)$.

We take $k = u+1$, $n = K$, $C = 1$ and $U_l := \{\xi = (\xi_i) : \|\xi\|_l := \sum_{i=0}^{\infty} \phi^l(\alpha_i)|\xi_i| \le 1\}$. Let $\alpha_{i_0} \le r < \alpha_{i_0+1}$, $\xi \in U_k$. We take $\xi = \eta + \zeta$ where

$$\eta := (\xi_1, \xi_2, \ldots, \xi_{i_0}, 0, \ldots), \quad \zeta := (0, 0, \ldots, 0, \xi_{i_0+1}, \xi_{i_0+2}, \ldots).$$

Obviously

$$\|\eta\|_K = \sum_{i=0}^{i_0} \phi^K(\alpha_i)|\xi_i| \le \phi^K(\alpha_{i_0}) \sum_{i=0}^{i_0} |\xi_i| \le \phi^K(r).$$

On the other hand, since $r\phi^u(\alpha_i) \le \alpha_i \phi^u(\alpha_i) \le (\phi^u(\alpha_i))^2 \le \phi^{u+1}(\alpha_i)$ for $i \ge i_0 + 1$, we get

$$\|\zeta\|_u = \sum_{i=i_0+1}^{\infty} \phi^u(\alpha_i)|\xi_i| = \sum_{i=i_0+1}^{\infty} \frac{\phi^u(\alpha_i)}{\phi^{u+1}(\alpha_i)} \phi^{u+1}(\alpha_i)|\xi_i| \le \frac{1}{r}.$$

3. Auxiliary short exact sequences. First we show that each locally projective Schwartz Fréchet space is a reduced projective limit of spaces $\lambda_\Phi(\alpha)$ for arbitrary Φ and suitably chosen α.

PROPOSITION 3.1. *Let (F_n) be a sequence of Schwartz Fréchet spaces which are reduced projective limits of l_1. Then there exists a tensor stable sequence α such that for each $n \in \mathbb{N}$, F_n is a reduced projective limit of $\lambda_\Phi(\alpha)$. In particular, there is a short exact sequence of the form*

$$0 \to F_n \to \prod_{i\in\mathbb{N}} \lambda_\Phi(\alpha) \to \prod_{i\in\mathbb{N}} \lambda_\Phi(\alpha) \to 0.$$

Proof. The last statement follows from the previous one by a Fréchet analogue of [V5, 1.3] or [V6, 1.1] (see also [P1, Cor. 5.1]).

Without loss of generality we may assume that each F_n is a reduced projective limit of spaces l_1 with compact linking maps.

Let $T : l_1 \to l_1$ be an arbitrary compact map. The image of the unit ball $T(B_{l_1})$ is relatively compact, thus contained in a closed absolutely convex hull of a null sequence (x_n). As easily seen, there is a real (monotonic) null sequence (α_n) such that $(\alpha_n^{-1}x_n)$ is still null. We take maps $R : l_1 \to l_1$ and $\sigma : l_1 \to l_1$ defined by

$$R(e_n) = \alpha_n^{-1}x_n \quad \text{and} \quad \sigma(e_n) = \alpha_n e_n.$$

Since $R\circ\sigma(B_{l_1}) \supseteq T(B_{l_1})$, there is a map $S : l_1 \to l_1$ such that $T = R\circ\sigma\circ S$. As is easily seen σ and hence also T factorize through a finite type power series space. Now, it suffices to show our proposition for a sequence (F_n) of Schwartz spaces $F_n = \lambda(A^n)$, where A^n are regular matrices.

We construct a tensor stable sequence α satisfying

$$(3.1) \qquad \forall n, k, l\ \exists C(n, k, l)\ \forall i \in \mathbb{N} : \quad \phi^l(\alpha_i) \le C(n, k, l)\frac{a_{i,k+1}^n}{a_{i,k}^n}.$$

First, we obtain easily a sequence $\beta = (\beta_i) \in c_0$ decreasing to zero and satisfying

$$\forall n, k\ \exists i(n, k)\ \forall i \ge i(n, k) : \quad \beta_i \ge \frac{a_{i,k}^n}{a_{i,k+1}^n}$$

and an increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\phi(1) = 1, \quad \forall k\ \exists C(k)\ \forall r \ge 1 : \quad \phi^k(r) \le C(k)\phi(r).$$

Now, we take simply $\alpha_i = \min(\phi^{-1}(1/\beta_i), 2\alpha_{2^{2^n}})$ for $2^{2^n} < i \le 2^{2^{n+1}}$.

A reduced projective spectrum representing F_n is given by $(\lambda_\Phi(\alpha))_{k\in\mathbb{N}}$ and linking maps $\sigma_k^{k+1} : \lambda_\Phi(\alpha) \to \lambda_\Phi(\alpha)$ are defined by

$$\sigma_k^{k+1}((x_i)_{i\in\mathbb{N}}) := ((a_{i,k}^n/a_{i,k+1}^n)x_i)_{i\in\mathbb{N}}.$$

The following map is an isomorphism:

$$W : \lambda(A^n) \to \operatorname{proj}(\lambda_\Phi(\alpha), \sigma_k^{k+1}) \subseteq \prod_{k\in\mathbb{N}} \lambda_\Phi(\alpha), \quad W((x_i)_{i\in\mathbb{N}}) := (a_{i,k}^n x_i)_{i,k}.$$

Indeed, W is continuous since

$$\|(a_{i,k}^n x_i)_{i\in\mathbb{N}}\|_p^{\lambda_\Phi(\alpha)} = \sum_{i=0}^{\infty} \phi^p(\alpha_i)|x_i|a_{i,k}^n \le C(n, k, p)\sum_{i=0}^{\infty} |x_i|a_{i,k+1}^n = C(n, k, p)\|(x_i)\|_{k+1}^{\lambda(A^n)}.$$

Moreover, W is open since

$$\|(x_i)\|_k^{\lambda(A^n)} = \sum_{i=0}^{\infty} a_{i,k}^n|x_i| \le \sum_{i=0}^{\infty} a_{i,k}^n\phi^p(\alpha_i)|x_i| = \|(a_{i,k}^n x_i)_{i\in\mathbb{N}}\|_p^{\lambda_\Phi(\alpha)}.$$

This completes the proof because the image of W contains the space of all finitely non-zero sequences which is dense in $\operatorname{proj}(\lambda_\Phi(\alpha))$.

PROPOSITION 3.2 [Kr, Proof of 2.2.4]. *Let α be tensor stable and let $\phi_i(r)/r$ increase monotonically for any $i \in \mathbb{N}$, $\Phi = (\phi_i)$. Then there exists a short exact sequence*

$$0 \to \lambda_\Phi(\alpha) \to \lambda_\Phi(\alpha) \to \lambda_\Phi(\alpha)^{\mathbb{N}} \to 0.$$

The first result of the type above was proved in [V1, proof of Lemma 1.6] for the space s. Now, there is a whole family of similar theorems very useful in the structural theory of Fréchet spaces (see [VW2, Th. 2.3 and 2.4], [V3], [V7, Th. 3.2]). Our result follows from the following theorem due to Apiola [A1, Prop. 2.3]:

THEOREM 3.3. *Let $\lambda(A)$ be a Schwartz Köthe sequence space with a regular matrix such that $(a_{i,k})_{i\in\mathbb{N}}$ increases for each $k \in \mathbb{N}$. Assume that there exists a bijection $\beta : \mathbb{N}^3 \to \mathbb{N}$ such that:*

(i) β *increases in each variable and* $j \le \beta(0,j,0)$.
(ii) $\forall k\ \exists p(k), C>0\ \forall n \le k$: $a_{\beta(n,j,i+1),k} \le C a_{\beta(n,j,i),p(k)}$.
(iii) $\forall k\ \exists p(k)$: $\sum_{i=0}^{\infty} a_{\beta(0,0,i),k}/a_{\beta(0,0,i),p(k)} < \infty$.
(iv) $\forall k\ \exists p(k), C>0\ \forall n \le k$: $a_{\beta(n,j,0),k} \le C a_{j,p(k)}$.

Then there exists a short exact sequence

$$0 \to \lambda(A) \to \lambda(A) \to \lambda(A)^{\mathbb{N}} \to 0.$$

Proof of 3.2. It suffices to check the assumptions of Th. 3.3 for $a_{i,k} = \phi^k(\alpha_i)$, $\alpha_0 = 1$.

First we have to define β. Assume that $\alpha_{2i} \le C\alpha_i$ for each $i \in \mathbb{N}$. For any i there is $l = 4^j$ such that $2C\alpha_i \le \alpha_l \le 2C^3\alpha_i$. We can construct inductively an increasing sequence $K_i = 4^{k_i}$, $K_0 = 0$, such that

$$(3.2)\qquad 2\alpha_{2K_i} \le \alpha_{K_{i+1}} \le S\alpha_{K_i} \quad \text{for } i \in \mathbb{N}, S = 2C^3.$$

We order all natural numbers not of the form $(2j+1)K_i$ for $i>0$ in an increasing sequence $(s(j))$. Obviously $j \le s(j) \le 2j+1$ for $j>0$. We define a bijection $\delta : \mathbb{N}^2 \to \mathbb{N}$:

$$\delta(i,j) := \begin{cases} s(j) & \text{for } i = 0,\ j \in \mathbb{N}, \\ (2j+1)K_i & \text{for } i \ge 1,\ j \in \mathbb{N}, \end{cases}$$

and

$$(3.3)\qquad \beta(n,j,i) := 2^n(2\delta(i,j)+1) - 1 = \begin{cases} 2^{n+1}s(j) + 2^n - 1 & \text{for } i = 0, \\ 2^{n+1}K_i(2j+1) + 2^n - 1 & \text{for } i \ge 1. \end{cases}$$

The condition (i) of 3.3 is obvious. In order to show (ii) we first observe that

$$(3.4)\qquad \beta(n,j,i+1) \le \max(8(\beta(n,j,0))^2, 2K_{i+1}^2).$$

Indeed, if $2^{n+1}(2j+1) \le K_{i+1}$, then $\beta(n,j,i+1) \le 2K_{i+1}^2$. Otherwise, if $2^{n+1}(2j+1) > K_{i+1}$, then $\beta(n,j,i+1) \le 8(\beta(n,j,0))^2$.

By the tensor stability of α we find a constant M such that

$$\alpha_{8(\beta(n,j,0))^2} \le M\alpha_{\beta(n,j,0)} \quad \text{and} \quad \alpha_{2K_{i+1}^2} \le M\alpha_{K_{i+1}}.$$

Since $\alpha_{K_i} \le \alpha_{\beta(n,j,i)}$, we find by (3.2) and (3.4) a constant $L>0$ such that

$$\alpha_{\beta(n,j,i+1)} \le L\alpha_{\beta(n,j,i)} \quad \text{for any } n,j,i.$$

Finally, for $\alpha_{\beta(n,j,i)} > L$ we get $a_{\beta(n,j,i+1),k} \le a_{\beta(n,j,i),k+1}$. Taking a suitable constant $C(k)$ we obtain (ii) for all $\beta(n,j,i)$.

We now prove (iii). Since $\phi_k(r)/r$ increases, $\phi_k(2r) \ge 2\phi_k(r)$ and $\phi^k(2r) \ge 2\phi^k(r)$. Moreover, $\beta(0,0,i) = 2K_i$ and, by (3.2),

$$\phi^k(\alpha_{\beta(0,0,i+1)}) = \phi^k(\alpha_{2K_{i+1}}) \ge \phi^k(2\alpha_{2K_i}) \ge 2\phi^k(\alpha_{2K_i}) = 2\phi^k(\alpha_{\beta(0,0,i)}).$$

Because $\phi_{k+1}(r) \ge r^2$, we obtain

$$\sum_{i=0}^{\infty} \frac{\phi^k(\alpha_{\beta(0,0,i)})}{\phi^{k+1}(\alpha_{\beta(0,0,i)})} \le \sum_{i=0}^{\infty} \frac{1}{\phi^k(\alpha_{\beta(0,0,i)})} \le \sum_{i=0}^{\infty} \frac{1}{2^i\phi^k(\alpha_{\beta(0,0,0)})} < \infty.$$

Finally, we prove (iv). We have $\beta(n,j,0) = 2^{n+1}s(j) + 2^n - 1 \le 2^{n+2}j + 2^{n+2}$. For $j>0$, $n \le k$ and some $M>0$ we have $\alpha_{\beta(n,j,0)} \le \alpha_{2^{n+3}j} \le M\alpha_j$. Thus for $\alpha_j \ge M$ we get

$$(3.5)\qquad \phi^k(\alpha_{\beta(n,j,0)}) \le \phi^{k+1}(\alpha_j).$$

For j such that either $\alpha_j < M$ or $j = 0$ we obtain (3.5) multiplying the right hand side by a suitable constant $C(k)$.

This completes the proof of 3.2.

4. Proof of the main result for locally projective Schwartz spaces. We assume that all F_n are reduced projective limits of l_1 spaces with compact linking maps (i.e., Schwartz spaces).

By Th. 2.1, we find easily $\Phi = (\phi_i)$, where $\phi_i(r)/r$ increases as r increases, such that each E_n and F_n satisfies (Ω_Φ). We define α according to Prop. 3.1. By 2.2 and 2.4,

$$(4.1)\qquad \operatorname{Ext}^1(\lambda_\Phi(\alpha), \lambda_\Phi(\alpha)) = 0, \quad \operatorname{Ext}^1(\lambda_\Phi(\alpha), E_n) = 0, \quad \operatorname{Ext}^1(\lambda_\Phi(\alpha), F_n) = 0.$$

Now, by 3.1 and 3.2, we obtain the first row and the last column of the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & F_n \arrow[r, "j_1"] & \lambda_\Phi(\alpha)^{\mathbb{N}} \arrow[r, "q_1"] \arrow[u] & \lambda_\Phi(\alpha)^{\mathbb{N}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F_n \arrow[r, "J_1"] \arrow[u, "\mathrm{id}"] & H \arrow[r, "Q_1"] \arrow[u, "Q_2"] & \lambda_\Phi(\alpha) \arrow[r] \arrow[u, "q_2"] & 0 \\
 & & \lambda_\Phi(\alpha) \arrow[r, "\mathrm{id}"] \arrow[u, "J_2"] & \lambda_\Phi(\alpha) \arrow[u, "j_2"] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $H := \{(x,y) \in \lambda_\Phi(\alpha)^{\mathbb{N}} \times \lambda_\Phi(\alpha) : q_1x = q_2y\}$,

$$J_1(x) := (j_1x, 0), \quad J_2(x) := (0, j_2x), \quad Q_1(x,y) := y, \quad Q_2(x,y) := x.$$

By (4.1), $H \simeq F_n \oplus \lambda_\Phi(\alpha)$. Similarly, we obtain another commutative diagram with exact rows and columns:

$$\begin{tikzcd}
& & 0 & 0 & \\
0 \arrow[r] & \lambda_\Phi(\alpha) \arrow[r] & \lambda_\Phi(\alpha) \oplus F_n \arrow[r] \arrow[u] & \lambda_\Phi(\alpha)^{\mathbb{N}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \lambda_\Phi(\alpha) \arrow[r] \arrow[u, "\mathrm{id}"'] & G \arrow[r] \arrow[u] & \lambda_\Phi(\alpha) \arrow[r] \arrow[u] & 0 \\
& & \lambda_\Phi(\alpha) \arrow[r, "\mathrm{id}"] \arrow[u] & \lambda_\Phi(\alpha) \arrow[u] & \\
& & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

By stability of α and (4.1), $G \simeq \lambda_\Phi(\alpha)$. Finally, we get a short exact sequence

$$0 \to \lambda_\Phi(\alpha) \to \lambda_\Phi(\alpha) \xrightarrow{q} \lambda_\Phi(\alpha) \oplus F_n \to 0.$$

Since $\operatorname{Ext}^1(\lambda_\Phi(\alpha), \lambda_\Phi(\alpha)) = 0$, the space $K := q^{-1}(F_n)$ is complemented in $\lambda_\Phi(\alpha)$ and

$$0 \to \lambda_\Phi(\alpha) \to K \xrightarrow{q|_K} F_n \to 0$$

is exact. Multiplying the above sequence by

$$0 \to \lambda_\Phi(\alpha) \xrightarrow{\mathrm{id}} \lambda_\Phi(\alpha) \to 0 \to 0$$

we obtain the sequence we are looking for (by Prop. 0.1, $K \oplus \lambda_\Phi(\alpha) \simeq \lambda_\Phi(\alpha)$). This completes the proof of the Main Theorem for locally projective Schwartz spaces.

We conclude this section with a simple consequence of the above case of the Main Theorem.

PROPOSITION 4.1. *If all F_n are of the form $l_1(J) \widetilde{\otimes}_\pi \lambda(D^n)$, where $\lambda(D^n)$ are Köthe Schwartz spaces, then the Main Theorem holds for $\lambda(A) = l_1(J) \widetilde{\otimes}_\pi \lambda(A^0)$, $\lambda(A^0)$ is a Köthe Schwartz space and the resolution $(*)$ is short as in (4).*

Proof. We apply the locally projective Schwartz case to the sequence of spaces $\lambda(D^n)$ instead of F_n. We find a Köthe Schwartz space $\lambda(A^0) = \lambda_\Phi(\alpha)$. Thus for any $n \in \mathbb{N}$ there exists a short exact sequence

$$0 \to \lambda(A^0) \xrightarrow{j} \lambda(A^0) \xrightarrow{q} \lambda(D^n) \to 0.$$

It is known ([J, 15.7.3]) that the tensored sequence

$$0 \to l_1(J) \widetilde{\otimes}_\pi \lambda(A^0) \xrightarrow{\mathrm{id} \otimes j} l_1(J) \widetilde{\otimes}_\pi \lambda(A^0) \to l_1(J) \widetilde{\otimes}_\pi \lambda(D^n) \to 0$$

is exact as well. Obviously (0) is satisfied for $\lambda(A) = l_1(J) \widetilde{\otimes}_\pi \lambda(A^0)$, whenever it is satisfied for $\lambda(A^0)$. By Th. 2.2 and Cor. 2.4, we get (1) and (2).

5. Proof in the general case. We will use the following two results:

THEOREM 5.1 (Vogt and Walldorf [VWd]). *Every Schwartz Fréchet space is isomorphic to a quotient of a Schwartz Köthe space.*

THEOREM 5.2 (Meise and Vogt [MV1, Prop. 7]). *Every quasinormable Fréchet space is isomorphic to a quotient of a space $l_1(J) \widetilde{\otimes}_\pi \lambda(D)$, where $\lambda(D)$ is a nuclear Köthe space.*

Proof of the Main Theorem. For every $n \in \mathbb{N}$ we define inductively, by use of 5.1 or 5.2, short exact sequences, setting $K_{n,0} = F_n$:

(5.1) $$0 \to K_{n,k+1} \to \lambda_{n,k} \to K_{n,k} \to 0,$$

where $\lambda_{n,k} = l_1(J) \widetilde{\otimes}_\pi \lambda(D^{n,k})$, $\lambda(D^{n,k})$ nuclear Köthe spaces, or (if all F_n are Schwartz spaces) $\lambda_{n,k}$ are Schwartz Köthe spaces.

We then apply the special case of the Main Theorem to the spaces E_n, $n \in \mathbb{N}$, on one side and $\lambda_{n,k}$, $n, k \in \mathbb{N}$ on the other side. We obtain $\lambda(A)$ fulfilling (0), (1) and (2) which in case of Schwartz spaces F_n is Schwartz as well. We proceed as follows. We set up the following diagram:

$$\begin{tikzcd}
& 0 & 0 & & \\
0 \arrow[r] & K_{n,k+1} \arrow[r] \arrow[u] & \lambda_{n,k} \arrow[r] \arrow[u] & K_{n,k} \arrow[r] & 0 \\
0 \arrow[r] & L_{n,k+1} \arrow[r] \arrow[u] & \lambda(A) \arrow[r] \arrow[u] & K_{n,k} \arrow[r] \arrow[u, "\cong"'] & 0 \\
& \lambda(A) \arrow[r] \arrow[u] & \lambda(A) \arrow[u] & & \\
& 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Here the upper row is (5.1), and the middle column is the short exact sequence obtained from (4).

For every $n, k \in \mathbb{N}$ we obtain exact sequences

$$0 \to \lambda(A) \to L_{n,k} \to K_{n,k} \to 0$$

and
$$0 \to L_{n,k+1} \to \lambda(A) \to K_{n,k} \to 0.$$
We use these to set up the following diagram (like the diagram (4.2)):
$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & \lambda(A) \arrow[r] & L_{n,k} \arrow[r] \arrow[u] & K_{n,k} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \lambda(A) \arrow[r] \arrow[u] & H_{n,k} \arrow[r] \arrow[u] & \lambda(A) \arrow[r] \arrow[u] & 0 \\
 & & L_{n,k+1} \arrow[r, "\mathrm{id}"] \arrow[u] & L_{n,k+1} \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$
Since the middle row splits on account of (1) and $\lambda(A) \oplus \lambda(A) \cong \lambda(A)$ we obtain for all $n, k \in \mathbb{N}$ an exact sequence
$$0 \to L_{n,k+1} \to \lambda(A) \to L_{n,k} \to 0.$$
Putting all these together we get a long exact sequence
$$\ldots \to \lambda(A) \xrightarrow{q_2} \lambda(A) \xrightarrow{q_1} \lambda(A) \xrightarrow{q_0} F_n \to 0,$$
where $\lambda(A)$ is either of the form $\lambda_\Phi(\alpha)$ or $l_1(J) \widetilde{\otimes}_\pi \lambda_\Phi(\alpha)$.

Now, assume that F_n is a reduced projective limit of Banach spaces l_1 and consider a short exact sequence
$$0 \to \ker q_0 \xrightarrow{j_0} \lambda(A) \xrightarrow{q_0} F_n \to 0.$$
We then obtain an exact sequence of the form (see the condition (III) in [V6] or [P1, p. 49])
$$\begin{aligned}
0 &\to L(F_n, \ker q_1) \to L(\lambda(A), \ker q_1) \to L(\ker q_0, \ker q_1) \\
&\to \mathrm{Ext}^1(F_n, \ker q_1) \to \mathrm{Ext}^1(\lambda(A), \ker q_1) \to \mathrm{Ext}^1(\ker q_0, \ker q_1) \\
&\to \mathrm{Ext}^2(F_n, \ker q_1) \to \mathrm{Ext}^2(\lambda(A), \ker q_1) \to \ldots
\end{aligned}$$
Since $\lambda(A)$ has property (Ω_Φ) which is inherited by quotients, the space $\ker q_1 = \operatorname{im} q_2$ has it as well. Thus, by [V6, Cor. 1.5] and Th. 2.2,
$$\mathrm{Ext}^2(F_n, \ker q_1) = 0 \quad \text{and} \quad \mathrm{Ext}^1(\lambda(A), \ker q_1) = 0.$$
Hence $\mathrm{Ext}^1(\ker q_0, \ker q_1) = 0$ and the sequence
$$0 \to \ker q_1 \to \lambda(A) \xrightarrow{q_1} \ker q_0 \to 0$$

splits. This means that $\ker q_0$ is isomorphic to a complemented subspace of $\lambda(A)$. Finally,
$$0 \to \ker q_0 \oplus \lambda(A) \xrightarrow{j_0 \oplus \mathrm{id}} \lambda(A) \oplus \lambda(A) \xrightarrow{q_0 \oplus 0} F_n \to 0$$
is the short exact sequence we are looking for because $\ker q_0 \oplus \lambda(A) \simeq \lambda(A)$, by Prop. 0.1.

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The splitting spectrum differs from the Taylor spectrum

by

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Abstract. We construct a pair of commuting Banach space operators for which the splitting spectrum is different from the Taylor spectrum.

Let $A_1,\ldots,A_n$ be mutually commuting operators in a Banach space X. The Koszul complex of the n-tuple $(A_1,\ldots,A_n)$ is the complex

$$0 \longrightarrow \Lambda^0(X,e) \xrightarrow{\delta_0} \Lambda^1(X,e) \xrightarrow{\delta_1} \ldots \xrightarrow{\delta_{n-1}} \Lambda^n(X,e) \longrightarrow 0$$

where $\Lambda^p(X,e)$ denotes the vector space of all forms of degree p in indeterminates $e_1,\ldots,e_n$ with coefficients in X and the linear mappings $\delta_p : \Lambda^p(X,e) \to \Lambda^{p+1}(X,e)$ are defined by

$$\delta^p(xe_{i_1}\wedge\ldots\wedge e_{i_p}) = \sum_{j=1}^n A_jxe_j\wedge e_{i_1}\wedge\ldots\wedge e_{i_p}.$$

It is well known that $\delta_{p+1}\delta_p = 0$ for every p. The *Taylor spectrum* $\sigma_{\mathrm T}(A_1,\ldots,A_n)$ is the set of all n-tuples $(\lambda_1,\ldots\lambda_n)$ of complex numbers for which the Koszul complex of $(A_1-\lambda_1,\ldots,A_n-\lambda_n)$ is not exact [5].

Instead of the Taylor spectrum it is sometimes useful to use the following variation (see e.g. [1], [3], [4]). We say that the n-tuple $(A_1,\ldots,A_n)$ is *splitting-regular* if its Koszul complex is exact and the ranges of the operators δ_p are complemented in $\Lambda^{p+1}(X,e)$. Equivalently, there exist operators $\varepsilon_p : \Lambda^{p+1}(X,e)\to\Lambda^p(X,e)$ ($p=0,\ldots,n-1$) such that $\varepsilon_p\delta_p+\delta_{p-1}\varepsilon_{p-1}$ is the identity operator on $\Lambda^p(X,e)$ for $p=0,\ldots,n$ (formally we set $\delta_{-1}=\delta_n=0$). The *splitting spectrum* $\sigma_{\mathrm S}(A_1,\ldots,A_n)$ is the set of all $(\lambda_1,\ldots,\lambda_n)\in\mathbb C^n$ such that the n-tuple $(A_1-\lambda_1,\ldots,A_n-\lambda_n)$ is not splitting-regular.

The splitting spectrum has similar properties as the Taylor spectrum. Clearly, $\sigma_{\mathrm T}(A_1,\ldots,A_n)\subset\sigma_{\mathrm S}(A_1,\ldots,A_n)$. For Hilbert space operators these two spectra coincide and the same is true for n-tuples of operators in ℓ_1

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